Abel-Jacobi maps and Double ramification cycles

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Shafarevich Seminar

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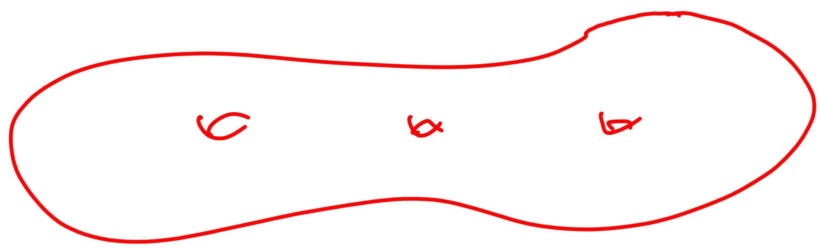
(ZOOM)

Let C be a nonsingular complex algebraic curve of genus g / $\mathbb{C}$

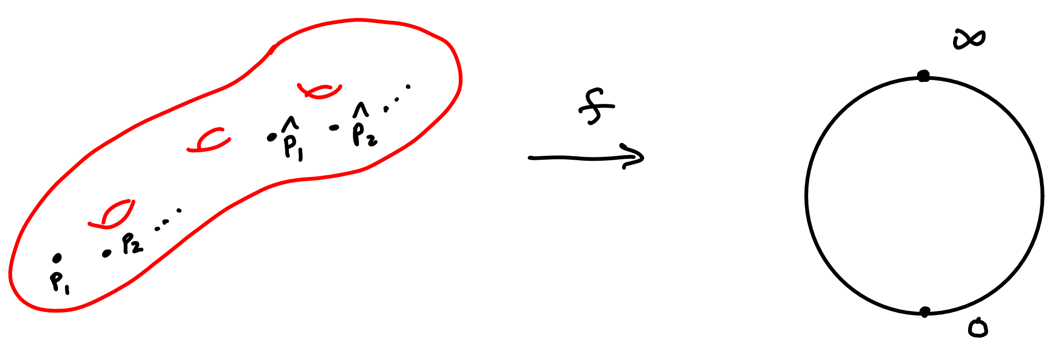


A function f can viewed

$f: C \to \mathbb{C}$ rational algebraic

$f: C \to \mathbb{C}$ meromorphic

$f: C \to \mathbb{CP}^1$ morphism



$$\operatorname{div}(f) = \sum_{p_i \in f^{-1}(0)} m_i p_i - \sum_{\hat{p}_j \in f^{-1}(\infty)} \hat{m}_j \hat{p}_j$$

The other way is to start with

$$A = (a_1, \dots, a_n) \in \mathbb{Z}^n \quad \text{with} \quad \sum_{i=1}^{n} a_i = 0$$

and distinct points $x_1, \dots, x_n \in C$.

Question: Does there exist a rational function

$$f : C \to \mathbb{C}$$

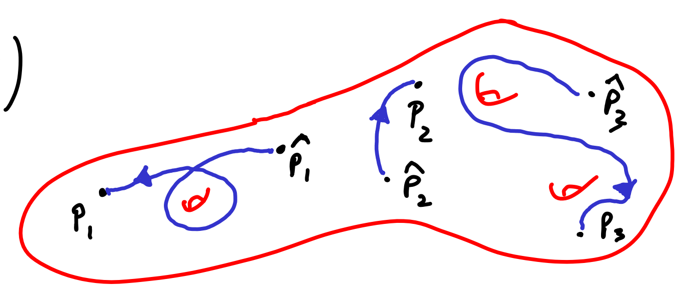
such that $\operatorname{div}(f) = \sum_{i=1}^{n} a_i x_i$?

Basic question in the history of algebraic geometry.

Solved by Abel and Jacobi in the 19^{th} Century.

New ideas: divisor, Jacobian, analytic methods

Solution $A = (1, 1, 1, -1, -1, -1)$



- Choose γ with

$$\partial \gamma = P_1 + P_2 + P_3 - \hat{P}_1 - \hat{P}_2 - \hat{P}_3$$

- Choose $\omega_1, \dots, \omega_g$ basis of holomorphic differentials on C
- Integrate : $\left(\int_\gamma \omega_1, \dots, \int_\gamma \omega_g \right) \in \mathbb{C}^g / H_1(C, \mathbb{Z})$

$$\mathbb{C}^g / H_1(C,\mathbb{Z}) = \mathrm{Jac}(C)$$

complex torus $\mathrm{Jac}(C)$

ambiguity in choice of γ

Answer (Abel - Jacobi)

$$\exists f : C \to \mathbb{C} \quad \text{with} \quad \mathrm{div}(f) = P_1 + P_2 + P_3 - \hat{P}_1 - \hat{P}_2 - \hat{P}_3$$

$$\Updownarrow$$

$$\left(\int_\gamma \omega_1, \dots, \int_\gamma \omega_g \right) = 0 \in \mathrm{Jac}(C)$$

Perfect gem from the 19th century.

~ 200 years later, we return to this question from a different point of view.

As before, $A = (a_1, a_2, \ldots, a_n) \in \mathbb{Z}^n$ with $\sum_{i=1}^{n} a_i = 0$.

$\mathcal{M}_{g,n}$ ⟵ moduli space of genus g curves with n distinct markings

Define an algebraic locus

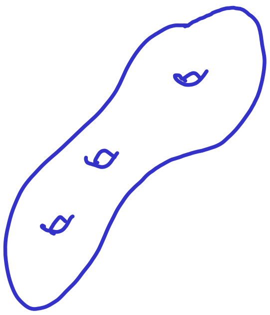
$$Z_{g,A} \subset \mathcal{M}_{g,n} = \left\{ (C, x_1, \ldots, x_n) \;\middle|\; \sum_i a_i x_i \sim 0 \right\}$$

Expect $\text{codim}_{\mathbb{C}} = g$ true (except in degenerate case)

$$\dim Z_{g,A} = 3g - 3 + n - g = 2g - 3 + n$$

We can approach $Z_{g,A}$ via Abel-Jacobi:

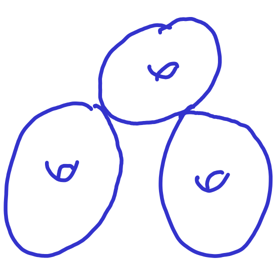
Smooth Curves



$$\begin{tikzcd} & \mathrm{Jac} \arrow[d] \\ \mathcal{M}_{g,n} \arrow[ur, "AJ"] \arrow[r, "\text{forget markings}"'] & \mathcal{M}_g \arrow[u, bend right, "O_{sect}"'] \end{tikzcd}$$

$$Z_{g,A} = AJ^{-1}(O_{sect}) \subset \mathcal{M}_{g,n}$$

Stable curve of compact type



$$\begin{tikzcd} & \mathrm{Jac} \arrow[d] \\ \mathcal{M}^{ct}_{g,n} \arrow[ur, "AJ"] \arrow[r, "\text{forget markings}"'] & \mathcal{M}^{ct}_g \arrow[u, bend right, "O_{sect}"'] \end{tikzcd}$$

$$\underset{H^{2g}(\mathcal{M}^{ct}_{g,n})}{\underset{\cap}{[Z^{ct}_{g,A}]}} = AJ^*([O_{sect}]) = AJ^*\left(\frac{1}{g!}\Theta^g\right)$$

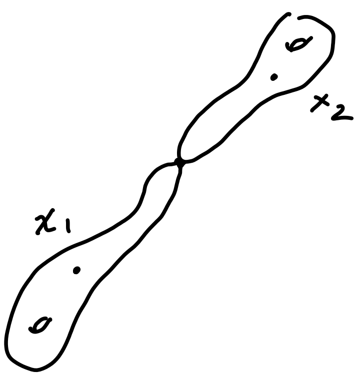
Hain
Grushevsky - Zakharov

The departure from Abel-Jacobi theory occurs when we ask the question for $\overline{\mathcal{M}}_{g,n}$

"What is $[Z_{g,A}] \in H^{2g}(\overline{\mathcal{M}}_{g,n})$"

<u>Not</u> well defined since the definition of $Z_{g,A}$ is <u>not</u> clear.

Why? $A = (2,-2)$



what does $2x_1 - 2x_2 \sim 0$ mean?

(a good answer in terms of twists)

Harder

(really not clear)

Some guiding questions:

Q1) Define and calculate $[Z_{g,A}] \in H^{2g}(\overline{M}_{g,n})$.

Different approaches to $Z_{g,A}$ have led to the same class

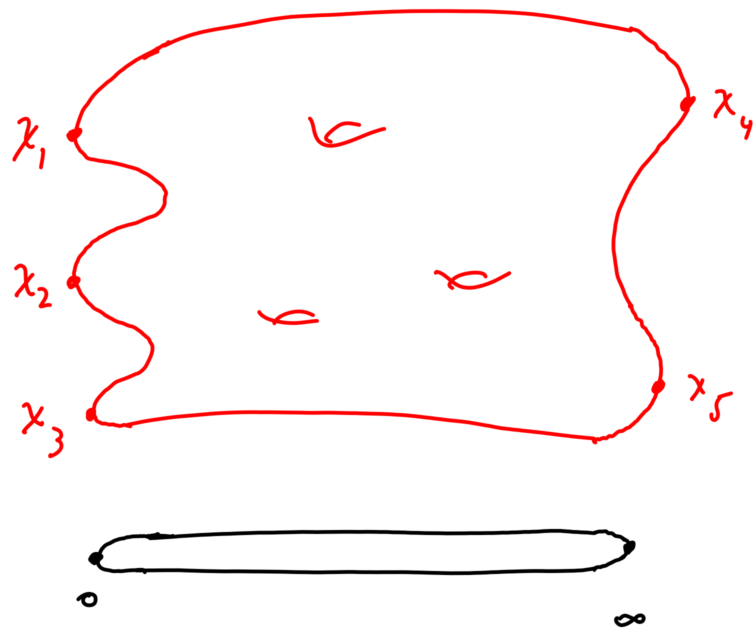
$$
\begin{array}{ccc}
\text{Li-Ruan, J. Li, Graber-Vakil} & & \text{log curves} \\
& & \text{Gross-Siebert, Marcus-Wise} \\
\text{Gromov Witten theory} \longrightarrow & [Z_{g,A}] & \swarrow \\
& & \nwarrow \\
& & \text{Holmes} \\
& & \text{Classical resolution of the Abel-Jacobi map}
\end{array}
$$

To calculate $[Z_{g,A}]$ is sometimes called Eliashberg's question ~ 2000 SFT

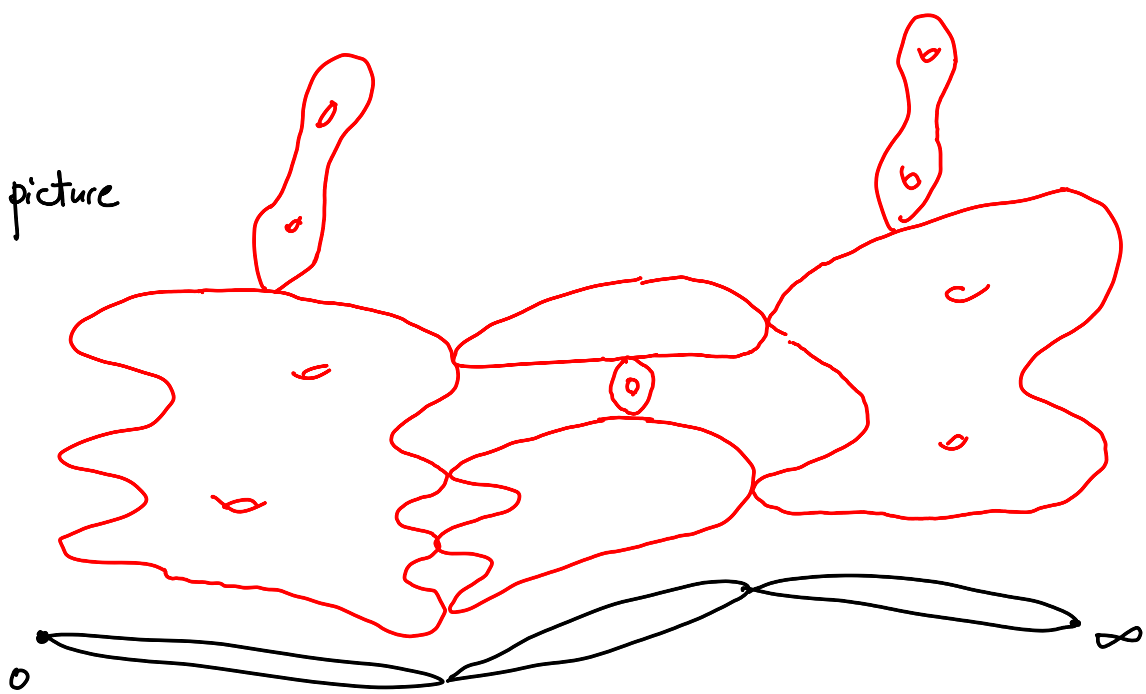
GW theory definition

$\overline{M}_g(\mathbb{P}^1, A)^{\sim}$ moduli of stable maps to $\mathbb{P}^1$ with ramifications over $0, \infty \in \mathbb{P}^1$ given by A.

ideal element



general picture



$\overline{\mathcal{M}}_g(\mathbb{P}^1, A)^\sim$ has 2 term obstruction theory

$\Rightarrow$ virtual fundamental class

$[\overline{\mathcal{M}}_g(\mathbb{P}^1, A)^\sim]^{vir}$ of dim $2g-3+n$ (no degenerate cases)

$$\overline{\mathcal{M}}_g(\mathbb{P}^1, A)^{\sim} \xrightarrow{\varepsilon} \overline{\mathcal{M}}_{g,n}$$ forgetful map

Definition: $[Z_{g,A}] = \varepsilon_*[\overline{\mathcal{M}}_g(\mathbb{P}^1, A)]^{\sim}$

Double ramification cycle

View $Z_{g,A}$ "=" $\mathrm{Im}(\varepsilon)$ $DR_g(A)$

Now we have a precise mathematical question.

Q2) Let $\begin{array}{c}\mathbb{E}_g \\ \downarrow \\ \overline{\mathcal{M}}_g\end{array}$ be the Hodge bundle with fiber $H^0(C, \omega_C)$ over $[C] \in \overline{\mathcal{M}}_g$

Over $\mathcal{M}_g^{ct}$ we have the universal Jac

$$\mathcal{M}_g^{ct} \xrightarrow{\tau} \mathcal{A}_g$$ moduli of PPAV

$$\mathbb{E}_g = \tau^*(\mathbb{E}_g)$$

GRR calculation $\Rightarrow$ $c_g(\mathbb{E}_g) = 0$ on $\mathcal{A}_g$

Van der Geer, Cycles on the moduli space of Abelian varieties

Conclusion : $\lambda_g = c_g(\mathbb{E}_g) \in CH^g(\overline{\mathcal{M}}_g)$ is

supported on $\overline{\mathcal{M}}_g \setminus \mathcal{M}_g^{ct} = \Delta_0$

Curves with a non disconnecting node

Question: find a formula for λ_g

supported on $\Delta_0 \subset \overline{\mathcal{M}}_g$.

Q3) Instead of functions, we can

consider differentials (or k-differentials)

Let $A = (a_1, \ldots, a_n)$ $\sum_{i=1}^{n} a_i = 2g-2$

$$\Omega_{g,A} \subset \mathcal{M}_{g,n} = \left\{ (C, x_1, \ldots, x_n) \,\middle|\, \sum a_i x_i \sim \omega_C \right\}$$

Question: Calculate $[\overline{\Omega}_{g,A}] \in H^{2g}(\overline{M}_{g,n})$

$\overline{\Omega}_{g,A}$ is the Zariski closure of $Z_{g,A}$

Bainbridge - Chen - Gendron - Grushevsky - Möller

Farkas - P, Schmitt, Sauvaget

The point of this talk is to explain the solution to all 3 questions.

Farkas - P 1508.07940

JPPZ DR 1602.04705

Holmes-Schmitt 1909.11981

JPPZ2 DR(x) 1812.10136

Schmitt 1607.08429

BHPSS Pic 2004.08676

JPPZ = Janda P Pixton Zvonkine

BHPSS = Bae Holmes P Schmitt Schwarz

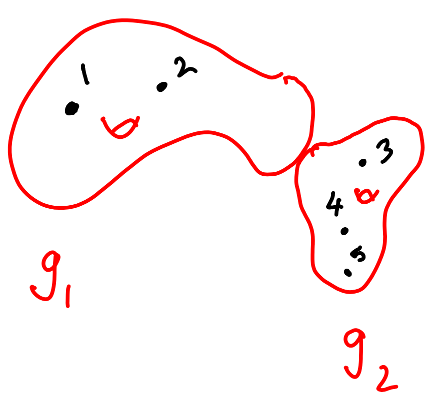
Q1) Eliashberg's question to calculate the double ramification cycle

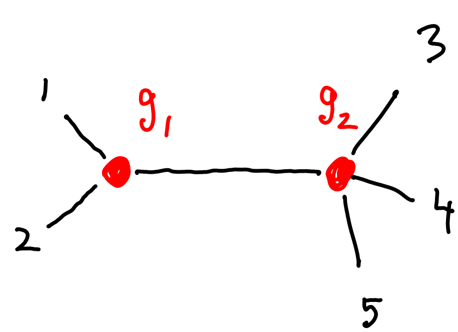
$$DR_g(A) \in H^{2g}(\overline{M}_{g,n})$$

Answer: Formula conjectured by Pixton (and proven in JPPZ).

Expressed in terms of tautological classes

Topological type $\longleftrightarrow$ Stable graph





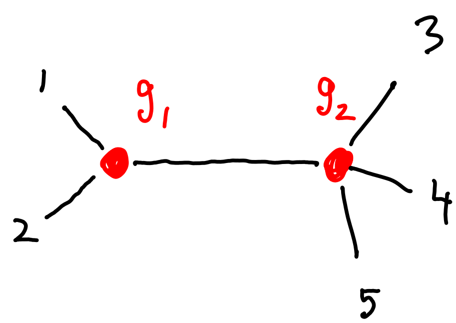
Let $G_{g,n}$ = set of stable graphs of genus g with n markings

finite set

for $\Gamma \in G_{g,n} \rightsquigarrow \overline{M}_{\Gamma} \xrightarrow{\xi_\Gamma} \overline{M}_{g,n}$

product of moduli spaces determinded by the vertices

for $\Gamma =$



$$\overline{M}_{\Gamma} = \overline{M}_{g_1,3} \times \overline{M}_{g_2,4}$$

Tautological classes are given by

$$\xi_{\Gamma *}\Big(\prod_{\text{markings}} \psi_i^{m_i} \prod_{\substack{\text{halves}\\ \text{of edges}}} \psi_j^{n_j} \prod_{\text{vertices}} \text{kappas} \Big) \in H^*(\overline{\mathcal{M}}_{g,n})$$

$$\overline{M}_\Gamma \xrightarrow{\xi_\Gamma} \overline{M}_{g,n}$$

The linear span of all such classes defines the tautological ring

$$RH^*(\overline{\mathcal{M}}_{g,n}) \subset H^*(\overline{\mathcal{M}}_{g,n})$$

We can also define the tautological ring

$$R^*(\overline{\mathcal{M}}_{g,n}) \subset CH^*(\overline{\mathcal{M}}_{g,n}).$$

Let $\Gamma \in G_{g,n}$ be a stable graph.

Let r be a positive integer

A **weighting mod r** of Γ is

$$w : H(\Gamma) \to \{0, 1, \dots, r-1\}$$

half edges

Remember

$A = (a_1, \dots, a_n)$

$\sum a_i = 0$

(I) $i \in$ Marking, $\quad w(i) = a_i \mod r$

(II) $e = (h, h') \in$ Edge, $\quad w(h) + w(h') = 0 \mod r$

(III) $v \in$ Vertex, $\quad \sum_{h \vdash v} w(h) = 0 \mod r$

$W_{\Gamma, r}$ is set of weightings mod r of Γ

$|W_{\Gamma,r}| = r^{h^1(\Gamma)}$

Definition (Pixton)

Let $P_g^{d,r}(A) \in R^d(\overline{M}_{g,n})$ be the degree d component of

$$\sum_{\Gamma \in G_{g,n}} \sum_{w \in W_{\Gamma,r}} \frac{1}{\mathrm{Aut}(\Gamma)} \frac{1}{r^{h^1(\Gamma)}} \cdot$$

$$\xi_{\Gamma *}\left[\prod_{1}^{n} \exp\left(\frac{a_i^2}{2}\psi_i\right) \cdot \prod_{e=(h,h')} \frac{1-\exp\left(-\frac{w(h)w(h')}{2}\cdot(\psi_h+\psi_{h'})\right)}{\psi_h+\psi_{h'}}\right]$$

Various motivations: Compact type restriction,
Givental - Teleman theory

Claim (Pixton):

$P_g^{d,r}(A) \in R^d(\overline{M}_{g,n})$ is polynomial in r for all $r >> 0$.

Definition (Pixton):

$P_g^d(A) \in R^d(\overline{M}_{g,n})$ is the constant term of $P_g^{d,r}(A)$

$r=0$

Theorem (Conjectured by Pixton, proven in JPPZ)

$$DR_g(A) = P_g^g(A) \in R^g(\overline{M}_{g,n}).$$

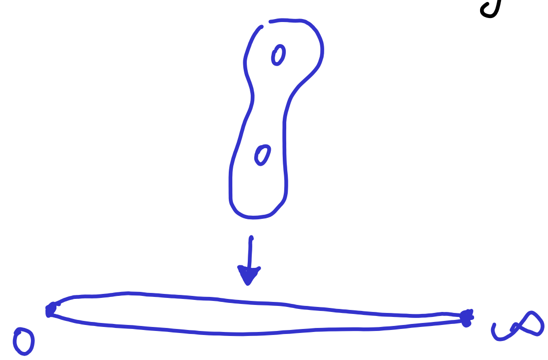
Q2) formula for $\lambda_g = c_g(\mathbb{E}_g)$ supported on $\Delta_0 \subset \overline{M}_g$

Answer: We view $\overline{M}_g = \overline{M}_{g,0}$

$n=0$

Let $A = \emptyset$

Geometry $\Rightarrow$ $\overline{M}_g(\mathbb{P}^1, \emptyset)^{\sim} \cong \overline{M}_g$



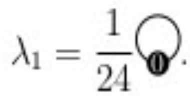
Moreover, $[\overline{M}_g(\mathbb{P}^1, \emptyset)^{\sim}]^{vir} = (-1)^g \lambda_g$

$$DR_g(\emptyset) = (-1)^g \lambda_g$$

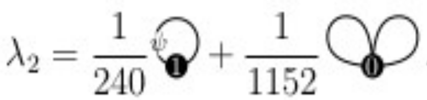
So we can apply the DR cycle formula:

from JPPZ

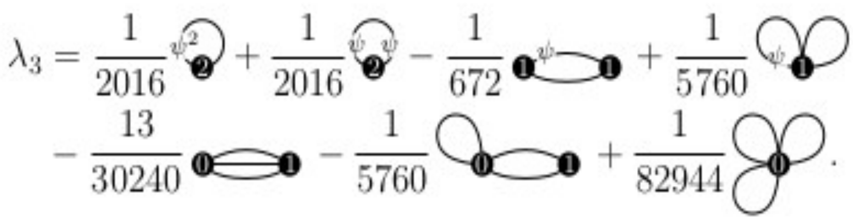
Genus 1.



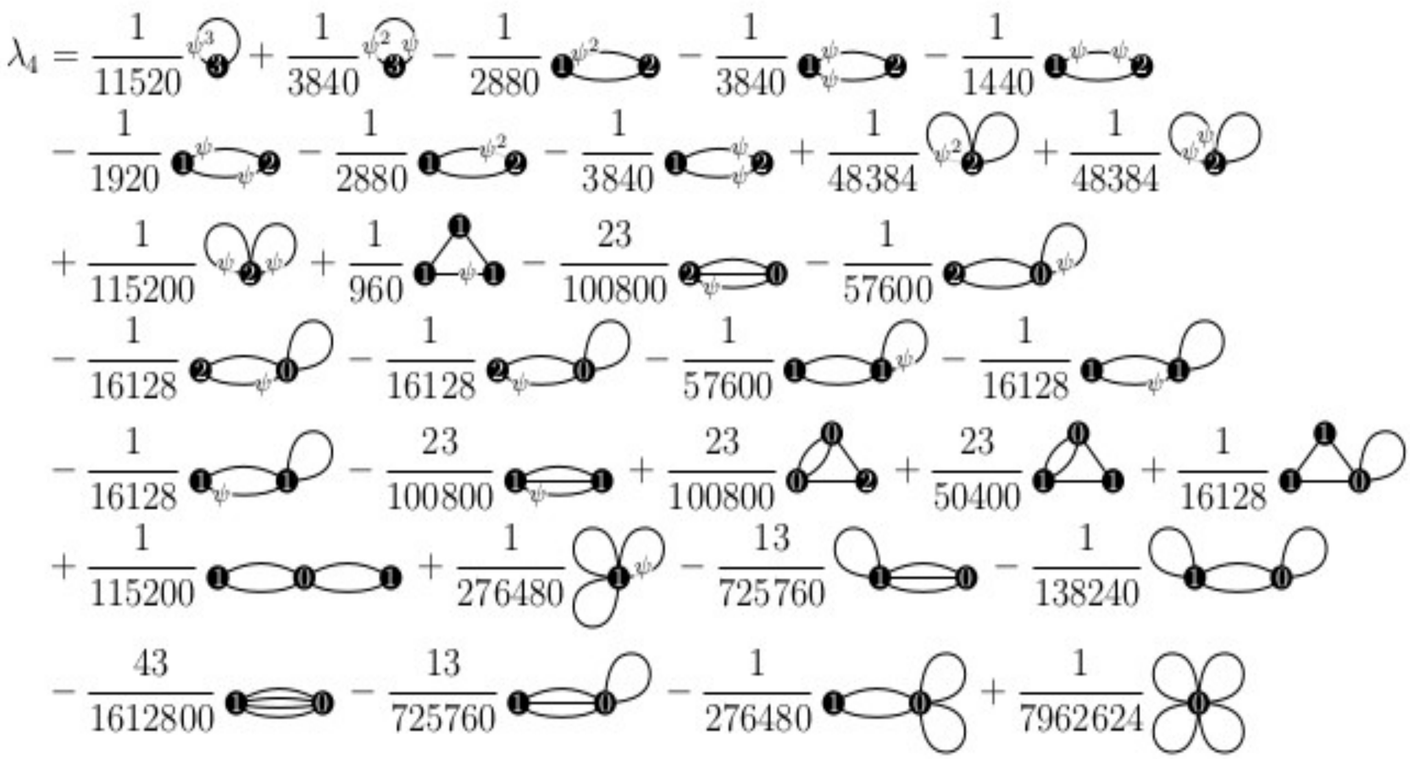
Genus 2.



Genus 3.



Genus 4.



Why only graphs with loops?

If Γ is an unpointed tree,

then $w = 0$.

1

$$\sum_{\Gamma \in G_{g,n}} \sum_{w \in W_{\Gamma,r}} \frac{1}{\mathrm{Aut}(\Gamma)} \frac{1}{r^{h^1(\Gamma)}} \cdot \xi_{\Gamma\star}\left[\prod_{e=(h,h')} \frac{1-\exp\left(-\frac{w(h)w(h')}{2}\cdot(\psi_h+\psi_{h'})\right)}{\psi_h+\psi_{h'}} \right]$$

0 0

0

The formula has no contributions in degree ≥ 1 !

$\boxed{\text{Can have no nodes}}$

Ideas in the proof of DR formula

(A) Pixton conjectured the complete formula.

(B) $P_g^{d,r}(A)$ is similar to a GRR calculation on $\overline{\mathcal{M}}_{g,A}(B\mathbb{Z}_r)$ for $r \gg 0$

orbifold GW theory (Ruan)

The expressions are not equal but differ by higher powers of r, so have the same constant term.

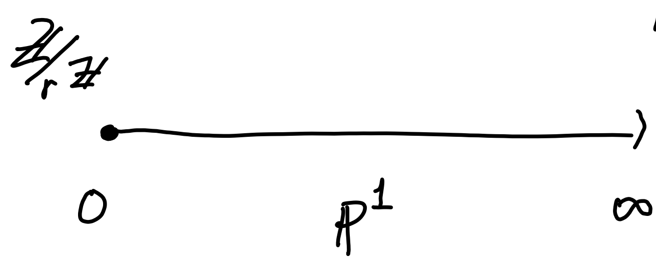
(C) Search for a target geometry in which both

$$\overline{\mathcal{M}}_{g,A}(B\mathbb{Z}_r) \quad \text{and} \quad \overline{\mathcal{M}}_g(\mathbb{P}^1, A)^{\sim}$$

play a role.

Gromov Witten theory of the target

relative / log



(D) Use virtual localization formula (Graber-P) in the limit $r >> 0$ to prove the DR cycle formula $\square$

for more details see JPPZ 1602.04705

Q3) Differentials (or k-differentials)

So far we have discussed the cycle defined by the condition

$$\mathcal{O}_C\left(\sum a_i p_i\right) \cong \mathcal{O}_C .$$

We would like now to consider

$$\mathcal{O}_C\left(\sum a_i p_i\right) \cong \omega_C$$

Idea is to develop a much more general perspective:

Universal twisted DR cycle on the Picard stack

BHPSS 2004.08676

$\mathcal{P}ic_{g,n,d}$ ⇜ Artin stack with objects:

$\mathcal{L}$ line bundle of degree d

C prestable curve of genus g

$$\begin{tikzcd} \mathcal{L} \arrow[dr] & \\ & C \arrow[d, "\pi"'] \\ & B \arrow[u, bend right, "s_1, \ldots, s_n"'] \end{tikzcd}$$

π flat

$s_1, \ldots, s_n$ sections are distinct and lie in the smooth locus

B base scheme

In words, $\mathcal{P}ic_{g,n,d}$ parameterizes

prestable n-pointed genus g curves with a degree d line bundle

- Let $A = (a_1, \dots, a_n) \in \mathbb{Z}^n$

with $\sum_{i=1}^{n} a_i = d$

We define $DR^{op}_{g,A} \in CH^{g}_{op}(\mathcal{P}ic_{g,n,d})$

$$DR^{op}_{g,A} : CH_k(B) \to CH_{k-g}(B)$$

for every family
$$\begin{tikzcd} \mathcal{L} \arrow[dr] & \\ & \mathcal{C} \arrow[d, "\pi"'] \\ & B \arrow[u, bend right, "s_1 \dots s_n"'] \end{tikzcd}$$

compatibly ...

see BHPSS

What is the definition of $DR^{op}_{g,A}$?

Heuristically, $DR^{op}_{g,A}$ acts on the base B by intersection with the locus where

$$\star \quad \text{"} \quad \mathcal{O}_C\Big(\sum_i a_i \Delta_i\Big) \cong \mathcal{L}_C \quad \text{"}$$

Technically, the definition is more subtle.

Simplest approach: use the closure in the versal deformation space of condition $\star$ on locus of smooth curves.

i

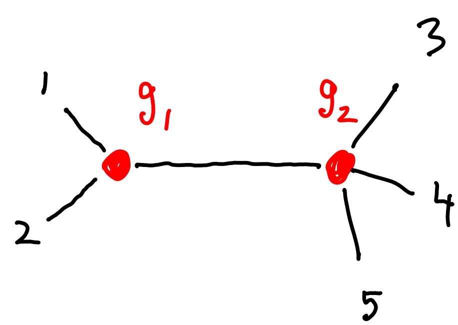
- What is Pixton's formula in $CH^g_{op}(\mathrm{Pic}_{g,n,d})$?

Another foundational discussion is required here to properly define the ring of tautological classes

— Graphs

Before stability

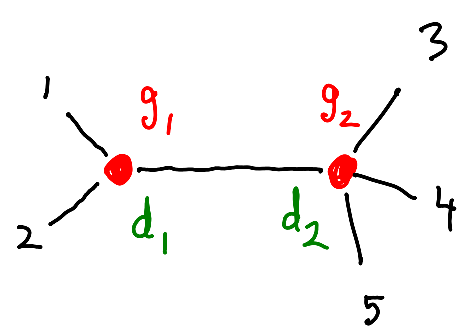
$\Gamma =$



$\in G_{g,n}$

Now no stability!

$\Gamma =$



$\in G_{g,n,d}$

— Classes

Before : ψ_i markings, ψ_j halves of edges

Now : ψ_i markings, ψ_j halves of edges

and $\xi_i = c_1(s_i^* \mathcal{L})$ ← marking i

$\eta(v) = \pi_*(c_1(\mathcal{L})^2)$ ← vertex v

— Weighting mod r (Condition III)

Before : $v \in \mathrm{Vertex}, \quad \sum_{h \vdash v} w(h) = 0 \mod r$

Now : $v \in \mathrm{Vertex}, \quad \sum_{h \vdash v} w(h) = d(v) \mod r$

Let $P_g^{k,r}(A) \in CH^k_{op}(Pic_{g,n,d})$ be the degree k component of

$$\sum_{\Gamma \in G_{g,n,d}} \sum_{w \in W_{\Gamma,r,d}} \frac{1}{\mathrm{Aut}(\Gamma)} \frac{1}{r^{h^1(\Gamma)}} \cdot$$

$$\xi_{\Gamma *}\Bigg[\prod_{i=1}^{n} \exp\left(\frac{a_i^2}{2}\psi_i + a_i \xi_i\right) \cdot \prod_{v} \exp\left(-\frac{1}{2}\eta(v)\right)$$

$$\cdot \prod_{e=(h,h')} \frac{1-\exp\left(-\frac{w(h)w(h')}{2}\cdot(\psi_h+\psi_{h'})\right)}{\psi_h+\psi_{h'}}\Bigg]$$

- $P_g^{k,r}(A) \in CH^k_{op}(Pic_{g,n,d})$ is polynomial in r for all $r >> 0$.

- $P_g^{k}(A) \in CH^k_{op}(Pic_{g,n,d})$ is the constant term of $P_g^{k,r}(A)$

$r=0$

Main Theorem (BHPSS 2020)

$$DR^{op}_{g,A} = P_g^g(A)$$

$$\in CH^g_{op}(Pic_{g,n,d})$$

We return to (Q3).

Let $A = (a_1, \ldots, a_n) \in \mathbb{Z}^n$

$$\sum_{i=1}^{n} a_i = 2g-2$$

Consider

$$\begin{tikzcd} \omega_\pi \arrow[dr] & & \\ & \mathcal{C} \arrow[d] & \text{universal curve} \\ & \overline{\mathcal{M}}_{g,n} \arrow[u, bend right, "s_1,\ldots,s_n"'] & \end{tikzcd}$$

By our Theorem

$$\mathrm{DR}^{op}_{g,A} = P^g_g(A) \in CH^g(\mathrm{Pic}_{g,n,2g-2}).$$

Both sides operate on Chow of $\overline{M}_{g,n}$:

$$DR^{op}_{g,A}\left[\overline{M}_{g,n}\right] = P^g_g(A)\left[\overline{M}_{g,n}\right]$$

formula in tautological classes

Holmes Schmitt $\quad \Big\| \quad$ if $\exists\, a_i < 0$

$$\left[\widetilde{H}_{g,A}\right] \in CH^g(\overline{M}_{g,n})$$

fundamental class of space of twisted differentials

$\Big\Updownarrow$ equivalent

$$\left[\overline{\Omega}_{g,A}\right] \in CH^g(\overline{M}_{g,n})$$

Farkas-P Schmitt

Effective : Software package by Delecroix, Schmitt, van Zelm

Steps in the proof of the Theorem

- after various transformations, consider the case

$$\begin{tikzcd} \mathcal{L} \arrow[dr] & \\ & \mathcal{C} \arrow[d, "\pi"'] \\ & B \arrow[u, bend right, "s_1 \cdots s_n"'] \end{tikzcd}$$

where $\mathcal{L}$ is sufficiently positive on the fibers of π

- further equivalences lead us to the case where

$$\begin{tikzcd} \mathcal{L} \arrow[r] \arrow[d] & \mathcal{O}(1) \arrow[d] \\ \mathcal{C} \arrow[r] \arrow[d, "\pi"'] & \mathbb{CP}^N \\ B \arrow[u, bend right, "s_1 \cdots s_n"'] & \end{tikzcd}$$

- finally, we apply the main result of JPPZ2 DR(x) 1812.10136

exactly calculates Double ramification cycles for target varieties

Universal twisted DR cycle for Pic

is calculated by studying

DR cycles for $\mathbb{CP}^N$ $N \to \infty$. $\square$

Various further directions...

(Q4) What is the algebra of tautological classes

$$R^*(\mathrm{Pic}_{g,n,d}) \subset CH^*_{op}(\mathrm{Pic}_{g,n,d}) \; ?$$

First result:

$$P^k_g(A) = 0 \quad \in CH^k_{op}(\mathrm{Pic}_{g,n,d})$$

for all $k > g$.

BHPSS, Bae for DR(x).

The End