HURWITZ/HODGE FORMULAS

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Recall the ELSV formula expresses Hurwitz numbers with arbitrary ramification over ∞ (and simple ramification elsewhere) as Hodge integrals:

$$H_{g,\mu} = \frac{(2g-2+|\mu|+\ell)!}{\mathrm{Aut}(\mu)} \prod_i \frac{\mu_i^{\mu_i}}{\mu_i!} \int_{\overline{M}_{g,\ell}} \frac{1-\lambda_1+\lambda_2-\lambda_3+\dots}{\prod_i(1-\mu_i\psi_i)}. \tag{1}$$

The ELSV formula is true because the Hurwitz number can be calculated as a single vertex contribution for an integral over a moduli space of maps to $\mathbf{P^1}$ relative to ∞.

Let $\mathbf{P^1}[a,b]$ denote the orbifold with stack points of order a, b at $0, \infty$. I believe there is an analogous Hurwitz/Hodge formula for each of these geometries — formula (1) is the case $[1,1]$. Here the structure a governs which integrals appear.

I will explain here the case $[2,1]$. The resulting formula provides a new calculation of Hodge integrals for the moduli stack of double covers of curves. From the point of view of integration over moduli space, I suspect the cases $[a,1]$ are the most important. The complexity grows with a.

Let $A_{g,b,n}$ be the moduli space of admissible double covers,

$$\pi : D \to C,$$

satisfying the following properties:

(i) the domain D is a nodal curve of genus $2g-1+\frac{b}{2}$,
(ii) the range C is a stable genus g curve with $b+n$ marked points,
(iii) the cover π is étale over C^{ns} except over the points b where π is ramified.

(iv) the distinct branches of each node η of D map to distinct branches of the node $\pi(\eta)$ of C with equal ramification orders over $\pi(\eta)$.

The moduli space $A_{g,b,n}$ is defined only if $2g - 2 + b + n > 0$ and is empty unless b is even. If $A_{g,b,n}$ is non-empty, every point has an automorphism obtained from the involution of the cover π.

A special case occurs when $b = 0$. Then $A_{g,0,n}$ has a decomposition:

$$A_{g,0,n} = \overline{M}_{g,n}/\mathbb{Z}_2 \ \bigcup \ \tilde{A}_{g,0,n},$$

where the first component is the moduli space of disconnected double covers (identified with the quotient of $\overline{M}_{g,n}$ by the trivial $\mathbb{Z}_2$-action). The second space $\tilde{A}_{g,0,n}$ parametrizes all the connected double covers. If $b > 0$, then all the covers are connected.

Define the Hodge bundle $\mathbb{K}$ over $A_{g,b,n}$ to have fiber over $\pi : D \to C$ given by the vector space

$$H^0(D, \omega_D)/\pi^* H^0(C, \omega_C).$$

Let $k_1, k_2, k_3, \ldots$ denote the Chern classes of $\mathbb{K}$.

The cotangent line classes ψ_i are well defined on $A_{g,b,n}$ for all the $b + n$ markings of the range curve C — the cotangent lines are the cotangent lines on C.

Note: it is perhaps more natural to place stack structure on C and take the cotangent lines to be the cotangent lines of the stack structure — I do *not* do that here (the difference is just some scalar factors throughout the formulas).

The bundle $\mathbb{K}$ has rank equal to $g - 1 + \frac{b}{2}$ *except* on the component $\overline{M}_{g,n}/\mathbb{Z}_2$ of $A_{g,0,n}$. On $\overline{M}_{g,n}/\mathbb{Z}_2$, the Hodge bundle reduces to the usual Hodge bundle $\mathbb{E}$ (with the trivial $\mathbb{Z}$-action) which is of rank g instead of $g - 1$. As a result, the component must often be treated separately.

We will now consider the Hurwitz problem for the target $\mathbf{P^1}[2,1]$. Let g and b denote a genus and branch number. Let μ be a partition of $d > 0$. Let ℓ denote the number of parts of μ, and let ℓ_E, ℓ_O denote the number of even and odd parts respectively. Let $\overline{M}_{g,b}(\mathbf{P^1}[2,1], \mu)$ be the moduli space of stable maps from genus g curves with b marked stack points of order 2 to $\mathbf{P^1}[2,1]$ of degree d with profile μ over ∞. The expected dimension of $\overline{M}_{g,b}(\mathbf{P^1}[2,1], \mu)$ is

$$r = 2g - 2 + \frac{1}{2}b + \frac{1}{2}d + \ell.$$

The moduli space is empty unless $d = b \mod 2$.

Let $H_{g,b,\mu}[2,1]$ denote the associated Hurwitz number: the count of maps from genus g curves with b marked stack points of order 2 to $\mathbf{P^1}[2,1]$ of degree d with profile μ over ∞ simply ramified over r fixed points of $\mathbf{A}$.

The Hurwitz problem for $\mathbf{P^1}[2,1]$ is easily seen to have a solution in terms of the usual Hurwitz problem:

- If $b > d$, then $H_{g,b,\mu}[2,1] = 0$.
- If $b \leq d$, then

  $$\frac{1}{b!} H_{g,b,\mu}[2,1] = H_{g,1^b 2^{(d-b)/2},\mu},$$

  where the latter is the standard Hurwitz number of $\mathbf{P^1}$ with profiles $1^b 2^{(d-b)/2}, \mu$ over $0, \infty$ and simple ramifications over r other fixed points. The factor of $1/b!$ is present because the order 2 points of the domain are labelled.

These Hurwitz problems are considered with connected domains.

The main point is the expression of the Hurwitz number $H_{g,b,\mu}[2,1]$ as a vertex integral over the moduli space $A_{g,b',n}$. As before, $H_{g,b,\mu}[2,1]$ may be expressed an equivariant Gromov-Witten integral over the moduli space $\overline{M}_{g,b}(\mathbf{P^1}[2,1], \mu)$ using a version of the ramification map I had defined with Barbara. After applying virtual localization, a formula is immediately obtained.

Of course, it is difficult to get all the factors, especially of 2, correct, but I think I have now succeeded. I will explain the formula and then show some sample calculations.

Case I: $2g-2+b+\ell>0$.

Here, the vertex term is non-degenerate. We find a direct fomula expressing the Hurwitz numbers as integrals over the moduli of double covers:

$$H_{g,b,\mu}[2,1] = \frac{r!}{\mathrm{Aut}(\mu)} \frac{1}{2^{g-1+\frac{b-\ell_O}{2}}} \cdot$$

$$\prod_{\mu_i \ even} \frac{\mu_i^{\frac{\mu_i}{2}}}{(\frac{\mu_i}{2})!} \prod_{\mu_i \ odd} \frac{\mu_i^{\frac{\mu_i-1}{2}}}{(\frac{\mu_i-1}{2}!} \int_{A_{g,b+\ell_O,\ell_E}} \frac{1-2k_1+4k_2-8k_3+\dots}{\prod_i(1-\mu_i\psi_i)}.$$

The cotangent line classes ψ_i correspond to the markings of $A_{g,b+\ell_0,\ell_E}$ via the obvious labelling by parts of μ.

Case II: $2g-2+b+\ell \leq 0$.

Here, $g=0$ and $b+\ell \leq 2$. In fact, the formula is unchanged with the following conventions for the unstable integrals:

$$\int_{A_{0,0,1}} \frac{1-2k_1+4k_2-8k_3+\dots}{(1-x\psi_1)} = \frac{1}{2}\cdot\frac{1}{x^2},$$

$$\int_{A_{0,2,0}} \frac{1-2k_1+4k_2-8k_3+\dots}{(1-x\psi_1)(1-y\psi_2)} = \frac{1}{2}\cdot\frac{1}{x+y},$$

$$\int_{A_{0,0,2}} \frac{1-2k_1+4k_2-8k_3+\dots}{(1-x\psi_1)(1-y\psi_2)} = \frac{1}{2}\cdot\frac{1}{x+y}.$$

These conventions fit the usual framework for unstable contributions. The factor of $\frac{1}{2}$ is for the automorphisms of the double cover moduli space. By parity considerations, the unstable spaces $A_{0,1,0}$ and $A_{0,1,1}$ do not appear.

As a final comment on the formula, we note if $b + \ell_0 = 0$, then the special moduli space $A_{g,0,\ell_E}$ arises. The formula may then be rewritten in terms of the components:

$$H_{g,0,\mu}[2,1] = \frac{r!}{\mathrm{Aut}(\mu)} \frac{1}{2^{g-1}} \prod_{\mu_i \ even} \frac{\mu_i^{\frac{\mu_i}{2}}}{(\frac{\mu_i}{2})!} \cdot$$
$$\Big(\frac{1}{2} \int_{\overline{M}_{g,\ell_E}} \frac{1 - 2\lambda_1 + 4\lambda_2 - \dots}{\prod_i (1 - \mu_i \psi_i)} + \int_{\tilde{A}_{g,O,\ell_E}} \frac{1 - 2k_1 + 4k_2 - \dots}{\prod_i (1 - \mu_i \psi_i)}\Big).$$

We now discuss several examples. These serve to check the formula in various basic cases:

(i) Let $g = 0$, $b = 0$, and $\mu = d^1$ where $d = 0 \mod 2$. Then, we find:

$$H_{0,0,d^1}[2,1] = H_{0,2^{\frac{d}{2}},d^1} = 2d^{\frac{d}{2}-3}.$$

I checked for $d = 2, 4, 6, 8$ that the answer is correct.

(ii) Let $g = 0$, $b = 1$, and $\mu = d^1$ where $d = 1 \mod 2$. Then, we find:

$$H_{0,1,d^1}[2,1] = H_{0,1^1 2^{\frac{d-1}{2}},d^1} = d^{\frac{d-3}{2}}.$$

I checked for $d = 1, 3, 5$ that the answer is correct.

(iii) Let $g = 0$, $b = 0$, and $\mu = 2^2$. Then, we find:

$$H_{0,0,2^2}[2,1] = H_{0,2^2,2^2} = 1,$$

which is the correct answer.

(iv) Let $g = 0$, $b = 0$, and $\mu = 1^1 3^1$. Then, we find:

$$H_{0,0,1^1 3^3}[2,1] = H_{0,2^2,1^1 3^1} = 3,$$

which is the correct answer.

(v) Let $g = 0$, $b = 2$, and $\mu = 1^2$. Then, we find:

$$H_{0,2,1^2} = 2! H_{0,1^2,1^2} = 1,$$

which is the correct answer.

This is the first case which involves integration. Expansion of the formula yields:

$$\begin{aligned} H_{0,2,1^2}[2,1] &= 2\int_{A_{0,4,0}} \frac{1-2k_1}{(1-\psi_1)(1-\psi_2)} \\ &= 2\frac{1}{2}\int_{\overline{M}_{0,4}} \frac{1}{(1-\psi_1)(1-\psi_2)} - 4\int_{A_{0,4,0}} k_1 \\ &= 2-4\int_{A_{0,4,0}} k_1. \end{aligned}$$

To evaluate the last integral, note the canonical,

$$A_{0,4,0} \to \overline{M}_{1,b_1},$$

where the first branch point is selected for the marking on the elliptic curve, is a stack theoretic map of degree 6. Moreover, k_1 is the pull-back of λ_1 under this map. Hence,

$$2-4\int_{A_{0,4,0}} k_1 = 2-4\cdot 6\cdot \frac{1}{24} = 1.$$

(vi) Let $g=0$, $b=3$, and $\mu = 1^1$. Then, we find:

$$H_{0,3,1^1} = 0,$$

which is the correct answer.

Expansion of the formula yields:

$$\begin{aligned} H_{0,3,1^1}[2,1] &= \int_{A_{0,4,0}} \frac{1-2k_1}{(1-\psi_1)} \\ &= \frac{1}{2}\int_{\overline{M}_{0,4}} \frac{1}{(1-\psi_1)} - 2\int_{A_{0,2,2}} k_1 \\ &= \frac{1}{2} - 2\cdot 6\cdot \frac{1}{24} \\ &= 0. \end{aligned}$$

(vii) Let $g=0$, $b=2\tilde{b}+1$, and $\mu=1^1$ for $\tilde{b}>0$. Then, we find:

$$H_{0,2\tilde{b}+1,1^1} = 0,$$

which is the correct answer.

Expansion of the formula yields:

$$H_{0,2\tilde{b}+1,1^1}[2,1] = \tilde{b}!\frac{1}{2^{-1+\tilde{b}}}\int_{A_{0,2\tilde{b}+2,0}} \frac{1-2k_1+4k_2-\cdots\pm 2^{\tilde{b}}k_{\tilde{b}}}{(1-\psi_1)}$$

The bundle $\mathbb{K}$ is simply given by the space of differential forms on the domain of the double cover (since the rational range has a vanishing space of differential forms).

The Chern roots of $\mathbb{K}$ can be identified by the vanishing sequence at a Weirstrass point of the domain. The Weirstrass point can be chosen to lie above the marking 1. The Chern roots of $\mathbb{K}$ are then $L, 3L, \ldots, (2\tilde{b}-1)L$ where L is the Chern class of the cotangent line of the Weirstrass point.

The class L on $A_{0,2\tilde{b}+2,0}$ is $\frac{1}{2}\psi_1$. Expanding the Chern roots of $\mathbb{K}$, we find:

$$\begin{aligned} H_{0,2\tilde{b}+1,1^1}[2,1] &= \tilde{b}!\frac{1}{2^{-1+\tilde{b}}}\int_{A_{0,2\tilde{b}+2,0}} \frac{\prod_{i=1}^{\tilde{b}}(1-(2i-1)\psi_1)}{(1-\psi_1)} \\ &= \tilde{b}!\frac{1}{2^{-1+\tilde{b}}}\int_{A_{0,2\tilde{b}+2,0}} \prod_{i=2}^{\tilde{b}}(1-(2i-1)\psi_1) \\ &= 0, \end{aligned}$$

where the last integral vanishes for dimension reasons for $\tilde{b} > 0$.

The argument here also provides an alternate computation of the vanishing in example (vi).