Hilbert Schemes of Singular Curves

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Let C be a reduced, irreducible Gorenstein curve.

For example, C could be a reduced, irreducible plane curve.

Let g = Arithmetic genus of C

$\tilde{g}$ = geometric genus of C

Let $\mathrm{Hilb}(C, n)$ be

the Hilbert scheme of length n

Subschemes of C.

Let

$$\mathcal{H}_C(q) = \sum_{n \geq 0} q^n \chi(\mathrm{Hilb}(C,n))$$

Where χ is the topological Euler

Characteristic.

Since $\mathrm{Hilb}(C, 0) = \text{point}$,

$$\mathcal{H}_C(q) = 1 + \cdots$$

In our 3rd paper (Stable pairs and BPS States), Richard Thomas and I prove

$$(\star) \quad \mathcal{H}_C(q) = \sum_{h=\tilde{g}}^{g_a} n_{h,C}\, q^{g_a-h} (1-q)^{2h-2}$$

for integers $n_{h,C}$.

If C is nonsingular then only one term appears in the sum $(\star)$ and $n_{g_a,C} = 1$.

Some examples.

(i) Let C be a rational curve with 1 node.

$g_a = 1 \quad , \quad \tilde{g} = 0$

$\chi(\mathrm{Hilb}(C,0)) = 1$

$\chi(\mathrm{Hilb}(C,1)) = \chi(C) = 1$

Hence

$$\mathcal{H}_C(q) = 1 + q + \cdots$$

We conclude $\quad n_{0,C} = 1$

$n_{1,C} = 1.$

(ii) Let C be a rational curve with 1 cusp

$g_a = 1$, $\tilde{g} = 0$

$\chi(\mathrm{Hilb}(C,0)) = 1$

$\chi(\mathrm{Hilb}(C,1)) = \chi(C) = 2$

Hence

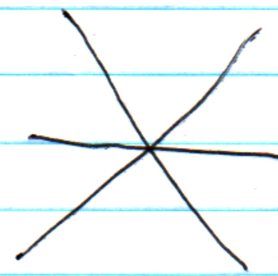
$\mathcal{H}_C(q) = 1 + 2q + \cdots$

We conclude

$n_{0,C} = 2$

$n_{1,C} = 1.$

(iii) Let C be a rational curve with 1 ordinary triple point



$g_a = 3$, $\tilde{g} = 0$

We need to calculate $\mathcal{H}_C(q)$ up to order q^3 to determine all $n_{h,C}$.

Let $C^* = C \setminus$ singular point.

$$\mathcal{H}_{C^*}(q) = (1-q)^{-\chi(C^*)} = (1-q)$$

Let $\mathcal{H}_{Tr}(q)$ be the generating series of Euler characteristics of subschemes of C supported on the triple point.

Certainly
$$\mathcal{H}_{C^*}(q) \cdot \mathcal{H}_{tr}(q) = \mathcal{H}_C(q).$$

Since a triple is locally defined by a product of 3 linear forms, all subschemes of $\mathbb{C}^2$ of length at most 3 can be placed on the triple point.

Thus

$$H_{tr}(q) = 1 + q + 2q^2 + 3q^3 + \cdots$$

Up to order q^3, these are just the Euler Chars of the punctual Hilbert Scheme

Putting all of this together:

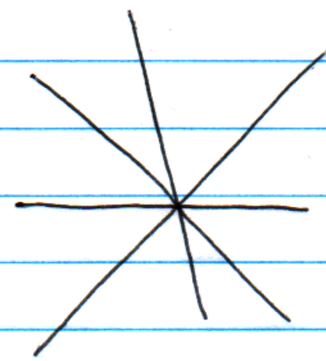
$$n_{0,c} = 1$$

$$n_{1,c} = 3$$

$$n_{2,c} = 4$$

$$n_{3,c} = 1.$$

(iv) Let C be a rational curve with 1 ordinary quadruple point



$g_a = 6$, $\bar{g} = 0$

As before

$$H_C(q) = H_{C^*}(q) \cdot H_{quad}(q)$$

$$= (1-q)^2 \cdot H_{quad}(q)$$

We must calculate $H_{quad}(q)$ up to order q^6.

All subscheme of $\mathbb{C}^2$ of length ≤ 4 can be placed on the quadruple point. Hence

$$H_{quad}(q) = 1 + q + 2q^2 + 3q^3 + 5q^4 + \cdots$$

but we need the q^5 and q^6 terms also.

If we let the quadruple point be given (locally analytically) by four linear form

$$0 = L_1(z_1, z_2) \cdot L_2(z_1, z_2) \cdot L_3(z_1, z_2) \cdot L_4(z_1, z_2)$$

Then there is a scaling torus action

$$t * (z_1, z_2) = (tz_1, tz_2)$$

which preserves the quadruple point and hence acts on the Hilbert Schemes.

To calculate $\chi(\mathrm{Hilb}(\mathrm{Quad}, 5))$ it suffices to examine the fixed points of the scaling torus.

The fixed points are simply subschemes defined by homogeneous polynomials in z_1, z_2.

For such a subscheme $S \subset$ Quadangle there is a natural degree grading on the length 5 ring $\mathcal{O}_S$.

The degree grading could be

$(1, 1, 1, 1, 1) \Rightarrow S$ is supported on one of the L_i

$\Rightarrow$ 4 possibilities.

$(1, 2, 2) \Rightarrow$ moduli space is $\mathbb{P}^2$

(choosing a quadratic form).

$(1, 2, 1, 1) \Rightarrow$ moduli space is $\mathbb{P}^1$

(The two quadratic forms in the ideal must have a common factor).

There are no other possibilities

$$\chi(\mathrm{Hilb}(\mathrm{Quad}, 5)) = 4 + 3 + 2 = 9$$

Now consider Hilb(Quad, 6)

The fixed points of the scaling torus are of grading:

(1, 1, 1, 1, 1, 1) $\Rightarrow$ moduli space is 4 points as before.

(1, 2, 3) $\Rightarrow$ moduli space is 1 point.

(1, 2, 2, 1) $\Rightarrow$ moduli space is a $\mathbb{P}^1$ bundle on $\mathbb{P}^2$

(1, 2, 1, 2) $\Rightarrow$ Impossible

(1, 2, 1, 1, 1) $\Rightarrow$ moduli space is 4 points.

No other possibilities

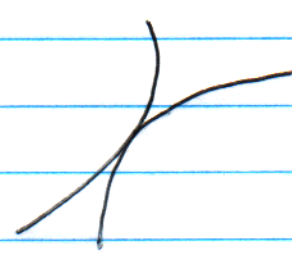
$$\chi(\mathrm{Hilb}(\mathrm{Quad}, 6)) = 4 + 1 + 6 + 0 + 4$$
$$= 15.$$

We can now calculate

$$n_{0,c} = 1$$
$$n_{1,c} = 6$$
$$n_{2,c} = 19$$
$$n_{3,c} = 36$$
$$n_{4,c} = 28$$
$$n_{5,c} = 9$$
$$n_{6,c} = 1.$$

If C is rational with normalization obtained <u>just</u> by separating branches, we seem to find $n_{0,C} = 1$.

(v) Let C be a rational curve with a tacnode



$g_a = 2 \quad \tilde{g} = 0$

$$\mathcal{H}_C(q) = \mathcal{H}_{C^*}(q) \cdot \mathcal{H}_{tac}(q)$$

$$= (1-q)^0 \, (1 + q + 2q^2 + \cdots$$

So $\quad n_{0,C} = 1$

$n_{1,C} = 3$

$n_{2,C} = 1.$

Question

(A) As we have seen in the calculations, the invariants $n_{h,C}$ depend upon the singularities of C.

How do they depend?

For example for the quadruple point, the answer was independent of the j-invariant of the 4-lines.

(B) Can the $n_{h,c}$ be easily calculated for some special classes of singularities?

(C) Can the genus h curves corresponding to $n_{h,c}$ be seen?