The quantum differential equation of the Hilbert scheme of points in the plane

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1 Introduction

1.1 Overview

In the study of the quantum cohomology of the Hilbert scheme of points in the plane [11], as well as in the Gromov-Witten/Donaldson-Thomas theories of threefolds [2, 10, 12], certain linear ODEs with remarkable properties arise naturally. These ODEs generalize the Schrödinger equation for the quantum Calogero-Sutherland operator and are the focus of the present paper. Following the tradition in their field of origin, we call them quantum differential equations, or QDEs for short, even though there is nothing quantum per se in these differential equations. They are linear ODEs with regular singularities, very classical objects indeed.

Two special values of the independent variable q play a special role. These are $q = -1$ and $q = 0$, which may be called the Gromov-Witten and Donaldson-Thomas points, respectively. The point $q = -1$ is nonsingular. We prove that the monodromy based at $q = -1$ is a polynomial in $e^{2\pi i t_i}$, where t_1 and t_2 are the parameters of the QDE.

Next we solve the connection problem for the points $q = -1$ and $q = 0$. The point $q = 0$ is singular with the residue being essentially the Calogero-Sutherland operator. Its eigenfunctions are known as the Jack symmetric functions. We prove that transported to $q = -1$ by the QDE, these become, up to normalization, Macdonald polynomials with parameters $e^{2\pi i t_i}$.

While both the objects and the results of this paper belong to the world of combinatorics and differential equations, our proofs require geometric input at several key points. Perhaps a deeper understanding of the integrable structures underlying the QDEs will lead to more direct proofs.

The appearance of Macdonald polynomials in the connection problem strongly suggests a relation to the equivariant K-theory of the Hilbert scheme. This relation is further pursued in [1].

1.2 Fock space and symmetric functions

The most natural way to write down our ODEs is in terms of creation and annihilation operators acting on the Fock space.

We follow the conventions of [11]. By definition, the Fock space $\mathcal{F}$ is freely generated over $\mathbb{C}$ by commuting creation operators α_{-k}, $k \in \mathbb{Z}_{>0}$, acting on the vacuum vector $v_\emptyset$. The annihilation operators α_k, $k \in \mathbb{Z}_{>0}$, kill the vacuum

$$\alpha_k \cdot v_\emptyset = 0, \quad k > 0\,,$$

and satisfy the commutation relations

$$[\alpha_k, \alpha_l] = k\,\delta_{k+l}\,.$$

A natural basis of $\mathcal{F}$ is given by the vectors

$$|\mu\rangle = \frac{1}{\mathfrak{z}(\mu)} \prod \alpha_{-\mu_i}\, v_\emptyset\,. \tag{1}$$

indexed by partitions μ. Here,

$$\mathfrak{z}(\mu) = |\operatorname{Aut}(\mu)| \prod \mu_i$$

is the usual normalization factor.

The linear map

$$p_\mu \mapsto \mathfrak{z}(\mu)\ |\mu\rangle\ , \tag{2}$$

where

$$p_\mu = \prod_k \sum_{i=1}^{\infty} z_i^{\mu_k}\,,$$

identifies $\mathcal{F}$ with symmetric functions of the variables

$$z_1, z_2, z_3, \dots\,.$$

Symmetric functions of fixed degree form eigenspaces of the *energy operator*:

$$|\cdot| = \sum_{k>0} \alpha_{-k}\,\alpha_k\,.$$

1.3 The QDE

The central object of this paper is the differential equation

$$q\frac{d}{dq}\,\Psi = \mathsf{M}\,\Psi\,, \quad \Psi \in \mathcal{F}\,, \tag{3}$$

where the operator M is given by

$$\begin{aligned}\mathsf{M}(q,t_1,t_2) = (t_1+t_2)\sum_{k>0}\frac{k}{2}\frac{(-q)^k+1}{(-q)^k-1}\,\alpha_{-k}\,\alpha_k + \\ \frac{1}{2}\sum_{k,l>0}\Big[t_1t_2\,\alpha_{k+l}\,\alpha_{-k}\,\alpha_{-l} - \alpha_{-k-l}\,\alpha_k\,\alpha_l\Big]\,.\end{aligned} \tag{4}$$

The variables t_1 and t_2 in (4) are parameters. Note that the q-dependence of M is only in the first sum in (4) which acts diagonally in the basis (1). The two terms in the second sum in (4) are known respectively as the splitting and joining terms.

The operator M commutes with the energy operator $|\cdot|$, therefore the equation splits into a direct sum of finite-dimensional ODEs. It is more convenient to study the equivalent equation

$$q\frac{d}{dq}\,\Psi = \mathsf{M}_D\Psi\,, \quad \Psi \in \mathcal{F}\,, \tag{5}$$

where

$$\mathsf{M}_D = \mathsf{M} - \frac{t_1+t_2}{2}\,\frac{(-q)+1}{(-q)-1}\,|\cdot|\,. \tag{6}$$

The equation (5) is the quantum differential equation for the Hilbert scheme of points, see [11]. One advantage of (5) is that $q=-1$ is a regular point for (5).

In this paper, we normalize everything exactly as in [11] even though some simplifications such as replacing q by $-q$ in (4) may seem obvious.

1.4 Calogero-Sutherland operator

The quantum-mechanical Calogero-Sutherland operator,

$$\mathsf{H}_{CS} = \frac{1}{2}\sum_i\left(z_i\frac{\partial}{\partial z_i}\right)^2 + \theta(\theta-1)\sum_{i<j}\frac{1}{|z_i-z_j|^2}\,, \tag{7}$$

describes particles moving on the torus $|z_i| = 1$ interacting via the potentials $|z_i - z_j|^{-2}$. The parameter θ adjusts the strength of the interaction. The function

$$\phi(z) = \prod_{i<j} (z_i - z_j)^\theta$$

is an eigenfunction of H_{CS}, and the operator $\phi\, \mathsf{H}_{CS}\, \phi^{-1}$ preserves the space of symmetric polynomials in the variables z_i. Therefore, via the identification (2), the operator $\phi\, \mathsf{H}_{CS}\, \phi^{-1}$ acts on the Fock space.

A direct computation shows the operator $\phi\, \mathsf{H}_{CS}\, \phi^{-1}$ equals

$$\Delta_{CS} = \frac{1-\theta}{2} \sum_k k\, \alpha_{-k}\, \alpha_k + \frac{1}{2} \sum_{k,l>0} \Big[\alpha_{-k-l}\, \alpha_k\, \alpha_l + \theta\, \alpha_{k+l}\, \alpha_{-k}\, \alpha_{-l} \Big] \qquad (8)$$

modulo scalars and a multiple of the momentum operator $\sum_i z_i \frac{\partial}{\partial z_i}$, see [13]. We find

$$\mathsf{M}(0) = -\, t_1^{\ell(\,\cdot\,)+1}\, \Delta_{CS}\Big|_{\theta=-t_2/t_1} t_1^{-\ell(\,\cdot\,)}\,, \qquad (9)$$

where $\ell(\mu)$ is the number of parts of the partition μ and $\ell(\,\cdot\,)$ is the diagonal operator with eigenvalues $\ell(\mu)$ in the basis $|\mu\rangle$.

The well-known duality $\theta \mapsto 1/\theta$ in the Calogero-Sutherland model corresponds to the permutation of t_1 and t_2.

Formula (9) implies the behavior of (3) near the regular singularities $q = 0, \infty$ is described by the Schrödinger equation for (7). The connection problem for these two points may be viewed as a scattering produced by nonstationary terms in (3). This problem will be considered in Section 4.6.

1.5 An application

The solution of the connection problem may be combined with the results of [10] to give a box-counting formula for triple Hodge integrals. While we will not reproduce the required formulas here, the essence summarized as follows.

In [10] a certain enumerative object, called the *capped vertex* was introduced and shown to satisfy the GW/DT correspondence. Its ingredients on the GW and DT sides are quite different. On the GW side, it encapsulates the general triple Hodge integrals, repackaged using fundamental solutions of our QDE, normalized at $q = -1$.

On the DT side, the expression is essentially combinatorial, its main ingredient being a certain weighted count of 3-dimensional partitions known as the *equivariant vertex*. It is similarly decorated by the fundamental solutions of the QDE, but this time normalized at $q = 0$. Therefore, it is precisely the connection formulas for the QDE that relate the two expressions.

1.6 Acknowledgments

We thank R. Bezrukavnikov, P. Etingof, N. Katz, D. Maulik, N. Nekrasov, and A. Oblomkov for valuable discussions. The results presented here date back several years. See, in particular, [1] for further developments.

Both authors were partially supported by the Packard foundation and the NSF.

2 Basic properties

2.1 Singularities

For $\Psi \in \mathcal{F}$ of energy n, the equation (5) is a linear first order ODE in $p(n)$ unknowns, where $p(n)$ is the number of partitions of n. It has regular singularities. These are $q = 0, \infty$, and solutions ζ of

$$(-\zeta)^m = 1, \quad m = 2, \dots, n\,,$$

excluding $q = -1$.

For example, for $n = 3$, in the basis (1) ordered lexicographically, the matrix M_D takes the form

$$\begin{bmatrix} 3(t_1+t_2)\dfrac{q^2-1}{q^2-q+1} & -3 & 0 \\ 2t_1t_2 & (t_1+t_2)\dfrac{q+1}{q-1} & -1 \\ 0 & 3t_1t_2 & 0 \end{bmatrix}.$$

2.2 Residues at $q = 0$

Equation (9) relates the residue of (5) at $q = 0$ to the Calogero-Sutherland operator. In particular, the eigenfunctions of $\mathsf{M}(0)$ are, up to normalization, Jack symmetric functions.

More precisely, let $J_\lambda \in \mathcal{F}$ be integral form of the Jack symmetric function depending on the parameter $\alpha = 1/\theta$ as in [8]. We define

$$\mathsf{J}^\lambda = t_2^{|\lambda|}\, t_1^{\ell(\cdot)}\, J_\lambda\Big|_{\alpha=-t_1/t_2}\,. \tag{10}$$

These are eigenfunctions of $\mathsf{M}_D(0)$ with eigenvalues

$$-c(\lambda; t_1, t_2) = -\sum_{(i,j)\in\lambda} \Big[(j-1)t_1 + (i-1)t_2\Big]\,. \tag{11}$$

The normalization is such that

$$\mathsf{J}^\lambda = |\lambda|!\,(t_1 t_2)^{|\lambda|}\,\left|1^{|\lambda|}\right\rangle + \dots\,. \tag{12}$$

For example, we have

$$\mathsf{J}^{(k)} = k!\, t_1^k \sum_{|\mu|=k} (-1)^{k-\ell(\mu)}\, t_2^{\ell(\mu)}\, |\mu\rangle\,.$$

In general, the coefficient of $|\mu\rangle$ in the expansion of J^λ is $(t_1 t_2)^{\ell(\mu)}$ times a polynomial in t_1 and t_2 of degree $|\lambda| - \ell(\mu)$.

Geometrically, J^λ correspond to classes of monomial ideals in the equivariant cohomology of the Hilbert scheme of points, see [14, 7]. From this point of view, the symmetry

$$\mathsf{J}^\lambda(t_2, t_1) = \mathsf{J}^{\lambda'}(t_1, t_2), \tag{13}$$

where λ' is the transpose of λ, is obvious.

By the general theory of ODEs, we can construct a solution

$$\Psi = \mathsf{Y}^\lambda(q)\, q^{-c(\lambda;t_1,t_2)}\,, \quad \mathsf{Y}^\lambda(q) \in \mathbb{C}[[q]]$$

of (5) which converges for $|q| < 1$ and satisfies

$$\mathsf{Y}^\lambda(0) = \mathsf{J}^\lambda\,.$$

The symmetry (13) generalizes to

$$\mathsf{Y}^\lambda(t_2, t_1) = \mathsf{Y}^{\lambda'}(t_1, t_2)\,.$$

2.3 Other residues

We find

$$\mathsf{M}_D(q^{-1}) = -(-1)^{\ell(\,\cdot\,)}\,\mathsf{M}_D(q)\,(-1)^{\ell(\,\cdot\,)}\,. \tag{14}$$

Therefore, the exponents at $q = \infty$ are the same as exponents at $q = 0$.

If $\zeta \neq -1$ is a root of unity, then

$$\operatorname{Res}_{q=\zeta} q^{-1}\,\mathsf{M}_D(q) = (t_1 + t_2) \sum_{\{k|(-\zeta)^k=1\}} \alpha_{-k}\,\alpha_k\,, \tag{15}$$

which is diagonal in the basis (1). In particular, the exponents at the roots of unity are positive integer multiples of the number $(t_1 + t_2)$, which we call the *level* of the equation.

For positive integer levels, the solutions will be seen to be regular at the roots of unity, implying that the Y^λ are polynomial.

2.4 Unitarity

The standard inner product on cohomology induces a certain bilinear inner product on $\mathcal{F}$ with respect to which M is symmetric.

As will be explained below, it is more convenient for us to work with a certain Hermitian inner product — an inner product *antilinear* in the second entry. This means that

$$\langle af, g\rangle = a\,\langle f, g\rangle\ , \quad a \in \mathbb{C}(t_1, t_2)$$

and

$$\langle f, g\rangle = \overline{\langle g, f\rangle}\,,$$

where, by definition

$$\overline{a(t_1, t_2)} = a(-t_1, -t_2)\,.$$

Specifically, we consider the Hermitian product defined on basis vectors by

$$\langle \mu|\nu\rangle = \frac{1}{(t_1t_2)^{\ell(\mu)}}\,\frac{\delta_{\mu\nu}}{\mathfrak{z}(\mu)}\,. \tag{16}$$

We then have

$$(\alpha_k)^* = (t_1t_2)^{\operatorname{sgn}(k)}\,\alpha_{-k}\,, \tag{17}$$

and hence M is skew-Hermitian:

$$\mathsf{M}^* = -\mathsf{M}\ . \tag{18}$$

The following is an immediate corollary

Proposition 1. *The connection*

$$\nabla(t_1,t_2) = q\frac{d}{dq} - \mathsf{M}_D(q;t_1,t_2)$$

is unitary.

The polynomials J^λ are orthogonal with respect to (16). Their norm is given by the product of the tangent weights:

$$\|\mathsf{J}^\lambda\|^2 = \prod_{\text{tangent weights } \mathsf{w}} \mathsf{w} \tag{19}$$

We recall that the tangent weights w to the Hilbert scheme at the monomial ideal indexed by λ are given by

$$\{\mathsf{w}\} = \left\{t_1(a(\square)+1) - t_2 l(\square), -t_1 a(\square) + t_2(l(\square)+1)\right\}_{\square\in\lambda}, \tag{20}$$

where $a(\square)$ and $l(\square)$ denote the arm-length $\lambda_i - j$ and leg-length $\lambda'_j - i$ of a square $\square = (i,j)$ in a the diagram λ. Proposition (1) implies the following

Corollary 2.

$$\left\langle \mathsf{Y}^\lambda(q), \mathsf{Y}^\mu(q)\right\rangle = \delta_{\lambda\mu}\,\|\mathsf{J}^\lambda\|^2 .$$

Proposition 1 may also be phrased as a relation between the fundamental solutions on opposite levels.

There is a geometric reason why the Hermitian inner product (16) is preferred. As explained in [10], it is natural do define inner products on $\mathcal{F}$ as equivariant GW/DT invariants of $\mathbf{P}^1 \times \mathbb{C}^2$ relative to the fibers over $0,\infty \in \mathbf{P}^1$. In that setting, our parameters t_1 and t_2 correspond to the ratios of torus weights in $\mathbb{C}^2$-direction to the weight in the $\mathbf{P}^1$-direction. But, clearly, the tangent spaces to $\mathbf{P}^1$ at 0 and ∞ have *opposite* torus weights.

3 Monodromy

3.1 Intertwiners

3.1.1

A crucial role in what follows will be played by certain operators

$$\mathsf{S}(a,b) \in \operatorname{End}(\mathcal{F}) \otimes \mathbb{Q}(q,t_1,t_2), \quad (a,b)\in\mathbb{Z}^2,$$

that intertwine the monodromy of $\nabla(t_1,t_2)$ and $\nabla(t_1-a,t_2-b)$, that is, satisfy

$$\nabla(t_1,t_2)\,\mathsf{S}(a,b) = \mathsf{S}(a,b)\,\nabla(t_1-a,t_2-b)\,. \tag{21}$$

The construction of these operators is geometric and is provided by the Gromov-Witten and Donaldson-Thomas theories of local curves. The equivalence of these two theories, a very special case of general GW/DT conjectures of [9], was proven in [12]. Specifically, the GW/DT partition function of the total space of the $\mathcal{O}(a)\oplus\mathcal{O}(b)$ bundle over $\mathbf{P}^1$, relative the fibers over $0,\infty\in\mathbf{P}^1$ defines an operator on $\mathcal{F}$. Up to normalization, these are the intertwiners S.

A formula for the intertwiners S in terms of the fundamental solution of QDE may be derived using equivariant localization in GW and DT theories, respectively. We refer to [12] for technical details and state only the final result here.

3.1.2

Let $\Phi(q;t_1,t_2)$ be the fundamental solution of the QDE

$$q\frac{d}{dq}\,\Phi(q;t_1,t_2) = \mathsf{M}_D\,\Phi(q;t_1,t_2)\,, \tag{22}$$

normalized by

$$\Phi(-1;t_1,t_2)=1\,.$$

Denote

$$g(x,t)=\frac{x^{tx}}{\Gamma(tx)}\,,\quad x>0\,.$$

and let G_{GW} denote the diagonal operator with the following eigenvalues

$$\mathsf{G}_{\mathrm{GW}}(t_1,t_2)\,|\mu\rangle = \prod_i g(\mu_i,t_1)\,g(\mu_i,t_2)\,|\mu\rangle\,. \tag{23}$$

3.1.3

Let Y denote the matrix formed by the vectors Y^λ. Let $\mathsf{G}_{\mathrm{DT}}(t_1,t_2)$ be the diagonal matrix with eigenvalues

$$q^{-c(\lambda,t_1,t_2)}\prod_{\text{tangent weights } \mathsf{w}}\frac{1}{\Gamma(\mathsf{w}+1)}\,,$$

where $c(\lambda, t_1, t_2)$ is the sum of (t_1, t_2)-contents defined in (11) and the product ranges over the tangent weights w to the Hilbert scheme at the monomial ideal labeled by the partition λ as in (20). We fix a branch of the multivalued functions $q^{-c(\lambda,t_1,t_2)}$ by

$$q^{-c(\lambda,t_1,t_2)}\Big|_{q=-1} = e^{\pi i c(\lambda,t_1,t_2)} \tag{24}$$

The matrix $\mathsf{Y}\,\mathsf{G}_{\mathrm{DT}}$ is a solution of the QDE with a particular normalization near $q = 0$.

3.1.4

Equivariant localization of relative GW/DT invariant of $\mathcal{O}(a) \oplus \mathcal{O}(b)$ yields the following result

Theorem 1. *For $(a, b) \in \mathbb{Z}^2$, there exists $\mathsf{S}(a, b) \in \mathrm{End}(\mathcal{F}) \otimes \mathbb{Q}(q, t_1, t_2)$ such that*

$$\begin{aligned} \mathsf{S}(a, b) &= \Phi(t_1, t_2)\, \mathsf{G}_{\mathrm{GW}}(t_1, t_2)\, \mathsf{G}_{\mathrm{GW}}(t_1 - a, t_2 - b)^{-1}\, \Phi(t_1 - a, t_2 - b)^{-1} \\ &= \mathsf{Y}(t_1, t_2)\, \mathsf{G}_{\mathrm{DT}}(t_1, t_2)\, \mathsf{G}_{\mathrm{DT}}(t_1 - a, t_2 - b)^{-1}\, \mathsf{Y}(t_1 - a, t_2 - b)^{-1}\,. \end{aligned} \tag{25}$$

3.1.5

Formulas (25) are derived as follows. Let $\mathsf{Z}'_{\mathrm{DT}}(\lambda, \mu)$ denote the reduced Donaldson-Thomas partition function of the total space of

$$\mathcal{O}(a) \oplus \mathcal{O}(b) \to \mathbf{P}^1,$$

relative to the fibers over $0, \infty \in \mathbf{P}^1$. The partitions λ and μ record the tangency to these fibers. The degree $d = |\lambda| = |\mu|$ of the curve is implicit in the notation. By definition $\mathsf{Z}'_{\mathrm{DT}}(\lambda, \mu)$ is a generating function counting 1-dimensional subschemes C with weight $q^{\chi(\mathcal{O}_C)}$. By Theorem 3 of [12],

$$(-q)^{-d(2+a+b)/2}\, \mathsf{Z}'_{\mathrm{DT}}(\lambda, \mu) = (-iu)^{d(a+b)+\ell(\lambda)+\ell(\mu)}\, \mathsf{Z}'_{\mathrm{GW}}(\lambda, \mu) \tag{26}$$

after the change of variables $q = -e^{iu}$. Here, $\mathsf{Z}'_{\mathrm{GW}}(\lambda, \mu)$ is the generating function for the Gromov-Witten counts weighting genus g curves by u^{2g-2}. Theorem 2 of [12] shows $\mathsf{Z}'_{\mathrm{DT}}(\lambda, \mu)$ is a rational function of q.

The equality (26) is proven in [12] in equivariant cohomology with respect to the fiberwise $(\mathbb{C}^*)^2$-action. A stronger result that involves the full $(\mathbb{C}^*)^3$ of automorphisms is proven in Proposition 1 of [10].

Both sides of (26) may be computed by equivariant localization, see for example, Section 2 of [10] for a summary. The formula involves the *rubber integrals* as well as certain *edge weights*. The relation of rubber integrals to the fundamental solution of our QDE, and hence to the matrix Y, is discussed in Section 11.2 of [12]. (Note a scalar difference between operators M and M_D.) Similarly, Gromov-Witten rubber integrals lead to the matrix Ψ.

The Donaldson-Thomas edge weights are described in Section 4 of [9]. In both theories, edge weights are rational function of equivariant parameters that factor into linear factors. They may be rewritten as ratios of Γ-functions, as they appear in the middle of (26). After making all terms explicit, cancelling a common prefactor, and scaling the equivariant weight of $T_0\mathbf{P}^1$ to equal 1, we arrive at the claim of the Theorem.

3.1.6

The intertwining property (21) is obvious from (25) as are the following composition property

$$\mathsf{S}(a_1, b_1; t_1, t_2)\, \mathsf{S}(a_2, b_2; t_1 - a_1, t_2 - a_2) = \mathsf{S}(a_1 + a_2, b_1 + b_2; t_1, t_2)$$

and symmetry

$$\mathsf{S}(a, b; t_1, t_2) = \mathsf{S}(b, a; t_2, t_1)\,.$$

We further have

Proposition 3. *The intertwiner* $\mathsf{S}(a, b)$ *is a Laurent polynomial in* q *when* $a + b \geq 0$.

Proof. The matrix $\mathsf{S}(a, b)$ is a rational solution of an ODE (21) with regular singularities. Its possible singularities are controlled by the integer exponents of (21). By (15) all integer exponents at roots of unity are nonnegative when $a + b \geq 0$. □

3.2

Since the matrix Φ is nonsingular away from the singularities of the differential equation, the additional singularities of S are determined by the diagonal matrix G_{GW}. We conclude the following result.

Theorem 2. *The connections $\nabla(t_1, t_2)$ and $\nabla(t_1, t_2 - 1)$ have isomorphic monodromy provided*

$$t_2 \neq \frac{r}{s}, \quad 0 < r \leq s \leq n\,.$$

At zero level, $t_1 + t_2 = 0$, the matrix M_D is constant in q and, hence, the monodromy is abelian. Furthermore, it is semisimple unless $t_1 = t_2 = 0$. We can use Theorem 2 repeatedly to get from an integer level to zero level.

Corollary 4. *At integer level $t_1 + t_2 = l \in \mathbb{Z}$, the monodromy is abelian. Additionally, the monodromy is semisimple provided*

$$t_1 \neq r/s \tag{27}$$

with $0 < s \leq n$ and

$$r = \begin{cases} 1, \dots, ls - 1\,, & l > 0\,, \\ ls, \dots, 0\,, & l \leq 0\,. \end{cases}$$

Proof. By Theorem 2, we need only prove the monodromy is semisimple for

$$(t_1, t_2) = (l, 0)\,, \quad l > 0\,.$$

For the rest of this proof we use $>$ to denote the following partial ordering on partitions of a fixed number n. It is the transitive closure of the following relation: $\lambda > \mu$ if $\lambda_i = \mu_j + \mu_k$ for some i, j, k, all other parts being identical in the two partitions. The matrix $\mathsf{M}_D(q; l, 0)$ is upper-triangular in the basis (1) with respect to this partial ordering.

We look for solutions of the form

$$B(q)\, q^{\mathsf{M}_D(0;l,0)}\,, \tag{28}$$

where $B(q)$ is holomorphic in q and upper-triangular with 1's on the diagonal. The matrix $\mathsf{M}_D(0; l, 0)$ is semisimple with eigenvalues of the form

$$-c(\lambda; l, 0) = -l \sum_i \frac{\lambda_i^2}{2}\,.$$

If $\lambda > \mu$ and $l > 0$ then,

$$c(\lambda; l, 0) - c(\mu; l, 0) > 0$$

because $(a+b)^2 > a^2 + b^2$ for $a, b > 0$. Thus, the eigenvalues of $\mathsf{M}_D(0; l, 0)$ are strictly increasing down the diagonal. Hence, by a straightforward argument, a formal solution (28) can always be calculated order by order in q. ∎

3.3

Consider the operator

$$\mathbf{\Gamma}\; |\mu\rangle = \frac{(2\pi i)^{\ell(\mu)}}{\prod \mu_i}\, \mathsf{G}_{\mathrm{GW}}(t_1,t_2)\, |\mu\rangle$$

and the connection

$$\nabla^{\Gamma} = \mathbf{\Gamma}\, \nabla\, \mathbf{\Gamma}^{-1}\,.$$

The monodromy of this connection with base point $q=-1$ defines a representation

$$\pi_1(\mathbb{C}\setminus\{\text{singularities}\},-1) \to \operatorname{Aut}(\mathcal{F}) \tag{29}$$

Theorem 3. *Matrix elements of* (29) *are Laurent polynomials in*

$$T_i = e^{2\pi i t_i}\,, \quad i=1,2\,.$$

Further, the monodromy (29) *is unitary with respect to the Hermitian form defined by*

$$\langle\!\langle \mu|\nu\rangle\!\rangle = \frac{\delta_{\mu\nu}}{\mathfrak{z}(\mu)} \prod_i \left(T_1^{\mu_i/2} - T_1^{-\mu_i/2}\right)\left(T_2^{\mu_i/2} - T_2^{-\mu_i/2}\right)\,. \tag{30}$$

The Hermitian form above is anti-linear with respect to the involution

$$\overline{T_i} = T_i^{-1}\,, \quad i=1,2\,.$$

We note that the Hermitian product (30) is essentially the Macdonald inner product as well as the natural inner product in the K-theory of the Hilbert scheme of points, see [4]. This will be revisited below and, more fully, in [1].

Proof. Let $\gamma(t_1,t_2)$ be the monodromy of ∇ along a loop based at $q=-1$. By (21), we have

$$\gamma(t_1-a,t_2-b) = S(-1;a,b)\,\gamma(t_1,t_2) S(-1;a,b)^{-1}\,.$$

On the other hand,

$$S(-1;a,b) = \mathbf{\Gamma}(t_1-a,t_2-b)^{-1}\,\mathbf{\Gamma}(t_1,t_2)\,.$$

Hence $\mathbf{\Gamma}\,\gamma\,\mathbf{\Gamma}^{-1}$ is invariant under $t_i \mapsto t_i+1$ and, therefore, is a meromorphic function of T_1 and T_2.

Since ∇ depends polynomially on t_1 and t_2, its monodromy is an entire function of t_1 and t_2. The matrix $\mathbf{\Gamma}$ is holomorphically invertible for $\Re t_i > 0$, hence the monodromy of ∇^Γ is holomorphic there. By periodicity, it is holomorphic for all $T_1, T_2 \in \mathbb{C}^*$.

The solutions of ∇ grow at most exponentially as $\Im t_i \to \infty$. From the Stirling formula, we have

$$\ln |\Gamma(x+iy)| = -\frac{\pi}{2}|y| + O(\ln |y|)\,, \quad y \to \pm\infty\,, \tag{31}$$

hence the monodromy of ∇^Γ grows at most polynomially as $T_i \to 0, \infty$. Therefore, $\mathbf{\Gamma}\,\gamma\,\mathbf{\Gamma}^{-1}$ is a Laurent polynomial in the T_i's.

By construction, the monodromy is unitary with respect to the new Hermitian form defined by

$$\langle\!\langle \mu|\nu\rangle\!\rangle = \langle \mu\,|\mathbf{\Gamma}(-t_1,-t_2)\mathbf{\Gamma}(t_1,t_2)|\,\nu\rangle$$

Using the formula

$$-\frac{2\pi i}{\Gamma(x)\Gamma(-x)} = x\left(e^{\pi i x} - e^{-\pi i x}\right)$$

we obtain (30). ☐

3.4 Γ-factors

This section is full of Γ-factors and one may wonder what is their deeper meaning. Technically, they arise as edge-weights in GW and DT localization formulas. Further, associated to the intertwiner operator $\mathsf{S}(a,b)$ is a system of *difference equations*

$$\mathsf{S}(a,b)\Psi(t_1-a,t_2-b) = \Psi(t_1,t_2) \tag{32}$$

which is compatible with the differential equation (3). The variable q of the original differential equation (3) is a parameter in the difference equation (32). The points $q = 0, -1$ are special for this difference equation in that for those values of the parameter it becomes abelian and may be solved explicitly in Γ-functions.

Also note that the formula

$$\Gamma(1+t)\,\Gamma(1-t) = \frac{2\pi i t}{e^{\pi i t} - e^{-\pi i t}}\,,$$

and its relatives that were used above, show that the Γ-factors play the role of the Mukai's vector $\sqrt{\mathrm{Td} X}$ for the anti-linear inner product used in this paper. Independently, parallel Γ-factors arose in the work of Iritani [5].

4 Connection Problem

4.1

In this section, we solve the connection problem for the QDE between special points $q = 0$ and $q = -1$, which may be called the DT and GW points, respectively. In plain English, the connection problem is to find the value of the matrix $\mathsf{Y}(q)$ at $q = -1$. We will see that the answer is given in terms of Macdonald polynomials.

We will use $\mathsf{P}^\lambda \in \mathcal{F} \otimes \mathbb{Q}(q,t)$ to denote the monic Macdonald polynomial as defined, for example, in the book [8]. Note that parameters q and t, that are traditionally used for P^λ are not our parameters q and t. The matching of parameters will be discussed below.

First, we explain the transformation that relates P^λ to the polynomials H^λ used by Haiman in his work on the K-theory of the Hilbert scheme, see [4]. It is given by the formula

$$\mathsf{H}^\lambda(q,t) = t^{\mathsf{n}(\lambda)} \prod_{\square \in \lambda} \left(1 - q^{a(\square)}\, t^{-l(\square)-1}\right) \Upsilon\, \mathsf{P}^\lambda(q, t^{-1}) \,, \tag{33}$$

where

$$\Upsilon\, |\mu\rangle = \prod_{\mu} (1 - t^{-\mu_i})^{-1}\, |\mu\rangle \ .$$

and

$$\mathsf{n}(\lambda) = \sum (i-1)\, \lambda_i \,.$$

We relate Haiman's parameters q and t to our parameters by the identification

$$(q,t) = (T_1, T_2) \,.$$

By their representation-theoretic meaning, they both define an element in the maximal torus of $GL(2)$ acting on $\mathbb{C}^2$ and its Hilbert schemes, so this identification is natural. Perhaps it would be even more natural to identify q and t with T_i^{-1}, but this amounts to a minor automorphism of symmetric functions.

4.2

Let H be the matrix with columns H^λ. We have the following

Theorem 4.

$$\mathbf{\Gamma}^{-1}\,\mathsf{Y}\,\mathsf{G}_{\mathrm{DT}}\Big|_{q=-1} = \frac{1}{(2\pi i)^{|\cdot|}}\,\mathsf{H} \tag{34}$$

The proof of this theorem will take several steps.

4.3

Let H be an $\mathrm{End}(\mathcal{F})$-valued meromorphic function of t_1 and t_2 that satisfies (34) in place of H. With this notation, $H = \mathsf{H}$ is what we need to show.

Formula (25) shows that H is periodic in t_i with period one, hence a meromorphic function of T_1 and T_2. Further, we claim that, in fact, H is a rational function of the T_i's. This follows from two following observations:

1. H has finitely many poles;
2. H grows at most polynomially as $T_i \to 0, \infty$.

Indeed, the poles of Y^λ may occur only when

$$c(\lambda; t_1, t_2) = c(\mu; t_1, t_2) + n\,, \quad \mu \neq \lambda, \quad n = 1, 2, \dots\,.$$

Since $c(\lambda; t_1, t_2)$ is a linear form with integer coefficients, these correspond to finitely many poles along divisors of the form

$$T_1^i T_2^j = 1\,.$$

In fact, these poles will be compensated by the zeros of G_{DT}, making H a polynomial in the T_i's, but we won't need this stronger result here.

The growth of H as $T_i \to 0, \infty$ is estimated as in the proof of Theorem 3.

4.4 Asymptotics in the connection problem

4.4.1

Since H is periodic, it may determined from studying its own asymptotics. Specifically, we will let $t_1 \to +\infty$ keeping the level

$$\kappa = t_1 + t_2$$

fixed. In this limit, we have

$$\mathbf{\Gamma}^{-1} \sim t_1^{\kappa|\,\cdot\,|-\ell(\,\cdot\,)}\, T_2^{-|\,\cdot\,|/2}\, \Upsilon\,.$$

Similarly,

$$\prod_{\text{tangent weights } \mathsf{w}} \frac{1}{\Gamma(\mathsf{w}+1)} \sim (-2\pi i)^{|\lambda|}\, h_\lambda^{-1-\kappa}\, t_1^{-(1+\kappa)|\lambda|} \times$$

$$T_1^{-\mathsf{n}(\lambda')/2}\, T_2^{(\mathsf{n}(\lambda)+|\lambda|)/2} \prod_{\square\in\lambda} \left(1 - T_1^{a(\square)} T_2^{-l(\square)-1}\right)\,,$$

where h_λ is the product of all hooklengths in the diagram of λ. Note that by Serre duality the tangent weights come in pairs $\mathsf{w}_1, \mathsf{w}_2$ such that $\mathsf{w}_1 + \mathsf{w}_2 = \kappa$.

4.4.2

Note that (24) implies

$$q^{-c(\lambda,t_1,t_2)}\Big|_{q=-1} = T_1^{\mathsf{n}(\lambda')/2}\, T_2^{\mathsf{n}(\lambda)/2}$$

Together with the above asymptotics, this shows we should prove

$$(-1)^{|\lambda|}\, t_1^{-|\lambda|-\ell(\lambda)}\, h_\lambda^{-1-\kappa}\, \mathsf{Y}^\lambda(-1) \sim \mathsf{P}^\lambda\,.$$

To adjust for the t_1 scaling above, we introduce

$$\begin{aligned} \mathbf{y}_\lambda &= (-1)^{|\lambda|}\, t_1^{-|\lambda|-\ell(\,\cdot\,)}\, \mathsf{Y}^\lambda\,, \\ \mathbf{j}_\lambda &= (-1)^{|\lambda|}\, t_1^{-|\lambda|-\ell(\,\cdot\,)}\, \mathsf{J}^\lambda\,, \\ \mathbf{m} &= t_1^{-\ell(\,\cdot\,)}\, \mathsf{M}_D\, t_1^{\ell(\,\cdot\,)}\,. \end{aligned}$$

Note that from (10) we have

$$\mathbf{j}_\lambda \to h_\lambda\, s_\lambda\,, \quad t_1 \to \infty\,. \tag{35}$$

The hook-length product in (35) appears from the difference between the monic and the integral form of the Jack polynomial.

4.4.3

At this point, we transformed the problem into showing that

$$h_\lambda^{-1-\kappa}\, \mathbf{y}_\lambda(-1) \sim \mathsf{P}^\lambda\,, \quad t_1 \to \infty\,. \tag{36}$$

By its definition, the matrix P is triangular in the basis $\{\mathbf{j}_\mu\}$ with respect to the dominance order of partitions. Our next step is to show that the asymptotics of $\mathbf{y}(-1)$ is similarly triangular.

This will be done through a reformulation of the QDE in terms of an integral equation, following the strategy behind Levinson's theorem [6], see for example Section 1.4 in [3]. This technique is standard and is presented here mostly to make the material more accessible to algebraic geometers.

4.4.4

It will be convenient to make a change of the independent variable via

$$q = -e^{-x}\,.$$

The new variable ranges from 0 to ∞ as q goes from -1 to 0.

We will write the differential equation

$$\frac{d}{dx}\,\Psi(x) = (-\mathbf{m} - c(\lambda; t_1, t_2))\,\Psi(x)$$

satisfied by $\mathbf{y}_\lambda$ in the form

$$\frac{d}{dx}\,\Psi(x) = (D(x) + R(x))\;\Psi\,, \tag{37}$$

where the matrix $D(x)$ is diagonal in the basis of Jack polynomials

$$D(x)\,\mathbf{j}_\mu = d_\mu(x)\,\mathbf{j}_\mu\,,$$

while the off-diagonal remainder matrix is linear in κ and decays exponentially as $x \to +\infty$.

4.4.5

Note that as $t_1 \to \infty$ we have

$$d_\lambda(x; t_1, t_2) = t_1\,\mathbf{f}_2(\lambda) + O(1)\,,$$

where $\mathbf{f}_2(\lambda)$ is the sum of contents of λ. Introduce a partial ordering $<_c$ on partitions by

$$\lambda <_c \mu \Leftrightarrow \mathbf{f}_2(\lambda) < \mathbf{f}_2(\mu)\,.$$

This is a refinement of the dominance order on partitions, that is,

$$\lambda < \mu \Rightarrow \lambda <_c \mu\,.$$

We denote by $\Pi_>$ the projection onto $\mathbf{j}_\mu$ with $\mu >_c \lambda$, that is

$$\Pi_>\,\mathbf{j}_\mu = \begin{cases} \mathbf{j}_\mu\,, & \mu >_c \lambda\,, \\ 0\,, & \text{otherwise}\,. \end{cases}$$

We define the operator $\Pi_<$ similarly and set

$$\Pi_\leq = 1 - \Pi_>\,, \quad \Pi_= = 1 - \Pi_> - \Pi_<\,.$$

Given an operator A, we set

$$A_> = \Pi_>\,A\,\Pi_>\,.$$

In English, this operator cuts out a corner of A corresponding to rows and columns indexed partitions μ such that $\mu >_c \lambda$.

4.4.6

Let the operator $A(x)$ be defined by

$$A(x)\,\mathbf{j}_\mu = e^{\int_0^x d_\mu(t)\,dt}\,\mathbf{j}_\mu\,.$$

It solves the equation

$$\frac{d}{dx}\,A(x) = D(x)\,A(x)\,.$$

One constructs a particular solution of the differential equation (37) as a solution of the integral equation

$$\begin{aligned}\Psi(x) = e^{-\int_x^\infty d_\lambda(t)\,dt}\,\mathbf{j}_\lambda \\ + A(x)_< \int_0^x A(t)^{-1}\,R(t)\,\Psi(t)\,dt \\ - A(x)_\leq \int_x^\infty A(t)^{-1} R(t)\,\Psi(t)\,dt\,. \quad (38)\end{aligned}$$

Note that the convergence of the integral in the first line is insured by the exponential decay of $d_\lambda(x)$ as $x \to \infty$. Since

$$\frac{d}{dx} A_< = D A_< \,,$$

it is immediate that a solution of (38) solves (37).

4.4.7

The fundamental role in the analysis of (38) is played by the estimates

$$\begin{aligned} \left\| A(x)_< A(t)^{-1} \right\| &\le K_1(\kappa)\, e^{-t_1(x-t)} \,, \\ \left\| A(x)_= A(t)^{-1} \right\| &\le K_1(\kappa) \,, \\ \left\| A(x)_> A(t)^{-1} \right\| &\le K_1(\kappa)\, e^{-t_1(t-x)} \,, \end{aligned} \tag{39}$$

satisfied for all sufficiently large t_1. Here $K_1(\kappa)$ is constant depending on κ. In particular, it follows that a bounded solution of (38) satisfies

$$\Psi(x) \to \mathbf{j}_\lambda \,, \quad x \to \infty \,.$$

4.4.8

Another crucial feature of the equation (38) is

$$\Pi_= \, R(x)\, \Pi_= = 0 \,. \tag{40}$$

This is a geometric property. The matrix $R(x)$ is proportional to $\kappa = t_1 + t_2$ and contains purely quantum parts of the quantum multiplication operator. The coefficient of $t_1 + t_2$ is nonvanishing only if a chain of unbroken curves exists between two torus fixed points and the won't be any such chains if $\mathbf{f}_2(\mu) = \mathbf{f}_2(\lambda)$, see Section 3.8.2 of [11].

4.4.9

A bounded solution of (38) may be constructed by successive approximations

$$\begin{aligned} \Psi_{n+1}(x) = e^{-\int_x^\infty d_\lambda(t)\,dt}\, \mathbf{j}_\lambda \\ + A(x)_< \int_0^x A(t)^{-1}\, R(t)\, \Psi_n(t)\, dt \\ - A(x)_\le \int_x^\infty A(t)^{-1} R(t)\, \Psi_n(t)\, dt \,, \end{aligned}$$

starting with

$$\Psi_0(x) = e^{-\int_x^\infty d_\lambda(t)\,dt}\,\mathbf{j}_\lambda\,.$$

It follows from (39) and (40) that

$$\begin{aligned} \|\Pi_=(\Psi_{n+1}-\Psi_n)\| &\le K_1K_2\,\|\Pi_{\neq}(\Psi_n-\Psi_{n-1})\|\ , \\ \|\Pi_{\neq}(\Psi_{n+1}-\Psi_n)\| &\le \frac{K_1K_3}{t_1}\,\|\Psi_n-\Psi_{n-1}\| \end{aligned} \tag{41}$$

in the norm of $C([0,\infty))$, where

$$K_2(\kappa) = \|R(x)\|_{L^1([0,\infty))}\ , \quad K_3(\kappa) = \|R(x)\|_{C([0,\infty))}\ .$$

For $t_1 \gg 0$ the iterations of (41) are contracting, whence the convergence to a bounded solution $\Psi(x)$ of (38). Further, we have

$$\Psi(x) \to \Psi_0(x)\,, \quad t_1 \to +\infty\,. \tag{42}$$

4.4.10

The asymptotics (42) implies that the connection matrix is diagonal in the $t_1 \to \infty$ limit modulo the terms that are exponentially small as $x \to \infty$. In other words, this proves that $\mathbf{y}(-1)$ is asymptotically triangular in the basis $\{s_\mu\}$ with respect to the partial order $>_c$.

Further, since $d_\mu(x)$ depends linearly on κ, the leading coefficients of $h_\lambda^{-1-\kappa}\,\mathbf{y}_\lambda(-1)$ are of the form $K_4K_5^\kappa$, where K_4 and K_5 are real and positive. At the same time, they should be periodic in κ, which yields $K_5 = 1$.

Since a periodic function of t_1 is uniquely determined by its asymptotics as $t_1 \to +\infty$, this implies the corresponding triangularity of $\Upsilon^{-1}H$.

4.5

We have

$$\mathsf{P}^\lambda = s_\lambda + \dots$$

where dots stand for a linear combination of s_μ with $\lambda >_c \mu$. Therefore,

$$H = \mathsf{H}\,U$$

where the matrix U is triangular with respect to $>_c$. Moreover, the diagonal entries of U are positive real numbers that are independent of t_1, t_2. The unitarity of the connection and the formulas for the norm squares of the Macdonald polynomial imply

$$U^* \, D \, U = D$$

where $U^* = \overline{U}^T$ and D is the Gram matrix of the inner product (30) in the basis H^λ. Since D is diagonal, it follows that U must be the identity matrix. This concludes the proof of Theorem 4.

4.6 Scattering

Relation (14) implies the matrix

$$(-1)^{\ell(\,\cdot\,)} \mathsf{Y}(q^{-1}) q^{c(\,\cdot\,)} \tag{43}$$

is another fundamental solution of the quantum differential equation. It is natural to ask for the relation between (43) and the solution $\mathsf{Y}(q) q^{-c(\,\cdot\,)}$. Since, as $q \to 0, \infty$, our ODE becomes the Calogero-Sutherland system, the $q \to q^{-1}$ transformation of the fundamental solution can be naturally interpreted as scattering by the nonstationary terms.

Since $q = -1$ is a fixed point of the involution $q \to q^{-1}$, we may use evaluation at $q = -1$ to compare the two solutions. From Theorem 4, we see that the scattering transformation essentially amounts to the action of the operator $(-1)^{\ell(\,\cdot\,)}$, i.e. the action of the standard symmetric function involution ω, in the basis $\{\mathsf{H}^\lambda\}$.

In particular, for integer levels $t_1 + t_2 \in \mathbb{Z}$, we get the action of ω on Schur functions and so, up to normalization, scattering simply transposes the diagram λ in that case.

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