Modern Enumerative Geometry

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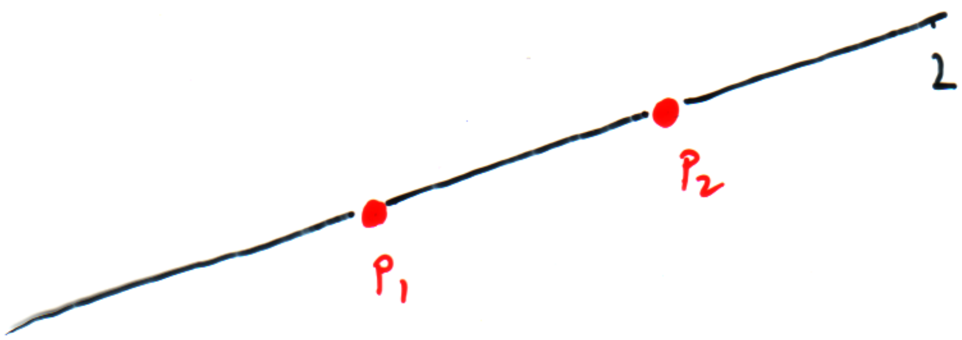
Enumerative Geometry :

Counting algebraic solutions of geometric questions

First question (Euclid)

How many lines pass through 2 points in the plane



Ans = 1

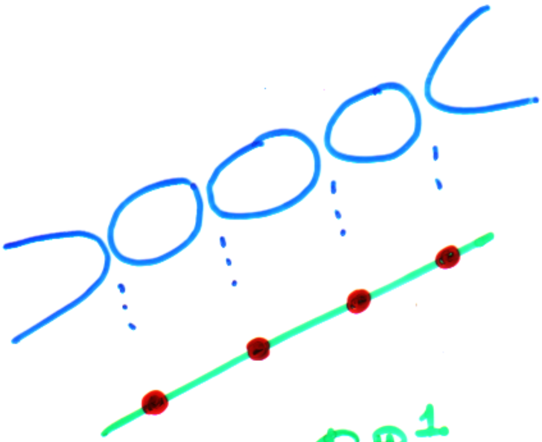
Two classical examples

(I) Hurwitz Covers 1890's

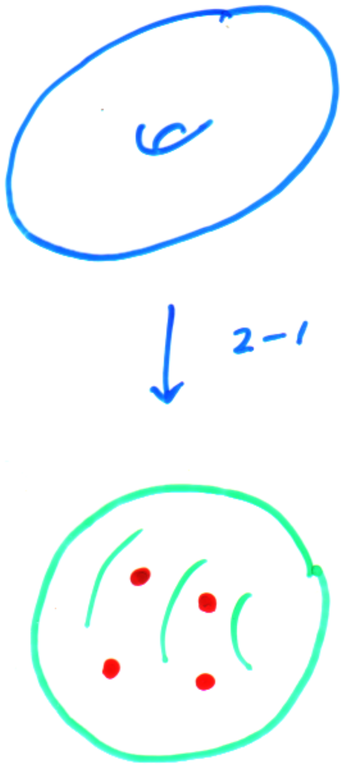
Algebraic geometer | Topologist

$g=1$ | $g=1$

$d=2$ | $d=2$



$\mathbb{CP}^1$



Riemann Sphere

DEF: $H_{g,d}$ = Number of genus g degree d Covers of $\mathbb{CP}^1$ with Simple ramification over $b = 2g-2+2d$ branch points

There are many way to Calculate Hurwitz numbers ...

A recent method

RP 99
A Okounkov 00

$$H(\lambda, y) = \sum_{g \geq 0} \sum_{d > 0} \lambda^{2g-2} e^{dy} \frac{H_{g,d}}{(2g-2+2d)!}$$

THRM :

$$\exp\left(H(\lambda, y+\lambda) + H(\lambda, y-\lambda) - 2H \right) = \lambda^2 e^{-y} H_{yy}$$

Toda equation uniquely determines $H_{g,d}$.

(II) Severi degrees (Schubert, Zeuthen, 19th century)

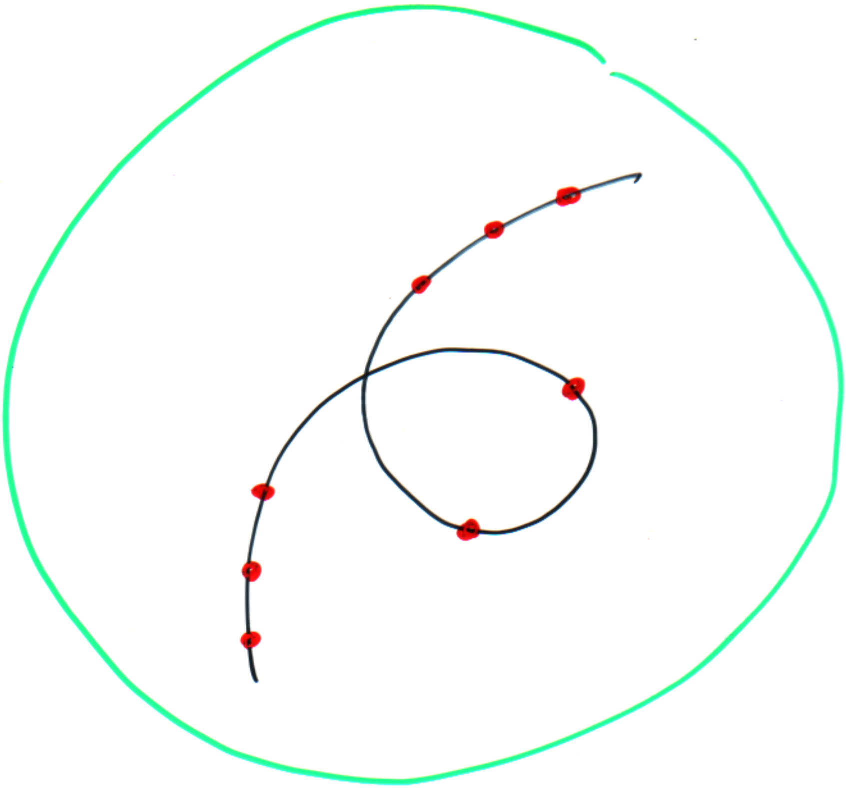
Generalization of Euclid's question:

$g=0$

$d=3$

8 points

11 others...



$\mathbb{CP}^2$

Severi:

How many plane curves of genus g and degree d pass through $3d+g-1$ points?

Def: $N_{g,d}$ = number solutions to the Severi problem

$N_{0,1} = 1$ Euclid

$N_{0,2} = 1$ # conic through 5 points

$N_{0,3} = 12$

$\vdots$

Very many methods available to compute $N_{g,d}$

Z. Ran 80's

Caporaso – Harris 90's

$\vdots$

No Answer as simple as Hurwitz solution

However, in case $g=0$, a very nice solution is available:

Let $\Gamma(y_1,y_2) = \sum_{d\geq 1} N_{0,d}\, e^{dy_1} \dfrac{y_2^{3d-1}}{(3d-1)!}$

By an application of the WDVV equation first noticed by Kontsevich:

$$(*) \qquad \Gamma_{222} = \Gamma_{112}^2 - \Gamma_{111}\Gamma_{122}$$

Together with $N_{0,1}=1$, $(*)$ determines all $N_{0,d}$.

Neither example is the topic of my talk today

Both are classical questions

Hurwitz	$\dim_{\mathbb{C}}\ 1$
Severi	$\dim_{\mathbb{C}}\ 2$

Classical Enumerative Geometry

(i) Start with a well-defined enumeration problem in algebraic geometry

(ii) Solve it.

The truly modern enumerative Geometry is the opposite :

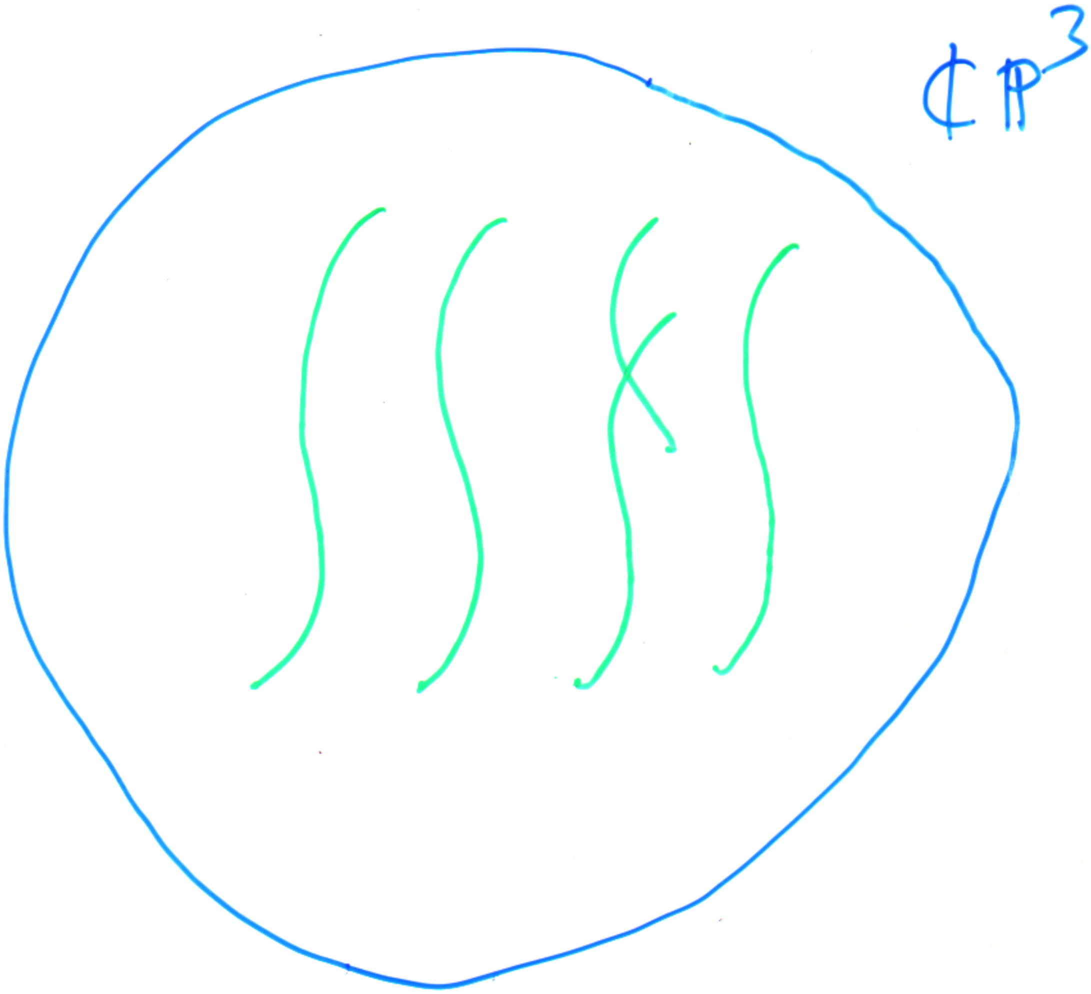
(i) Start with answer

(ii) Try to discover what is being counted

I will explain this perspective for the enumerative geometry of curves in dim$_{\mathbb{C}}$ 3

Curves in $\mathbb{CP}^3$ have been studied since the late 19^{th} century

M. Noether, Halphen



Ask:

What does the family of genus g, degree d space curves look like?

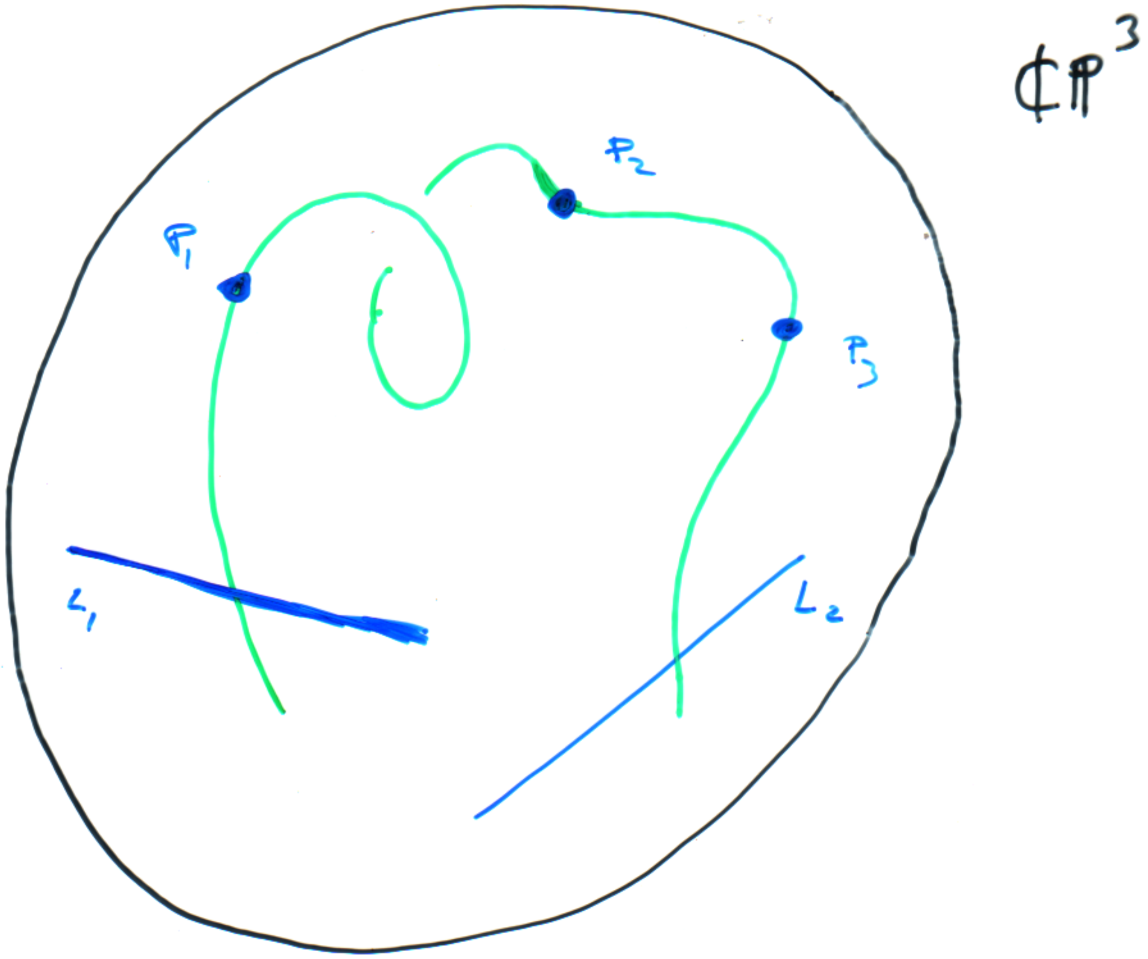
Ans: Impossibly bad

- hard to predict nonemptiness
- not known how to find dimension
- can be very singular, have nilpotents, etc.

Mumford

The situation appears unfavorable for an enumerative geometry

We would like to ask:



How many space curves of genus g and degree d meet a given configuration of lines and points.

Of course we can consider arbitrary 3-folds X also

(in fact X = Calabi-Yau 3-fold

$$K_X \cong \mathcal{O}_X$$
$$\shortparallel$$
$$\Lambda^3 \Omega_X$$

is very interesting case)

Enumerative geometry in $\dim_{\mathbb{C}} 3$ is not classical question

To even define the problem, very new ideas are required.

Amazingly, we now have 3 different definitions.

3 approaches to enumerative geometry in $\dim_{\mathbb{C}}$ 3

- Gromov-Witten theory — roots late 80s

  moduli of maps

- Donaldson-Thomas theory — 90s

  moduli of sheaves

- Moduli of complexes in the derived Category — 00s

  RP + R. Thomas

I will explain each.

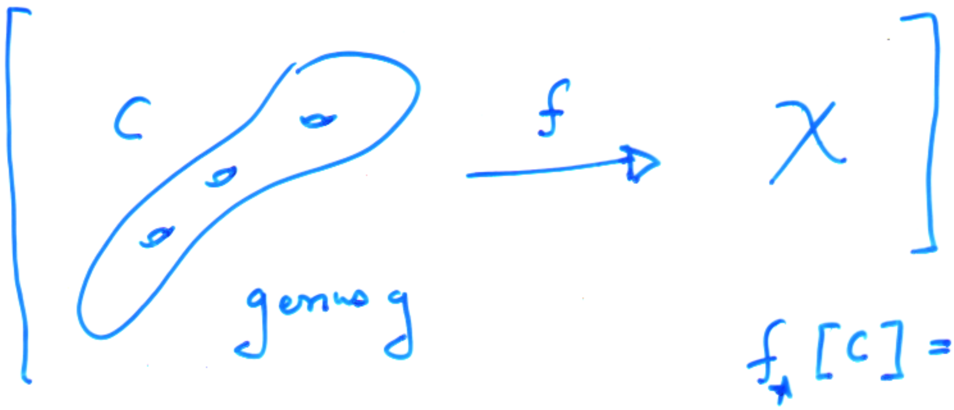
All very different.

Miracle: All arrive at same definition (Conjecturally)

(i) GW theory

$\overline{\mathcal{M}}_g(X,\beta)$ moduli space of stable maps

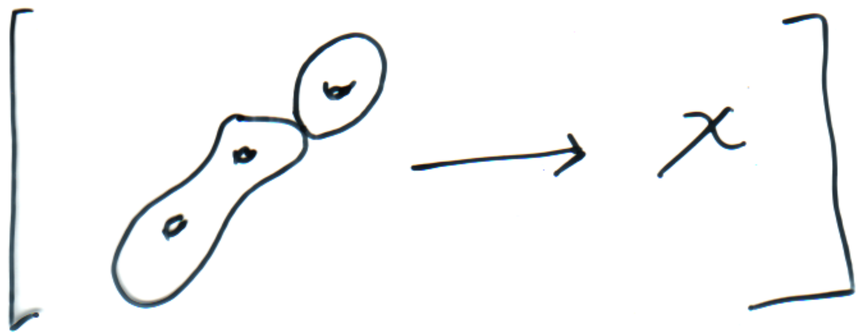
$\Downarrow$



f algebraic

$f_*[C] = \beta \in H_2(X,\mathbb{Z})$

for compactness: need to add nodal degenerations



Kontsevich 94

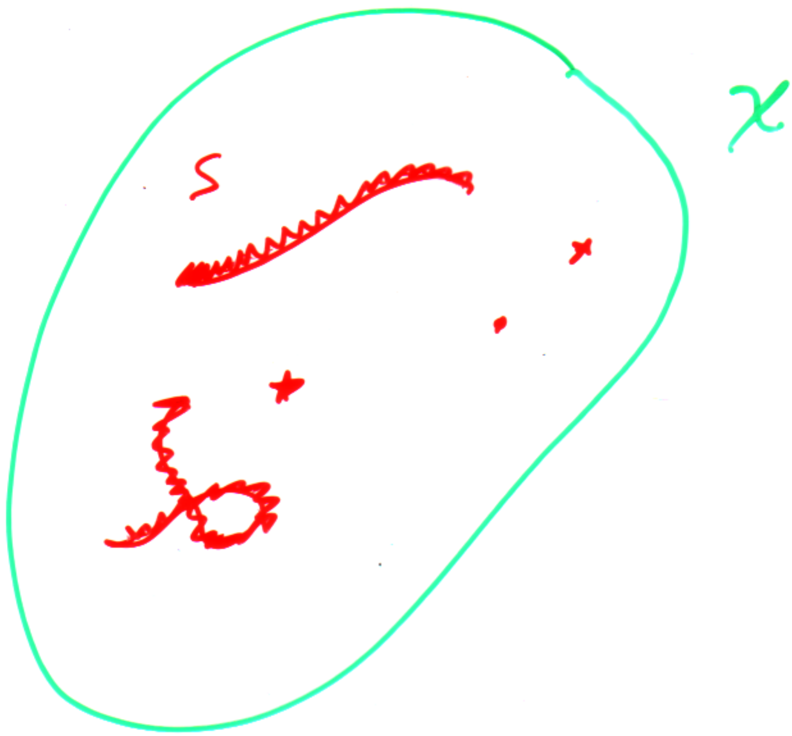
$\overline{\mathcal{M}}_g(X,\beta)$ compact algebraic variety (stack)

(ii) Donaldson - Thomas theory

$I_n(X, \beta)$ Hilbert scheme of curves

Grothendieck 60

parameterizes subschemes



$0 \to \mathcal{I}_S \to \mathcal{O}_X \to \mathcal{O}_S \to 0$

$[\mathcal{I}_S] \in I_n(X, \beta)$

$n = \chi_{hol}(\mathcal{O}_S)$

$\beta = [S] \in H_2(X, \mathbb{Z})$

(iii) Complexes

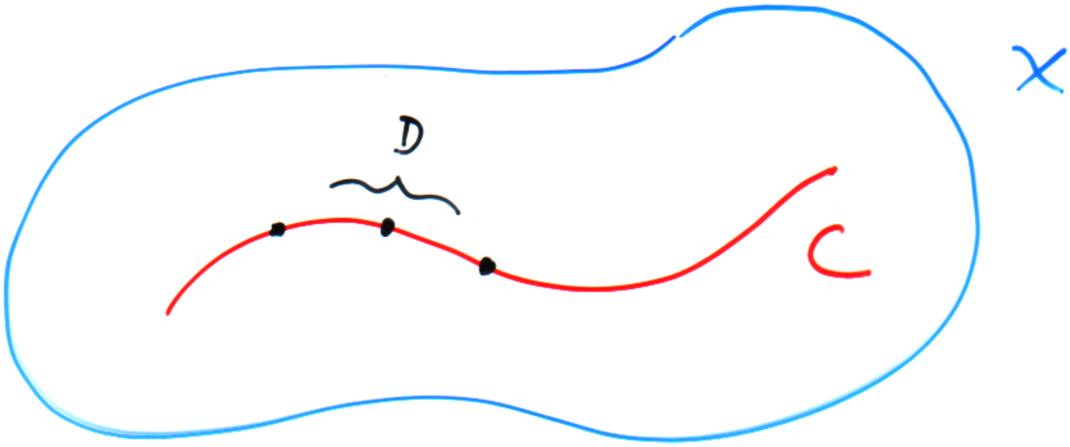
RP + R. Thomas

$P_n(X, \beta)$ moduli space of complexes

Bridgeland
Inabi
Lieblich

idea: would like moduli space of pairs (C, D)

- $C \subset X$ curves
- $D \subset C$ divisor



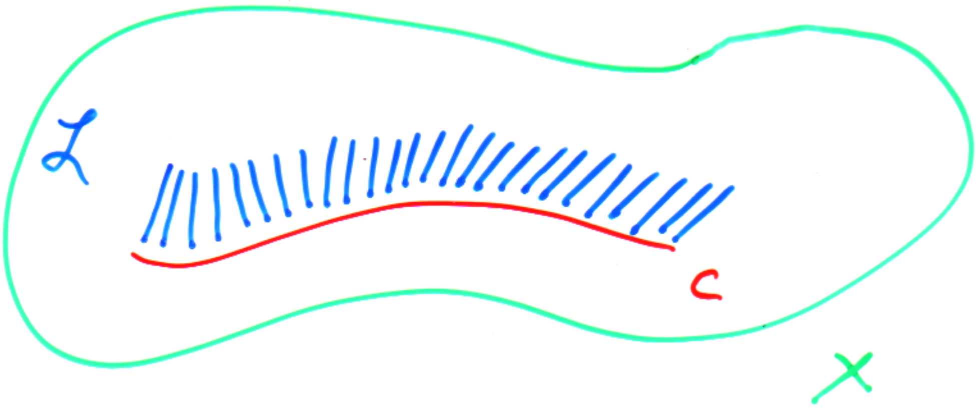
A divisor $D \subset C \Rightarrow$ line bundle

$$\mathcal{L} = \mathcal{O}(D) \text{ on } C$$

$$+ \text{ section } H^0(C, \mathcal{L})$$

so $(C, D) \Rightarrow \left[\mathcal{O}_X \to \mathcal{L}|_C \right]$

complex in $D^b(X)$



$P_n(X, \beta)$ parameterizes complexes

$$\left[\mathcal{O}_X \to \mathcal{F} \right]$$

$$n = \chi_{hol}(\mathcal{F})$$

$$\beta = [\mathcal{F}] \in H_2(X, \mathbb{Z})$$

X is 3-fold

- $\overline{M}_g(X, \beta)$ stable maps
- $I_n(X, \beta)$ Hilbert scheme
- $P_n(X, \beta)$ moduli of pairs in $D^b(X)$

How does this help for enumerative geometry?

As expected all three moduli spaces are very badly behaved in all respects except one: deformation theory

The best deformation theories in algebraic geometry are controlled by two groups:

- Def group $\leftarrow$ Tangent space to moduli
- Obs group

All three of these moduli spaces have such two-term def theories

(i) $\overline{M}_g(X, \beta)$

for fixed domain $C \xrightarrow{f} X$

Def $\quad H^0(C, f^*T_X)$

Obs $\quad H^1(C, f^*T_X)$

(ii) $I_n(X, \beta)$ Hilbert Scheme

$$0 \to \mathcal{I} \to \mathcal{O}_X \to \mathcal{O}_S \to 0$$

Def $\quad \mathrm{Ext}^1_0(\mathcal{I}, \mathcal{I})$

Obs $\quad \mathrm{Ext}^2_0(\mathcal{I}, \mathcal{I})$

$_0$ ← traceless

$\mathrm{Ext}^0_0(\mathcal{I}, \mathcal{I}) =$
$\mathrm{Ext}^3_0(\mathcal{I}, \mathcal{I}) = 0$

R. Thomas PhD thesis

Def theory obtained from viewing $I_n(X, \beta)$ as moduli of ideal sheaves.

(iii) $P_n(X,\beta)$ pairs in $D^b(X)$

$$I^\bullet = \{ \mathcal{O}_X \to F \}$$

Def $\quad Ext^1_0(I^\bullet, I^\bullet)$ traceless

Obs $\quad Ext^2_0(I^\bullet, I^\bullet)$

$Ext^0_0 = Ext^3_0 = 0$

Def theory obtained by deforming $[\mathcal{O}_X \to F]$ in moduli of complexes in $D^b(X)$

Def theory (if 2 term)

$\Rightarrow$ Virtual class

J. Li – G. Tian
K. Behrend–
B. Fantechi

$$[\bar{M}_g(X,\beta)]^{vir} \in H_{expected}(\bar{M}_g(X,\beta), \mathbb{Q})$$

$$[I_n(X,\beta)]^{vir} \in H_{expected}(I_n(X,\beta), \mathbb{Z})$$

$$[P_n(X,\beta)]^{vir} \in H_{expected}(P_n(X,\beta), \mathbb{Z})$$

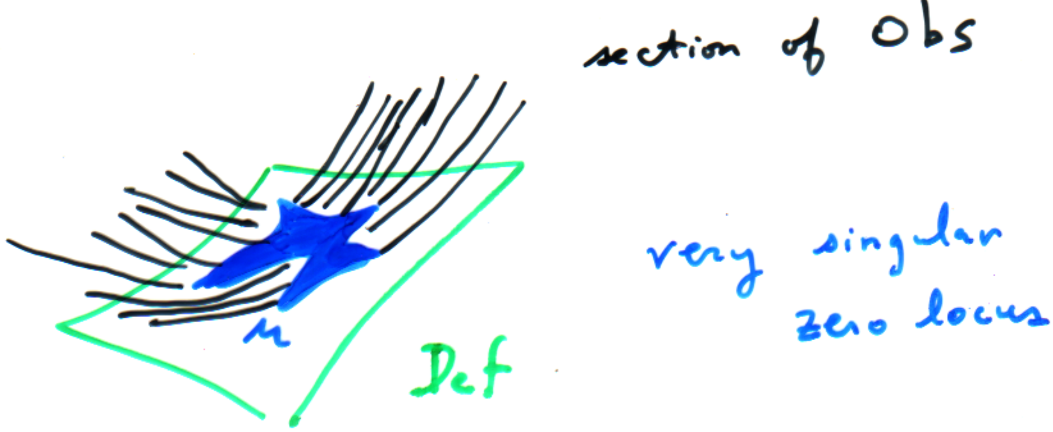
$$\text{expected} = 2 \cdot (\dim_{\mathbb{C}} \mathrm{Def} - \dim_{\mathbb{C}} \mathrm{Obs})$$

$$= 2 \int_\beta c_1(X) \qquad \text{all cases}$$

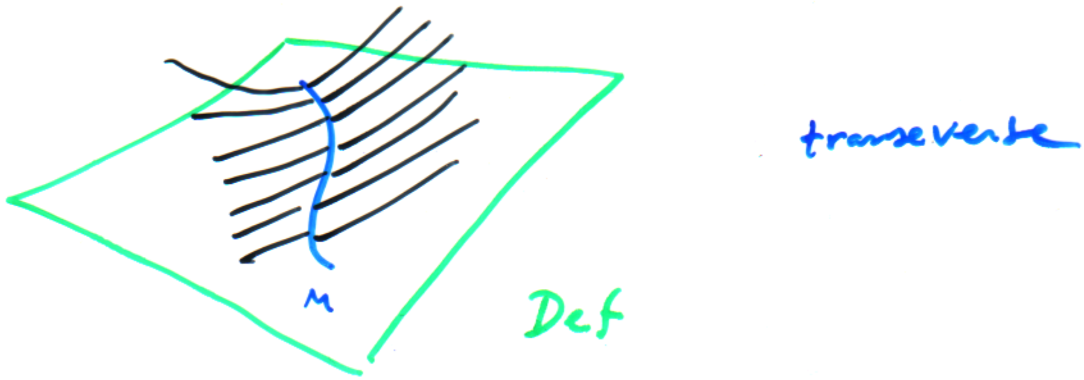
$$\beta \in H_2(X, \mathbb{Z}) \qquad c_1(X) = c_1(T_X) \in H^2(X, \mathbb{Z})$$

idea of the virtual class:

Moduli space M

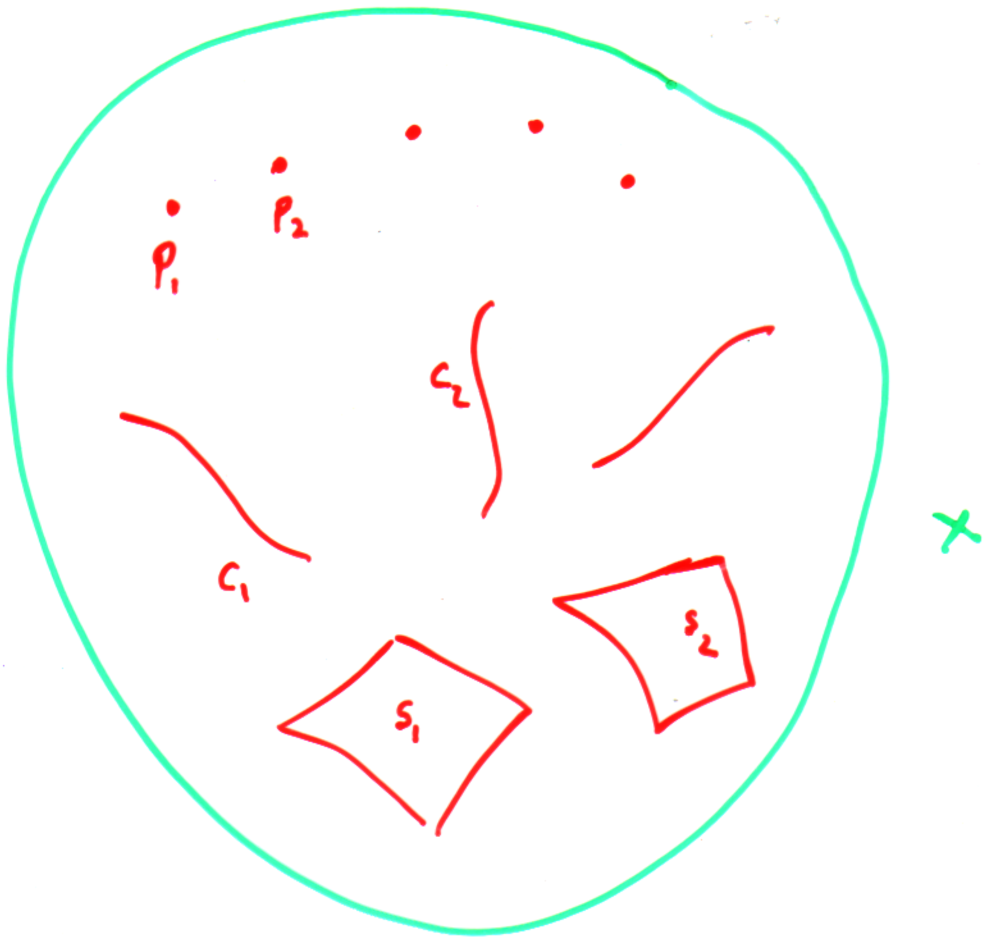


By perturbation, hope



In general, can't achieve such a good perturbation. But def theory tells us what the fundamental class would be if we could.

fix incidence conditions in X



$$\gamma = (\underbrace{p_1, \ldots}_{\text{Points}}, \underbrace{C_1, \ldots}_{\text{Curves}}, \underbrace{S_1, \ldots}_{\text{Surfaces}})$$

Now easy to define enumerative invariants

$$GW_{g,\beta}(\gamma), \quad I_{n,\beta}(\gamma), \quad P_{n,\beta}(\gamma)$$

via the virtual class.

In each of the theories

$$[\quad]^{vir} \cap (\text{incidence } \gamma \text{ conditions})$$

Homology $\qquad\qquad$ cohomology

$\Downarrow$

number

$$GW_{g,\beta}(\gamma), \quad I_{n,\beta}(\gamma), \quad P_{n,\beta}(\gamma)$$

Generating Series : $\qquad$ fix $\beta \in H_2(X)$

$$Z^{GW}_\beta(u) = \sum_g GW_{g,\beta}^{(\gamma)}\, u^{2g-2}$$

$$Z^{I}_\beta(q) = \sum_n I_{n,\beta}^{(\gamma)}\, q^{n}$$

$$Z^{P}_\beta(q) = \sum_n P_{n,\beta}^{(\gamma)}\, q^{n}$$

Conjectures:

All theories are equivalent

$$Z_\beta^{GW}(u)$$

$$\boxed{-q = e^{iu}}$$

Maulik
Nekrasov
Okounkov
P

$$\|$$

$$Z_\beta^{I}(q) / Z_0^{I}(q)$$

$$\|$$

P
R. Thomas

$$Z_\beta^{P}(q)$$

Proven?

GW $\Longleftrightarrow$ I

True for $\mathbb{CP}^3$, toric

Maulik
Oblomkov
Okounkov
P

I $\Longleftrightarrow$ P

True for toric CY

P
R. THOMAS

GW theory
Moduli of Curves
matrix model / int systems

Enumerative
geometry of
3 folds

DT theory
moduli of Sheaves
box Counting /
random Surface theory
Commutative alg

moduli of
Complexes
Derived
Category Stability
Derived
Geometry