Enumerative Geometry of Calabi-Yau 5-Folds

Appendix

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A Explicit Formulas for the Local Example

A.1 Recursions

The recursive description of the number $n_1(d)$ is simplified below from the original one using Lemmas 1 and 2 below.[1] The main statement of interest is Conjecture 1, which implies that $n_1(d)=S(d)V(d)$ and thus confirms the BPS integrality prediction for the local Calabi-Yau 5-fold $3\mathcal{O}(-1)\longrightarrow\mathbb{P}^2$.

For each d, denote by $\mathcal{P}(d)$ the set of *odd* prime numbers dividing d and by $\ell(d)$ the exponent of the largest power of 2 dividing d. If $d\geq 1$, define

$$A_1(d) = \frac{1}{4}\prod_{x\in\mathcal{P}(d)}(1-x^2)\times\begin{cases}4, & \text{if } d=1;\\ 1, & \text{if } \ell(d)=0,\ d\neq 1;\\ -4, & \text{if } \ell(d)=1;\\ 0, & \text{if } \ell(d)\geq 2;\end{cases} \tag{A.1}$$

$$A_2(d) = -\frac{3d}{2}\prod_{x\in\mathcal{P}(d)}(1-x)\cdot\begin{cases}0, & \text{if } d=1;\\ -1, & \text{if } d=2;\\ 1, & \text{if } \ell(d)=0,\ d\neq 1;\\ -\frac{3}{2}, & \text{if } \ell(d)=1,\ d\neq 2;\\ -\frac{1}{2}, & \text{if } \ell(d)\geq 2.\end{cases} \tag{A.2}$$

For $d_1,d_2\geq 1$, let

$$B_1(d_1,d_2) = -3d_2\prod_{x\in\mathcal{P}(\langle d_1,d_2\rangle)}(1-x)\cdot\begin{cases}1, & \text{if } \ell(\langle d_1,d_2\rangle)=0;\\ -\frac{3}{2}, & \text{if } \ell(\langle d_1,d_2\rangle)=1;\\ -\frac{1}{2}, & \text{if } \ell(\langle d_1,d_2\rangle)\geq 2.\end{cases} \tag{A.3}$$

[1] The original recursions for $A_2(d)\equiv n_d(\tilde{\psi}^2,H^2)$ and $B_1(d_1,d_2)\equiv n_{d_1d_2}(|;H^2)$ do not involve any of the other terms. Using Lemma 1, the formulas (A.2) and (A.3) can be shown to satisfy the two recursions. The original formula for $A_1(d)\equiv n_d(\tilde{\psi}H^2)$ expresses $A_1(d)$ in terms of $B_1(d_1,d_2)$. The formula (A.1) follows from (A.3) and Lemma 2. Originally, there are also the terms $n_d(\tilde{\psi}H,H^2)$, but it is not hard to see that

$$n_d(\tilde{\psi}H,H^2) = d\,n_d(\tilde{\psi}H^2) \qquad \forall\ d\geq 3;$$

so we are able to eliminate these terms.

The following is the initial data for the remaining A-terms:

$$A_3(1)=-3,\qquad A_4(1)=6,\qquad A_5(1)=6;$$
$$A_3(2)=0,\qquad A_4(2)=3,\qquad A_5(2)=-72.$$

The following equations apply to $d\geq 3$ (the last two hold for all d); the sums are taken over $d_1,d_2\geq 1$:

$$\begin{aligned}
d^2A_3(d) &= -dA_1(d)+\sum_{d_1+d_2=d} d_1^2 B_4(d_1,d_2);\\
d^2A_4(d) &= A_2(d)-2d\,A_3(d)+\sum_{d_1+d_2=d} d_1^2\big(B_3(d_1,d_2)+B_5(d_1,d_2)\big);\\
A_5(d) &= d\bigg(-\frac{9}{2}A_1(d)-6\,A_2(d)+\frac{1}{2}A_4(d)-\sum_{d_1+d_2=d}\Big(\frac{5}{2}B_3(d_1,d_2)+2B_5(d_1,d_2)\Big)\bigg)\\
&\qquad + A_3(d)-\sum_{d_1+d_2=d}\bigg(\frac{1}{2}B_2(d_1,d_2)+2B_4(d_1,d_2)\bigg).
\end{aligned}$$

The following equations apply to $d_1,d_2\geq 1$; the sums are taken over $d,d'\geq 1$:

$$\begin{aligned}
B_2(d_1,d_2) &= -\begin{cases}
B_4(d_2-d_1,d_1)+\sum\limits_{d<d_1,d_2} d\,C(d_1-d,d,d_2-d) & \\
\quad +d_1\Big(B_6(d_2-d_1,d_1)+\frac{1}{2}\sum\limits_{d+d'=d_2-d_1} C(d,d_1,d')\Big), & \text{if } d_2>d_1;\\
A_5(d_1)+\sum\limits_{d<d_1,d_2} d\,C(d_1-d,d,d_2-d), & \text{if } d_1=d_2;\\
-B_2(d_2,d_1), & \text{if } d_2<d_1;
\end{cases}\\
d_2^2B_3(d_1,d_2) &= B_1(d_1,d_2)-2d_2B_2(d_1,d_2)+\sum_{d+d'=d_2} d^2C(d_1,d',d);\\
d_2B_4(d_1,d_2) &= d_2B_2(d_1,d_2)-B_1(d_1,d_2)+\sum_{d+d'=d_2} dd'C(d_1,d',d);\\
B_5(d_1,d_2) &= -\sum_{d+d'=d_2} C(d_1,d',d)\,;\\
B_6(d_1,d_2) &= -3B_1(d_1,d_2)+2B_5(d_1,d_2)+B_3(d_1,d_2)+B_3(d_2,d_1).
\end{aligned}$$

The following formulas apply to $d_1, d_2, d_3 \geq 1$:

$$\mathtt{C}^{(1)}(d_1, d_2, d_3) = \begin{cases} C(d_3 - d_1, d_1, d_2), & \text{if } d_3 > d_1; \\ C(d_1 - d_3, d_3, d_2), & \text{if } d_3 < d_1; \\ B_6(d_2, d_1), & \text{if } d_3 = d_1; \end{cases}$$

$$\mathtt{C}^{(2)}(d_1, d_2, d_3) = \begin{cases} C(d_1, d_2, d_3 - d_2), & \text{if } d_3 > d_2; \\ C(d_1, d_3, d_2 - d_3) + C(d_1, d_2 - d_3, d_3), & \text{if } d_3 < d_2; \\ -3B_1(d_1, d_2) + 2B_5(d_1, d_2), & \text{if } d_3 = d_2; \end{cases}$$

$$\mathtt{C}^{(12)}(d_1, d_2, d_3) = \begin{cases} C(d_3 - d_1 - d_2, d_1, d_2), & \text{if } d_3 > d_1 + d_2; \\ C(d_1 + d_2 - d_3, d_3 - d_2, d_2), & \text{if } d_2 < d_3 < d_1 + d_2; \\ B_6(d_2, d_1), & \text{if } d_3 = d_1 + d_2; \\ 0, & \text{otherwise}; \end{cases}$$

$$C(d_1, d_2, d_3) = -\mathtt{C}^{(1)}(d_1, d_2, d_3) + \mathtt{C}^{(2)}(d_1, d_2, d_3) + \mathtt{C}^{(12)}(d_1, d_2, d_3).$$

A.2 Main Conjecture

For each $d \leq 1$, let

$$A_6(d) = 3A_1(d) - A_4(d) + \sum_{d_1 + d_2 = d} B_3(d_1, d_2);$$

$$\sum_{d=1}^{\infty} \sigma(d) q^d = \sum_{d=1}^{\infty} d \frac{q^d}{1 - q^d};$$

$$\mu(d) = \begin{cases} (-1)^r, & \text{if d is the product of } r \text{ distinct primes}; \\ 0, & \text{otherwise}; \end{cases}$$

$$S(d) = \begin{cases} \mu(d), & \text{if } d \ncong 4 \pmod 8; \\ \mu(d/4), & \text{if } d \cong 4 \pmod 8; \end{cases}$$

$$V(d) = \frac{k^2 - 1}{8} \times \begin{cases} \frac{k^2 - 1}{8}, & \text{if } d = k,\ 2 \not| k; \\ \frac{17k^2 + 7}{8}, & \text{if } d = 2k,\ 2 \not| k; \\ 2k^2 + 1, & \text{if } d = 4k,\ 2 \not| k. \end{cases}$$

Conjecture 1 (G. Martin) *For all $d \geq 1$,*

$$\frac{(-1)^d}{8d} = \sum_{r \leq d, r|d} \frac{\sigma(r)}{r} S(d/r) V(d/r) + \frac{1}{24} \sum_{r \leq d, r|d} \frac{1}{r} A_6(d/r).$$

A.3 Some General Identities

For each d, denote by $\mathcal{P}(d)$ the set of *odd* prime numbers dividing d and by $\ell(d)$ the exponent of the largest power of 2 dividing d. If $d\geq 1$, define

$$\eta_1(d) = \prod_{x\in\mathcal{P}(d)}(1-x)\cdot\begin{cases}1, & \text{if } \ell(d)=0;\\ -3/2, & \text{if } \ell(d)=1;\\ -1/2, & \text{if } \ell(d)\geq 2;\end{cases}$$

$$\eta_2(d) = \prod_{x\in\mathcal{P}(d)}(1-x^2)\times\begin{cases}1, & \text{if } \ell(d)=0;\\ -4, & \text{if } \ell(d)=1;\\ 0, & \text{if } \ell(d)\geq 2.\end{cases}$$

Lemma 1 *If $d\geq 3$,*

$$\sum_{r=1}^{r=d}\eta_1\big(\gcd(r,d)\big) = 0.$$

Lemma 2 *If $d\geq 3$,*

$$d\,\eta_2(d) = -6\sum_{\substack{d_1+d_2=d\\ d_1,d_2>0}} d_1 d_2 \eta_1\big(\gcd(d_1,d_2)\big).$$

Proof of Lemma 1: (1) We first show that Lemma 1 is satisfied if $d=p^a$ for some prime p and $a\geq 1$ ($a\geq 2$ if $p=2$). If p is odd, then

$$\sum_{r=1}^{r=d}\eta_1\big(\gcd(r,d)\big) = p^{a-1}\cdot\eta_1(p) + \big(p^a-p^{a-1}\big)\cdot\eta_1(1) = p^{a-1}\cdot(1-p) + \big(p^a-p^{a-1}\big)\cdot 1 = 0.$$

If $p=2$, then

$$\begin{aligned}\sum_{r=1}^{r=d}\eta_1\big(\gcd(r,d)\big) &= 2^{a-2}\cdot\eta_1(4) + \big(2^{a-1}-2^{a-2}\big)\cdot\eta_1(2) + \big(2^a-2^{a-1}\big)\cdot\eta_1(1)\\ &= -\frac{1}{2}\cdot 2^{a-2} - \frac{3}{2}\cdot 2^{a-2} + 2^{a-1} = 0.\end{aligned}$$

(2) Let $\phi\colon\mathbb{Z}^+\longrightarrow\mathbb{Z}^+$ denote the Euler ϕ-function:

$$\phi(d) = \big|\big\{r\in\mathbb{Z}^+ : r\leq d,\ \gcd(r,d)=1\big\}\big|.$$

Since $\phi(d)$ is the number of generators of the cyclic group $\mathbb{Z}_d\equiv\mathbb{Z}/d\mathbb{Z}$,

$$\phi(ab) = \phi(a)\cdot\phi(b) \qquad \text{if} \quad \gcd(a,b)=1.$$

The function η_1 satisfies the same identity. Suppose $a, b \in \mathbb{Z}^+$ are relatively prime and Lemma 1 is satisfied for $d = a$: Then,

$$\begin{aligned}
\sum_{r=1}^{r=ab} \eta_1\big(\gcd(r,d)\big) &= \sum_{c|d} \sum_{\substack{\gcd(r,d)=c \\ 1\le r\le d}} \eta_1(c) = \sum_{c|d} \phi(d/c)\eta_1(c) = \sum_{c_1|a}\sum_{c_2|b} \phi\big((a/c_1)(b/c_2)\big)\eta_1(c_1c_2) \\
&= \bigg(\sum_{c_1|a} \phi(a/c_1)\eta_1(c_1)\bigg)\bigg(\sum_{c_2|b} \phi(b/c_2)\eta_1(c_2)\bigg) \\
&= \bigg(\sum_{r=1}^{r=a} \eta_1\big(\gcd(r,a)\big)\bigg)\bigg(\sum_{r=1}^{r=b} \eta_1\big(\gcd(r,b)\big)\bigg) = 0.
\end{aligned}$$

Lemma 1 now follows by induction on the number of factors in d.

Proof of Lemma 2: (1) This will be a direct computation. Note that for any $d \in \mathbb{Z}^+$ and a finite nonempty set $\mathcal{S}$,

$$\sum_{r=1}^{r=d} r(d-r) = \frac{d}{6}\big(d^2-1\big), \qquad \sum_{J\subset\mathcal{S}} (-1)^{|J|} = 0. \tag{A.4}$$

If $d \in \mathbb{Z}^+$, let

$$\bar{\mathcal{P}}(d) = \begin{cases} \mathcal{P}(d), & \text{if } \ell(d)=0; \\ \mathcal{P}(d)\cup\{2\}, & \text{if } \ell(d)=1; \\ \mathcal{P}(d)\cup\{2,4\}, & \text{if } \ell(d)\ge 2; \end{cases} \qquad \begin{aligned} \mathcal{S}(d) &= \big\{I\subset\bar{\mathcal{P}}(d) : 2\in I \text{ if } 4\in I\big\}; \\ \mathcal{S}^*(d) &= \big\{I\in\mathcal{S}(d) : 4\notin I\big\}. \end{aligned}$$

If $I \subset \bar{\mathcal{P}}(d)$, let

$$x_I = \prod_{x\in I} x \cdot \begin{cases} 1, & \text{if } 4\notin I; \\ \frac{1}{2}, & \text{if } 4\in I; \end{cases} \qquad d_I = d/x_I, \qquad (1-x)_I = \prod_{x\in I\cap\mathcal{P}(d)} (1-x).$$

Note that for any $J \subset \mathcal{P}(d)$,

$$\sum_{I\subset J} (-1)^{|J-I|}(1-x)_I = (-1)^{|J|}x_J, \qquad \sum_{J\subset\mathcal{P}(d)} (-1)^{|J|}x_J^2 = \prod_{x\in\mathcal{P}(d)} (1-x^2). \tag{A.5}$$

Furthermore, for any $I \in \mathcal{S}(d)$,

$$\sum_{\substack{\bar{\mathcal{P}}(\gcd(r,d))=I \\ 0<r<d}} r(d-r) = \frac{d}{6}x_I \sum_{I\subset J\in\mathcal{S}'(d)} (-1)^{|J-I|}x_{J-I}\big(d_J^2-1\big), \quad \text{where} \quad \mathcal{S}' = \begin{cases} \mathcal{S}^*(d), & \text{if } 2\notin I; \\ \mathcal{S}(d), & \text{if } 2\in I; \end{cases} \tag{A.6}$$

this follows from (A.4).

(2) We break the sum in the statement Lemma 2 into sub-sums with fixed $\bar{\mathcal{P}}(\gcd(r,d)) \in \mathcal{S}(d)$, as $\eta_1(\gcd(r,d))$ is determined by $\bar{\mathcal{P}}(\gcd(r,d))$. If $I \subset \mathcal{P}(d)$, by (A.6)

$$\sum_{\substack{\bar{\mathcal{P}}(\gcd(r,d))=I \\ 0<r<d}} r(d-r)\eta_1\big(\gcd(r,d)\big) = \frac{d}{6}x_I(1-x)_I \sum_{I\subset J\in\mathcal{S}^*(d)} (-1)^{|J-I|}x_{J-I}\big(d_J^2-1\big). \tag{A.7}$$

If $J\subset\mathcal{P}(d)$, the coefficient of (d_J^2-1) in the sum of (A.6) over all $I\subset\mathcal{P}(d)$ is

$$\frac{d}{6}x_J\sum_{I\subset J}(-1)^{|J-I|}(1-x)_I=(-1)^{|J|}x_J^2\frac{d}{6}$$

by the first identity in (A.5). Summing over all $J\subset\mathcal{P}(d)$, we obtain

$$\frac{d}{6}\sum_{J\subset\mathcal{P}(d)}(-1)^{|J|}x_J^2\big(d_J^2-1\big)=-\frac{d}{6}\cdot\begin{cases}\prod_{x\in\mathcal{P}(d)}(1-x^2), & \text{if } \mathcal{P}(d)\neq\emptyset;\\ 1-d^2, & \text{if } \mathcal{P}(d)=\emptyset;\end{cases}\tag{A.8}$$

see the second identities in (A.4) and (A.5). This concludes the proof of Lemma 2 in the $\ell(d)=0$ case.

(3) Suppose $\ell(d)\geq 1$. If $2\in I\in\mathcal{S}^*(d)$, by (A.6)

$$\sum_{\substack{\bar{\mathcal{P}}(\gcd(r,d))=I\\ 0<r<d}}r(d-r)\eta_1\big(\gcd(r,d)\big)=-\frac{3}{2}\cdot\frac{d}{6}x_I(1-x)_I\sum_{I\subset J\in\mathcal{S}(d)}(-1)^{|J-I|}x_{J-I}\big(d_J^2-1\big).\tag{A.9}$$

If $J\subset\mathcal{P}(d)$, the coefficient of $(d_J^2/4-1)$ in the sum of (A.6) and (A.9) over all possibilities for I is

$$\frac{d}{6}\Big(-2-\frac{3}{2}\cdot 2\Big)\sum_{J\subset\mathcal{P}(d)}(-1)^{|J|}x_J^2\big(d_J^2/4-1\big)=5\frac{d}{6}\cdot\begin{cases}\prod_{x\in\mathcal{P}(d)}(1-x^2), & \text{if } \mathcal{P}(d)\neq\emptyset;\\ 1-d^2/4, & \text{if } \mathcal{P}(d)=\emptyset.\end{cases}\tag{A.10}$$

Along with (A.8), (A.10) implies Lemma 2 in the $\ell(d)=1$ case.

(4) Finally, suppose $\ell(d)\geq 2$. If $I\in\mathcal{S}(d)-\mathcal{S}^*(d)$, by (A.6)

$$\sum_{\substack{\bar{\mathcal{P}}(\gcd(r,d))=I\\ 0<r<d}}r(d-r)\eta_1\big(\gcd(r,d)\big)=-\frac{1}{2}\cdot\frac{d}{6}x_I(1-x)_I\sum_{I\subset J\in\mathcal{S}(d)}(-1)^{|J-I|}x_{J-I}\big(d_J^2-1\big).\tag{A.11}$$

If $J\subset\mathcal{P}(d)$, the coefficient of $(d_J^2/16-1)$ in the sum of (A.9) and (A.11) over all possibilities for I is

$$\frac{d}{6}\Big(\frac{3}{2}\cdot 4-\frac{1}{2}\cdot 4\Big)\sum_{J\subset\mathcal{P}(d)}(-1)^{|J|}x_J^2\big(d_J^2/16-1\big)=-4\frac{d}{6}\cdot\begin{cases}\prod_{x\in\mathcal{P}(d)}(1-x^2), & \text{if } \mathcal{P}(d)\neq\emptyset;\\ 1-d^2/16, & \text{if } \mathcal{P}(d)=\emptyset.\end{cases}\tag{A.12}$$

The $\ell(d)=2$ case follows from (A.8), (A.10), and (A.12).