Moduli Spaces of differentials on Curves

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(ZOOM)

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Recent progress on several questions related to the moduli spaces of holomorphic / meromorphic differentials on curves:

- New perspectives on Twisted differentials
  Farkas - P 2015, Holmes - Schmitt

- New ideas about the connections to Witten's r-spin class
  JPPZ 2016, Q. Chen - J - Ruan - Sauvaget - Zvonkine, Janda - Pixton - Schmitt

- New results about connections to volumes D. Chen, Möller, Sauvaget, Zagier

I. The space of twisted differentials

Let $g \in \mathbb{Z}_{\geq 0}$ be the genus

Let $\mu = (m_1, \ldots, m_\ell)$ $m_i \in \mathbb{Z}$

with $\sum_{i=1}^{\ell} m_i = 2g-2$

Consider the locus

$$\mathcal{H}_{g,\mu} \subset \mathcal{M}_{g,\ell} \ni [C, p_1, \ldots, p_\ell]$$

defined by the condition:

$$\omega_C \cong \mathcal{O}_C\left(\sum m_i p_i\right)$$

It is well known that $\mathcal{H}_{g,\mu}$ is

- pure codim $g-1$ in the <u>holomorphic case</u>
  (all $m_i \geq 0$)

- pure codim g in the strictly <u>meromorphic case</u>
  ($\exists i \quad m_i < 0$)

Moreover $\mathcal{H}_{g,\mu}$ is nonsingular (as a DM stack)

Polishchuk 2006 and others ...

The moduli space of <u>twisted differentials</u>

$$\widetilde{\mathcal{H}}_{g,\mu} \subset \overline{\mathcal{M}}_{g,\ell}$$

was defined in Farkas-P.

Not the closure!

We defined $\widetilde{\mathcal{H}}_{g,\mu}$ set-theoretically

via an explicit condition on

$$[C, p_1, \dots, p_\ell] \in \overline{\mathcal{M}}_{g,\ell}$$

via <u>twists</u>. I dont review the precise definition.

Results 2015:

- $\widetilde{\mathcal{H}}_{g,\mu} \cap \mathcal{M}_{g,\ell} = \mathcal{H}_{g,\mu}$
- μ meromorphic: $\widetilde{\mathcal{H}}_{g,\mu} \subset \overline{\mathcal{M}}_{g,\ell}$ pure codim g.

  Rule by JPPZ to assign multiplicities

  I dont review the rules.
- μ holomorphic: $\widetilde{\mathcal{H}}_{g,\mu} \subset \overline{\mathcal{M}}_{g,\ell}$ codim g, codim $g-1$

  (multiplicities ?)

Completely New view of twisted differentials

Holmes, Holmes - Schmitt, Bae - H - P - Sch - Schwarz

Let $\mathrm{Pic}_{g,\ell,2g-2}$ be the stack parameterizing

$$[C, p_1, \ldots, p_\ell, \mathcal{L}]$$ where

- C is a connected nodal of genus g with ℓ markings (prestable)
- $\mathcal{L} \to C$ is a line bundle of degree $2g-2$

$\mathrm{Pic}_{g,\ell,2g-2}$ is nonsingular of

$$\dim_{\mathbb{C}} = 3g-3+\ell+g$$

Let $AJ_\mu \subset \mathrm{Pic}_{g,\ell,2g-2}$ be

the <u>closure</u> of the locus of

$$[C, p_1, \ldots, p_\ell, \mathcal{L}] \quad \text{Satisfying:}$$

C nonsingular and $\mathcal{L} \cong \mathcal{O}_C\left(\sum_{i=1}^{\ell} m_i p_i\right)$

$AJ_\mu \subset \mathrm{Pic}_{g,\ell,2g-2}$ is pure codim <u>g</u>

We have a canonical morphism.

$$\phi_\mu : \overline{\mathcal{M}}_{g,\ell} \to \mathrm{Pic}_{g,\ell,2g-2},$$

$$\phi_\mu [C, p_1, \ldots, p_\ell] = [C, p_1, \ldots, p_\ell, \omega_C]$$

Define $\widetilde{\mathcal{H}}_{g,\mu}$ via the fiber square:

$$
\begin{tikzcd}
\widetilde{\mathcal{H}}_{g,\mu} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & AJ_{\mu} \arrow[d] \\
\overline{\mathcal{M}}_{g,\ell} \arrow[r, "\phi_{\mu}"'] & \mathcal{P}ic_{g,\ell,2g-2}
\end{tikzcd}
$$

universal AJ theory

Actually, the diagram puts a <u>scheme</u> structure on $\widetilde{\mathcal{H}}_{g,\mu}$.

- Exercise: At the level of points, we match Farkas–P.

- Much harder (Holmes–Schmitt): For μ meromorphic, the multiplicities match!

From the fiber square, the class

$$[\widetilde{\mathcal{H}}_{g,\mu}] \in A^g(\overline{\mathcal{M}}_{g,\ell})$$

is a very natural intersection product.

Calculating $[\widetilde{\mathcal{H}}_{g,\mu}]$ becomes an obvious question which is solved by Pixtons formula.

BHPSS 2020

There is a long story

I will not tell it here.

Instead I provide only a road map:

Orbi, Rel GW theory

JPPZ DR 1602.04705

JPPZ2 DR(x) 1812.10136

Farkas-P 1508.07140

Holmes-Schmitt 1909.11981

Schmitt 1607.08429

BHPSS Pic 2004.08676

Whole theory also for k-diff

JPPZ = Janda P Pixton Zvonkine

BHPSS = Bae Holmes P Schmitt Schwarz

Are there any questions left

from the perspective I have explained?

(Q1) Calculate $[\widetilde{\mathcal{H}}_{g,\mu}] \in A^g(\overline{\mathcal{M}}_{g,\ell})$

via classical intersection theory.

(Q2) For μ holomorphic,

geometrically identify the excess cycle

on the main codim $g-1$ component.

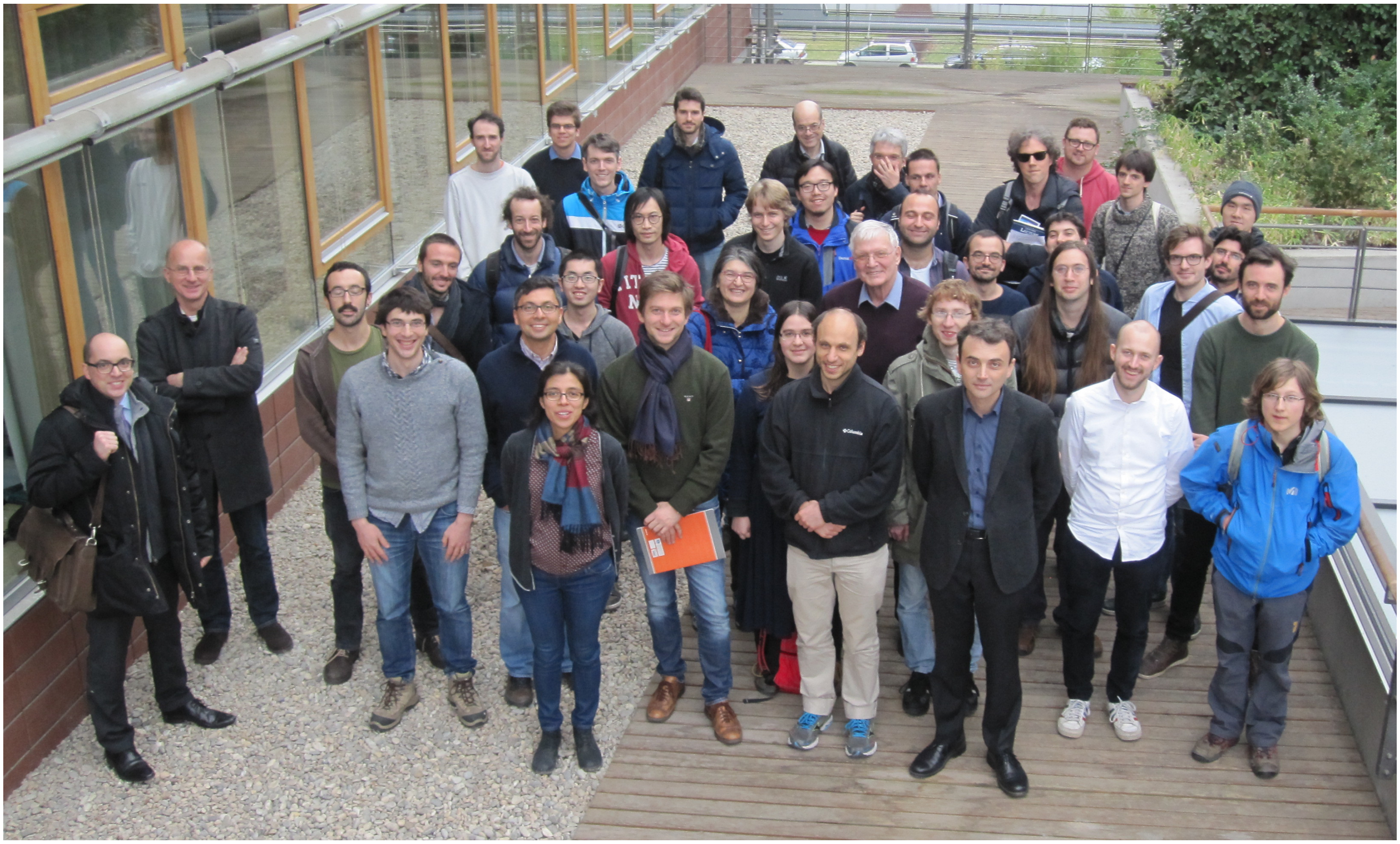
(Q3) Use the formula of [BHPSS] for the universal DR cycle applied to moduli space of curves with even theta characteristics.

Sauvaget, Schmitt, ...

H. H. Tseng

(Q4) Find a natural desingularization of $AJ_\mu \subset \mathrm{Pic}_{g,\ell,2g-2}$.

See Bainbridge, D. Chen, Gendron, Grushevsky, Möller for a desingularization of the closure of $\mathcal{H}_{g,\mu}$.



Moduli of holomorphic differentials

9-11 Feb 2016

II. Connections to Witten's r-spin class

Let $r \geq 2$ be an integer

Witten's r-spin class : $a_i \in \{0, 1, \ldots, r-2\}$

$$W^r_{g,n}(a_1, a_2, \ldots, a_n) \in H^{2 \cdot D}(\overline{\mathcal{M}}_{g,n}).$$

$$D = \frac{(r-2)(g-1) + \sum_{i=1}^n a_i}{r}$$

For V_r of dim $r-1$, we have a CohFT

$$W^r_{g,n} : V_r^{\otimes n} \to H^*(\overline{\mathcal{M}}_{g,n}).$$

Geometric idea (Witten, Polishchuk, Polishch-Vaintrob ...)

Look on the moduli of r^{th}-roots $\overline{M}^r_{g,n}(a_1,\dots,a_n)$

$$\mathcal{L}^{\otimes r} \cong \omega_C\left(-\sum_{i=1}^n a_i p_i\right)$$

Imagine $H^1(\mathcal{L})$ is a bundle

$H^0(\mathcal{L})=0$ (True in $g=0$)

so

$$\begin{array}{c} H^1(\mathcal{L}) \\ \downarrow \\ \overline{M}^r_{g,n}(a_1,\dots,a_n) \end{array}$$

Rank of $H^1(\mathcal{L})$ is then exactly

$$D = \frac{(r-2)(g-1)+\sum_{i=1}^n a_i}{r}.$$

Then $W^r_{g,n}(a_1, \ldots, a_n)$ is (up to an r-factor)

just the push-forward under

$$\pi : \overline{M}^r_{g,n}(a_1, \ldots, a_n) \to \overline{M}_{g,n}$$

of $c_{top}\left(H^1(\mathcal{L})^* \right)$

Long study of the theory ...

Complete solution via semisimplicity, R-matrices, Verlinde algebras ...

[using Givental-Teleman classification

- PPZ 2016 arxiv: 1607.00978
- P 2018 Cohomological Field Theory Calculations (overview)

Conjecture (JPPZ 2016):

Let $a_1, \ldots, a_n$ be non-negative integers with $\sum_{i=1}^{n} a_n = 2g-2$.

Let $\mu = (a_1, \ldots, a_n)$. Then,

$$\left[\mathrm{Closure}\left(\mathcal{H}_{g,\mu}\right)\right] =$$

$$-(-r)^{g-1} \, \mathcal{W}^{r}_{g,n}(a_1, \ldots a_n) \Big|_{r=0} \in H^{2g-2}\left(\overline{M}_{g,n}\right)$$

By PPZ, the above is a polynomial in r (for all $r \gg 0$).

$$\left[\dim_{\mathbb{C}} D = \frac{(r-2)(g-1) + \sum_{i=1}^{n} a_i}{r} = g-1 \right]$$

The Conjecture from 2016 covers the holomorphic case.

Let $\mu = (\underbrace{a_1, \dots, a_n}_{a_i \geq 0}, \underbrace{-b_1, \dots, -b_m}_{b_j > 0})$, $|\mu| = 2g-2$. Meromorphic

Conjecture (Janda-Pixton-Schmitt, June 2020)

$$\left[\mathrm{Closure}\left(\mathcal{H}_{g,\mu}\right)\right] =$$

$$(-r)^g \, W^r_{g,n}\left(a_1, \dots a_n, \frac{r}{m} - b_1, \dots, \frac{r}{m} - b_m\right)\Bigg|_{r=0} \in H^{2g}\left(\overline{M}_{g,n}\right)$$

And there are other variations ...

Proof in progress: Q. Chen - J - Ruan - Sauvaget - Zvonkine, ...

(Q5) Geometrically understand all $\eta = 0$ restrictions of such limits of Witten's r-spin class.

(Q6) The twisted differentials theory governs all higher differentials also. Is there some variant of Witten's theory for k-differentials?

III. Volumes (D. Chen, Möller, Sauvaget, Zagier)

Let $\mu = (m_1, \dots, m_\ell)$ with $m_i \geq 0$ [holomorphic case

$\mathcal{H}_{g,\mu}$ is a nonsingular DM stack

with a natural volume form [defined using period coords, normalization conventions

$\mathrm{Vol}(\mathcal{H}_{g,\mu})$ is called the <u>Masur-Veech</u> volume,

first computed by Eskin-Okounkov

using a study of Hurwitz covers of $\circled{\text{a}}$.

Let $\mathbb{E} \to \overline{\mathcal{M}}_{g,\ell}$ be the Hodge bundle

We have

$$\begin{tikzcd} & \mathbb{P}(\mathbb{E}|_{\mathcal{M}_{g,\ell}}) \arrow[d] \\ \mathcal{H}_{g,\mu} \arrow[ur, hook] \arrow[r, hook] & \mathcal{M}_{g,\ell} \end{tikzcd},$$

let

$$\begin{tikzcd} & \mathbb{P}(\mathbb{E}) \arrow[d] \\ \widehat{\mathcal{H}}_{g,\mu} \arrow[ur, hook] \arrow[r, hook] & \overline{\mathcal{M}}_{g,\ell} \end{tikzcd}$$

be the closure.

Tautological divisor classes:

$$\xi = c_1(\mathcal{O}_{\mathbb{P}(\mathbb{E})}(1)) \in H^2(\mathbb{P}(\mathbb{E}))$$

$\psi_1, \dots, \psi_\ell \in H^2(\overline{\mathcal{M}}_{g,\ell})$ cotangent lines

Main result (D. Chen, Möller, Sauvaget, Zagier 2019)

$$\mathrm{vol}\left(\mathcal{H}_{g,\mu}\right) =$$

$$-\frac{2(2i\pi)^{2g}}{(2g-3+n)!} \int_{\widehat{\mathcal{H}}_{g,\mu}} \xi^{2g-2} \prod_{i=1}^{n} \psi_i$$

Idea of proof: volumes + integrals satisfy the same recursion equations.

(Q7) Is there a formula for

$$[\hat{\mathcal{H}}_{g,\mu}] \in A^{2g-2}(\mathbb{P}(\mathbb{E})) \; ?$$

Using divisor formulas for

$$\xi \in \mathrm{Pic}(\hat{\mathcal{H}}_{g,\mu}),$$

We can calculate $[\hat{\mathcal{H}}_{g,\mu}]$ in terms of

- Pixton's formula for $[\tilde{\mathcal{H}}_{g,\mu}]$
- r-Spin formula for $[\mathrm{Closure}(\mathcal{H}_{g,\mu})]$

But the result is not a formula yet...

(Q8) What is $\mathrm{Pic}(\mathcal{H}_{g,\mu})$?

Speculation: In the irreducible <u>holomorphic</u> case,

$$\mathrm{Pic}(\mathcal{H}_{g,\mu}) \otimes \mathbb{Q} = \mathbb{Q}$$

for $g \geq 2$.

Maybe a topological approach is needed here.

The End