

Enumerative geometry of Calabi-Yau 4-folds

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Abstract

Gromov-Witten theory is used to define an enumerative geometry of curves in Calabi-Yau 4-folds. The main technique is to find exact solutions to moving multiple cover integrals. The resulting invariants are analogous to the BPS counts of Gopakumar and Vafa for Calabi-Yau 3-folds. We conjecture the 4-fold invariants to be integers and expect a sheaf theoretic explanation.

Several local Calabi-Yau 4-folds are solved exactly. Compact cases, including the sextic Calabi-Yau in \mathbb{P}^5 , are also studied. A complete solution of the Gromov-Witten theory of the sextic is conjecturally obtained by the holomorphic anomaly equation.

0 Introduction

0.1 Gromov-Witten theory

Let X be a nonsingular projective variety over \mathbb{C} . Let $\overline{M}_{g,n}(X, \beta)$ be the moduli space of genus g , n -pointed stable maps to X representing the class $\beta \in H_2(X, \mathbb{Z})$. The Gromov-Witten theory of primary fields¹ concerns the integrals

$$N_{g,\beta}^X(\gamma_1, \dots, \gamma_n) = \int_{[\overline{M}_{g,n}(X, \beta)]^{vir}} \prod_{i=1}^n \text{ev}_i^*(\gamma_i), \quad (1)$$

where

$$\text{ev}_i : \overline{M}_{g,n}(X, \beta) \rightarrow X$$

¹We consider only primary Gromov-Witten theory in the paper.

is the i^{th} evaluation map and $\gamma_i \in H^*(X, \mathbb{Z})$. The notation

$$N_{g,\beta}^X = \int_{[\overline{M}_g(X,\beta)]^{\text{vir}}} 1$$

is used in case there are no insertions. Since the moduli space $\overline{M}_{g,n}(X, \beta)$ is a Deligne-Mumford stack, the Gromov-Witten invariants (1) are \mathbb{Q} -valued.

0.2 Enumerative geometry

The relationship between Gromov-Witten theory and the enumerative geometry of curves in X is straightforward in three cases:

- (i) X is convex (in genus 0),
- (ii) X is a curve,
- (iii) X is \mathbb{P}^2 or $\mathbb{P}^1 \times \mathbb{P}^1$.

For (i-iii), the Gromov-Witten theory with primary insertions equals the classical enumerative geometry of curves. A discussion of convex varieties (i) in genus 0 can be found in [11]. Examples (ii) and (iii) hold for all genera and recover the étale Hurwitz numbers and the classical Severi degrees respectively. Case (iii) certainly extends in some form to all rational surfaces viewed as generic blow-ups. The genus 0 case is treated in [16].

The first nontrivial cases occur for irrational surfaces. When X is a minimal surface of general type, Taubes' results exactly determine the primary Gromov-Witten invariant for the adjunction genus in the canonical class,

$$N_{g_X, K_X}^X = (-1)^{\chi(X, \mathcal{O}_X)},$$

see [36, 37, 38, 39]. While much is known about surfaces of general type [23, 28], surfaces in between are more mysterious. For example, many questions about the relationship of Gromov-Witten theory to the enumerative geometry of the $K3$ and Enriques surfaces remain open [5, 20, 22, 28].

The enumerative significance of Gromov-Witten theory in dimension 3 has been studied since the beginning of the subject. For Calabi-Yau 3-folds, essentially all Gromov-Witten invariants, even in genus 0, have large denominators. The Aspinwall-Morrison formula [1] was conjectured to produce

integer invariants in genus 0. A full integrality conjecture for the Gromov-Witten theory of Calabi-Yau 3-folds in terms of BPS states was formulated by Gopakumar and Vafa [14, 15]. Later, integral invariants for all 3-folds were conjectured in [32, 33]. Various mathematical attempts to capture the BPS counts in terms of the cohomologies of associated moduli of sheaves on X were put forward without a definitive treatment. However, the integral invariants of [14, 15, 33] can be conjecturally interpreted in terms of the sheaf enumeration of Donaldson-Thomas theory [25, 26, 40].

Our main point here is to show the integrality of Gromov-Witten theory persists in higher dimensions as well. We speculate there exist universal transformations in every dimension which express Gromov-Witten theory in terms of \mathbb{Z} -valued invariants. We conjecture the exact form of the transformation for Calabi-Yau 4-folds. A sheaf theoretic interpretation of the resulting invariants remains to be found.

0.3 Calabi-Yau 4-folds

Let X be a nonsingular, projective, Calabi-Yau 4-fold, and let $\beta \in H_2(X, \mathbb{Z})$ be a curve class. Since

$$\text{vir dim } \overline{M}_g(X, \beta) = \int_{\beta} c_1(X) + (\dim X - 3)(1 - g) = 1 - g,$$

Gromov-Witten theory vanishes for $g \geq 2$. We need only consider genus 0 and 1.

We measure the degree of β with respect to a fixed ample polarization L on X ,

$$\text{deg}(\beta) = \int_{\beta} c_1(L).$$

All effective curve classes² satisfy $\text{deg}(\beta) > 0$. We abbreviate the latter condition by $\beta > 0$. We are only interested here in Gromov-Witten invariants for classes satisfying $\beta > 0$.

²Integrality constraints for Gromov-Witten theory always exclude constant maps. The constant contributions are easily determined in terms of the classical cohomology of X . For $D \in H^2(X, \mathbb{Z})$, the genus 1 invariant

$$N_{1,0}(D) = -\frac{1}{24} \int_X c_3(X) \cup D$$

has denominator bounded by 24.

Integrality in genus 0 is expressed by the following generalization of the Aspinwall-Morrison formula. Invariants $n_{0,\beta}(\gamma_1, \dots, \gamma_n)$ virtually enumerating rational curves of class β incident to cycles dual to the classes γ_i are uniquely defined by

$$\sum_{\beta > 0} N_{0,\beta}(\gamma_1, \dots, \gamma_n) q^\beta = \sum_{\beta > 0} n_{0,\beta}(\gamma_1, \dots, \gamma_n) \sum_{d=1}^{\infty} d^{-3+n} q^{d\beta}. \quad (2)$$

A justification for the definition via multiple coverings is given in Section 1.1.

Conjecture 0: *The invariants $n_{0,\beta}(\gamma_1, \dots, \gamma_n)$ are integers.*

Let S_1, \dots, S_s be a basis of $H^4(X, \mathbb{Z})$ mod torsion. Let

$$g_{ij} = \int_X S_i \cup S_j$$

be the intersection form, and let

$$\sum_{i,j} g^{ij} [S_i \otimes S_j] \in H^8(X \times X, \mathbb{Z})$$

be the $H^4(X, \mathbb{Z}) \otimes H^4(X, \mathbb{Z})$ part of the Künneth decomposition of the diagonal (mod torsion).

For $\beta_1, \beta_2 \in H_2(X, \mathbb{Z})$, we define invariants m_{β_1, β_2} virtually enumerating rational curves of class β_1 meeting rational curves of class β_2 . The meeting invariants are uniquely determined by the following rules.

(i) The invariants are symmetric,

$$m_{\beta_1, \beta_2} = m_{\beta_2, \beta_1}.$$

(ii) If either $\deg(\beta_1) \leq 0$ or $\deg(\beta_2) \leq 0$, then $m_{\beta_1, \beta_2} = 0$.

(iii) If $\beta_1 \neq \beta_2$, then,

$$m_{\beta_1, \beta_2} = \sum_{i,j} n_{0,\beta_1}(S_i) g^{ij} n_{0,\beta_2}(S_j) + m_{\beta_1, \beta_2 - \beta_1} + m_{\beta_1 - \beta_2, \beta_2}.$$

(iv) In case of equality,

$$m_{\beta,\beta} = n_{0,\beta}(c_2(T_X)) + \sum_{i,j} n_{0,\beta}(S_i) g^{ij} n_{0,\beta}(S_j) - \sum_{\beta_1+\beta_2=\beta} m_{\beta_1,\beta_2}.$$

A geometric derivation of the rules (i-iv) is presented in Section 1.2. The conjectural integrality of the invariants $n_{0,\beta}(\gamma)$ implies the integrality of the meeting invariants m_{β_1,β_2} .

In genus 1, we need only consider Gromov-Witten invariants $N_{1,\beta}$ of X with *no* insertions since the virtual dimension is 0. The invariants $n_{1,\beta}$ virtually enumerating elliptic curves are uniquely defined by

$$\begin{aligned} \sum_{\beta>0} N_{1,\beta} q^\beta &= \sum_{\beta>0} n_{1,\beta} \sum_{d=1}^{\infty} \frac{\sigma(d)}{d} q^{d\beta} \\ &+ \frac{1}{24} \sum_{\beta>0} n_{0,\beta}(c_2(T_X)) \log(1 - q^\beta) \\ &- \frac{1}{24} \sum_{\beta_1,\beta_2} m_{\beta_1,\beta_2} \log(1 - q^{\beta_1+\beta_2}). \end{aligned} \tag{3}$$

The function σ is defined by

$$\sigma(d) = \sum_{i|d} i.$$

The number of automorphism-weighted, connected, degree d , étale covers of an elliptic curve is $\sigma(d)/d$.

Conjecture 1: *The invariants $n_{1,\beta}$ are integers.*

The explicit form of (3) is derived from studying a particular solvable local Calabi-Yau 4-fold in Section 2.

0.4 Examples

The last four Sections of the paper are devoted to the calculation of basic examples of Calabi-Yau 4-folds. The two local cases,

$$\mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(-2) \rightarrow \mathbb{P}^2,$$

$$\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-1, -1) \oplus \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-1, -1) \rightarrow \mathbb{P}^1 \times \mathbb{P}^1,$$

are solved in closed form by virtual localization in Section 3. The local case

$$\mathcal{O}_{\mathbb{P}^3}(-4) \rightarrow \mathbb{P}^3$$

and the compact Calabi-Yau 4-fold hypersurfaces

$$X_6 \subset \mathbb{P}^5,$$

$$X_{10} \subset \mathbb{P}^5(1, 1, 2, 2, 2, 2), \quad X_{2,5} \subset \mathbb{P}^1 \times \mathbb{P}^4,$$

$$X_{24} \subset \mathbb{P}^5(1, 1, 1, 1, 8, 12)$$

are solved by the conjectural holomorphic anomaly equation³ in Sections 4-6. The compact cases are much more interesting than the local toric examples. In all calculations, the integralities of Conjectures 0 and 1 are verified.

0.5 Physical interpretation

Type IIA string compactifications on Calabi-Yau 4-folds give rise to massive theories with $(2, 2)$ supergravity in 2 dimensions. Such theories and their BPS states were extensively studied in general [6, 7] and in particular for type IIA on Calabi-Yau 4-folds in [13, 18]. The effective action, worked out in [13], contains an $\int d^2z R^{(2)}$ term, and the topological string at genus 1 calculates a 1-loop correction to this term. The latter comes from the famous 1-loop term in 10 dimensional type IIA theory that was discovered in the context of heterotic type II duality in [41] and gives the following contribution to the 10 dimensional effective action

$$\delta S = - \int d^{10}x B Y_8(R). \quad (4)$$

Here, B is the $NS - NS$ 2-form of type IIA coupling to the string and $Y_8(R)$ is an 8-form constructed as a quartic polynomial in the curvature. In 10 dimensions, the term can be directly calculated from the 1-loop amplitude with 4 gravitons and the antisymmetric B -field as external legs. If the latter is in the 2 non-compact dimensions, in the absence of further flux terms, the tadpole condition that $-\frac{\chi(X)}{24}$ vanishes is obtained. The topological string

³While mathematical approaches to the genus 1 invariants in the compact case are available [12, 24, 27], the methods are much less effective than the anomaly equation.

computes the correction to the $\int d^2z R^{(2)}$ term calculated from a loop with 1 external graviton, 3 internal gravitons, and the B -field.⁴

As in [14, 15], the loop integral receives only contributions from BPS states. The behavior of the topological string amplitude in the large volume limit appears as the zero mass contribution and supports the claim that the amplitude computes the reduction of (4). BPS states with a $D2$ -brane charge β contribute $\sim \log(1 - q^\beta)$ to the integral. The integer expansion (3) can be alternatively written as

$$\begin{aligned} \sum_{\beta>0} N_{1,\beta} q^\beta &= - \sum_{\beta>0} \tilde{n}_{1,\beta} \log(1 - q^\beta) \\ &+ \frac{1}{24} \sum_{\beta>0} n_{0,\beta}(c_2(T_X)) \log(1 - q^\beta) \\ &- \frac{1}{24} \sum_{\beta_1,\beta_2} m_{\beta_1,\beta_2} \log(1 - q^{\beta_1+\beta_2}). \end{aligned}$$

The integrality condition for the invariants $\tilde{n}_{1,\beta}$ is equivalent to the conjectured integrality for $n_{1,\beta}$. We interpret the $\tilde{n}_{1,\beta}$ as counting BPS states. Furthermore, the structure of the

$$m_{\beta_1,\beta_2} \log(1 - q^{\beta_1+\beta_2})$$

term suggest that the invariants m_{β_1,β_2} count bound states at the threshold of BPS states with $D2$ -brane charge β_1 and β_2 respectively.

0.6 Outlook

The meeting invariants make integrality in genus 1 for Calabi-Yau 4-folds considerably more subtle than the corresponding integrality for Calabi-Yau 3-folds. The integrality transformations in the higher dimensional Calabi-Yau cases should include all genus 0 meeting configurations. In the non Calabi-Yau cases, higher genus meeting configurations should occur as well. Finding the correct coefficients for such a universal transformation is an interesting problem.

⁴Work in progress.

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1 Genus 0

1.1 Aspinwall-Morrison

Let π and ι denote the universal curve and map over the moduli space,

$$\begin{aligned}\pi : \mathcal{C} &\rightarrow \overline{M}_{0,0}(\mathbb{P}^1, d), \\ \iota : \mathcal{C} &\rightarrow \mathbb{P}^1.\end{aligned}$$

The Aspinwall-Morrison formula is

$$\int_{\overline{M}_{0,0}(\mathbb{P}^1, d)} c_{top} (R^1 \pi_* \iota^* (\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1))) = \frac{1}{d^3}.$$

By the divisor equation, we obtain

$$\int_{\overline{M}_{0,n}(\mathbb{P}^1, d)} c_{top} (R^1 \pi_* f^* (\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1))) \cup \prod_{i=1}^n \text{ev}_i^*([P]) = d^{-3+n}, \quad (5)$$

where $[P] \in H^2(\mathbb{P}^1, \mathbb{Z})$ is the class of a point.

Let X be a Calabi-Yau 4-fold, and let

$$V_1, \dots, V_n \subset X$$

be cycles imposing a 1-dimensional incidence constraint for curves. Let

$$C \subset X$$

be a nonsingular rational curve transversely incident to the cycles V_i . If the rational curve has generic normal bundle splitting,

$$N_{X/C} \cong \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1},$$

the contribution of C to the genus 0 Gromov-Witten theory of X is

$$\sum_{d=1}^{\infty} d^{-3+n} q^{d[C]}$$

by (5). The constraints kill the trivial normal direction.

The justification for definition (2) for the virtually enumerative invariants $n_{0,n}(\gamma_1, \dots, \gamma_n)$ is complete. Of course, since transversality and genericity were assumed in the justification, we do not have a proof of Conjecture 0.

1.2 Meeting invariants

1.2.1 Rules (i) and (ii)

Let X be a Calabi-Yau 4-fold. The meeting invariant m_{β_1, β_2} virtually enumerates rational curves of class β_1 meeting rational curves of class β_2 . Rules (i) and (ii) have clear geometric motivation. In fact, rule (i) is consequence of rules (ii-iv). Rule (ii) may be viewed as a boundary condition.

Ultimately, m_{β_1, β_2} is defined by rules (i-iv). Rules (iii) and (iv) are derived by assuming the best possible behavior for rational curves. However, the ideal assumptions are typically false. As in Section 1.1, our derivation can be viewed, rather, as a justification for the definitions.

1.2.2 Boundary divisor

For nonzero classes $\beta_1, \beta_2 \in H_2(X, \mathbb{Z})$, let $\Delta_{\beta_1, \beta_2}$ denote the virtual boundary divisor

$$\Delta_{\beta_1, \beta_2} \xrightarrow{\epsilon} \overline{M}_{0,0}(X, \beta_1 + \beta_2)$$

corresponding to reducible nodal curves with degree splitting of type (β_1, β_2) . In the balanced case $\beta_1 = \beta_2$, an ordering is taken in $\Delta_{\beta_1, \beta_2}$, and ϵ is of degree 2.

The virtual dimension of $\Delta_{\beta_1, \beta_2}$ is 0. Let

$$M_{\beta_1, \beta_2} = \int_{[\Delta_{\beta_1, \beta_2}]^{vir}} 1 \in \mathbb{Q}$$

be the associated Gromov-Witten invariant. By the splitting axiom of Gromov-Witten theory,

$$M_{\beta_1, \beta_2} = \sum_{i,j} N_{0, \beta_1}(S_i) g^{ij} N_{0, \beta_2}(S_j),$$

following the notation of Section 0.3. The meeting invariants m_{β_1, β_2} , defined by rules (i-iv), may be viewed as an integral version of M_{β_1, β_2} .

1.2.3 Rule (iii)

Ideally, the *embedded* rational curves in X of class β_i occur in complete, nonsingular, 1-dimensional families

$$F_i \subset \overline{M}_{0,0}(X, \beta_i).$$

Let π_i and ι_i denote the universal curve and map over F_i ,

$$\pi_i : S_i \rightarrow F_i,$$

$$\iota_i : S_i \rightarrow X.$$

Since $\beta_1 \neq \beta_2$, the families F_1 and F_2 are distinct. Ideally, the surfaces S_i are nonsingular and the morphisms π_i are smooth except for finitely many 1-nodal fibers. The meeting number m_{β_1, β_2} is related to the intersection

$$\iota_{1*}(S_1) \cap \iota_{2*}(S_2) \subset X. \tag{6}$$

However, the intersection (6) is not transverse (even ideally). A fiber of π_1 may be a component of a reducible fiber of π_2 or vice versa.

The meeting number m_{β_1, β_2} is defined to count the ideal number of isolated points of the intersection (6). Hence,

$$m_{\beta_1, \beta_2} + \delta = \int_X \iota_{1*}[S_1] \cap \iota_{2*}[S_2]$$

where the correction δ is determined by the non-transversal intersection loci.

The number of times a fiber of π_1 occurs as a component of a reducible fiber of π_2 is simply $m_{\beta_1, \beta_2 - \beta_1}$. Similarly, the opposite event occurs $m_{\beta_1 - \beta_2, \beta_2}$ times. The contribution to δ of each non-transversal is easily determined. Let

$$C \subset X$$

be a fiber of π_1 and a component of a reducible fiber of π_2 . Then

$$\delta(C) = \int_C c_1(E)$$

where

$$0 \rightarrow N_{S_1/C} \oplus N_{S_2/C} \rightarrow N_{X/C} \rightarrow E \rightarrow 0$$

is the normal bundle sequence. Certainly $N_{S_1/C}$ is trivial and $N_{S_2/C}$ has degree -1 . By the Calabi-Yau condition, $N_{X/C}$ is of degree -2 . Hence,

$$\delta(C) = -1.$$

Rule (iii) is obtained by expanding the intersection (6) via the Künneth decomposition of the diagonal. We have

$$\begin{aligned} m_{\beta_1, \beta_2} &= \int_X \iota_*[S_1] \cap \iota_*[S_2] - \delta \\ &= \sum_{i,j} n_{0, \beta_1}(S_i) g^{ij} n_{0, \beta_2}(S_j) + m_{\beta_1, \beta_2 - \beta_1} + m_{\beta_1 - \beta_2, \beta_2}. \end{aligned}$$

1.2.4 Rule (iv)

In case of equality, the meeting number is more subtle. While the surface S_β is ideally nonsingular and

$$\iota : S_\beta \rightarrow X$$

is ideally an immersion, ι is not (even ideally) an embedding. The correct interpretation of $m_{\beta, \beta}$ is twice the number of ideal double points of ι . The factor of 2 arises from ordering.

The double point formula [10] yields a calculation of $m_{\beta, \beta}$ as a correction to the self-intersection,

$$m_{\beta, \beta} = \int_X \iota(S_\beta) \cap \iota(S_\beta) - \int_{S_\beta} \frac{c(T_X)}{c(T_{S_\beta})}$$

where $c(T_X)$ and $c(T_{S_\beta})$ denote the total Chern classes of the respective bundles. Expanding the correction term (and using the Calabi-Yau condition) we find

$$\int_{S_\beta} \frac{c(T_X)}{c(T_{S_\beta})} = \int_S c_2(T_X) + c_1(T_{S_\beta})^2 - c_2(T_{S_\beta}).$$

Certainly,

$$n_{0, \beta}(c_2(T_X)) = \int_{S_\beta} c_2(T_X).$$

There is a decomposition

$$c_1(T_{S_\beta}) = -\psi + c_1(F_\beta)$$

where ψ is the cotangent line on S_β viewed as the 1-pointed space. Hence

$$\int_{S_\beta} c_1(T_{S_\beta})^2 = \int_{S_\beta} \psi^2 + 4\chi(F_\beta).$$

An elementary geometric argument shows

$$\int_{S_\beta} \psi^2 = -\frac{1}{2} \sum_{\beta_1+\beta_2=\beta} m_{\beta_1,\beta_2}$$

where the right side is number of reducible fibers of π_β . Since

$$\int_{S_\beta} c_2(T_{S_\beta}) = \chi(S_\beta) = 2\chi(F_\beta) + \frac{1}{2} \sum_{\beta_1+\beta_2=\beta} m_{\beta_1,\beta_2},$$

only a calculation of the topological Euler characteristic $\chi(F_\beta)$ remains. The formula

$$\chi(F_\beta) = -n_{0,\beta}(c_2(T_X)) + \sum_{\beta_1+\beta_2=\beta} m_{\beta_1,\beta_2}$$

is obtained via a Grothendieck-Riemann-Roch calculation applied to the deformation theoretic characterization of T_{F_β} ,

$$0 \rightarrow R^0\pi_*(\omega_\pi^\vee) \rightarrow R^0\pi_*\iota^*(T_X) \rightarrow T_{F_\beta} \rightarrow \mathcal{O}_{F_\beta}(D) \rightarrow 0,$$

where $D \subset F_\beta$ is the divisor corresponding to nodal fibers of

$$\pi : S \rightarrow F_\beta,$$

see [31] for a similar discussion.

Rule (iv) is obtained by expanding the self-intersection of $\iota_*[S_\beta]$ via the Künneth decomposition of the diagonal and putting together the above surface calculations. We have

$$\begin{aligned} m_{\beta,\beta} &= \int_X \iota_*[S_\beta] \cap \iota_*[S_\beta] - \int_S \frac{c(T_X)}{c(T_{S_\beta})} \\ &= \sum_{i,j} n_{0,\beta}(S_i) g^{ij} n_{0,\beta}(S_j) + n_{0,\beta}(c_2(T_X)) - \sum_{\beta_1+\beta_2=\beta} m_{\beta_1,\beta_2}. \end{aligned}$$

2 Genus 1

2.1 Step I

The equation defining the virtually enumerative invariants $n_{1,\beta}$,

$$\begin{aligned} \sum_{\beta>0} N_{1,\beta} q^\beta &= \sum_{\beta>0} n_{1,\beta} \sum_{d=1}^{\infty} \frac{\sigma(d)}{d} q^{d\beta} \\ &+ \frac{1}{24} \sum_{\beta>0} n_{0,\beta}(c_2(T_X)) \log(1 - q^\beta) \\ &- \frac{1}{24} \sum_{\beta_1,\beta_2} m_{\beta_1,\beta_2} \log(1 - q^{\beta_1+\beta_2}), \end{aligned} \tag{7}$$

is justified in three steps — one for each term on the right.

The first term is the easiest since the contribution of an embedded, super-rigid, elliptic curve $E \subset X$ of class β to the genus 1 Gromov-Witten theory of X is

$$\sum_{d=1}^{\infty} \frac{\sigma(d)}{d} q^{d[E]}.$$

See [32] for a discussion of super-rigidity and multiple covers of elliptic curves.

2.2 Step II

The second term of (7) is obtained from the contributions of families of rational curves to the genus 1 Gromov-Witten invariants of X .

Let $F \subset \overline{M}_{0,0}(X, \beta)$ be the ideal nonsingular family of embedded rational curves (as considered in Section 1.2.3). We make two further hypotheses.

Condition 1. *The family F contains no nodal elements.*

Hence, the morphism

$$\pi : S \rightarrow F$$

is a \mathbb{P}^1 -bundle. In fact, Condition 1 is rarely valid, even ideally, and will be corrected in Step III.

The contribution of F to $N_{1,d\beta}$ is expressed as an excess integral over the moduli space of maps $\overline{M}_1(S, d)$ to the \mathbb{P}^1 -bundle representing d times the

fiber class. Let $\widehat{\pi}$ and $\widehat{\iota}$ denote the universal curve and map over the moduli space,

$$\begin{aligned}\widehat{\pi} &: C \rightarrow \overline{M}_1(S, d), \\ \widehat{\iota} &: C \rightarrow S.\end{aligned}$$

Condition 2. $R^0\widehat{\pi}_*\widehat{\iota}^*(N_{X/S})$ vanishes.

With the vanishing of Condition 2,

$$\text{Cont}_F(N_{1,\beta}) = \int_{[\overline{M}_1(S,d)]^{vir}} c_{top} (R^1\widehat{\pi}_*\widehat{\iota}^* N_{X/S}).$$

Lemma 1. Under the above hypotheses,

$$\text{Cont}_F(N_{1,\beta}) = -\frac{1}{24d} \int_S c_2(T_X).$$

Proof. Consider the relative moduli space of maps to the fibers of $\widehat{\pi}$,

$$\overline{M}_1(\widehat{\pi}, d) \rightarrow F. \quad (8)$$

We will use the isomorphism

$$\overline{M}_1(S, d) \cong \overline{M}_1(\widehat{\pi}, d).$$

The two virtual classes are easily compared,

$$\begin{aligned}[\overline{M}_1(S, d)]^{vir} &= \\ & - \lambda \cap [\overline{M}_1(\widehat{\pi}, d)]^{vir} + \chi(F) \cdot [\overline{M}_1(\mathbb{P}^1, d)]^{vir} \in H_*(\overline{M}_1(S, d), \mathbb{Q}).\end{aligned} \quad (9)$$

On the right, λ is the Chern class of the Hodge bundle, $\chi(F)$ is the topological Euler characteristic, and $\overline{M}_1(\mathbb{P}^1, d)$ a fiber of (8).

Consider first the integral

$$- \int_{[\overline{M}_1(\widehat{\pi}, d)]^{vir}} \lambda \cdot c_{top} (R^1\widehat{\pi}_*\widehat{\iota}^* N_{X/S}). \quad (10)$$

Using the basic boundary relation⁵

$$\lambda = \frac{1}{12} \Delta_0 \in H^2(\overline{M}_{1,1}, \mathbb{Q})$$

⁵The required marked point can be added and removed by the divisor equation.

and the normalization sequence, we can rewrite (10) as

$$-\frac{1}{24} \int_{[\overline{M}_{0,2}(\widehat{\pi}, d)]^{vir}} (\text{ev}_1 \times \text{ev}_2)^*([\Delta_{Diag}]) \cdot \text{ev}_1^*(c_2(N_{X/S})) \cdot c_{top}(R^1\widehat{\pi}_*\widehat{\mathcal{L}}^*N_{X/S}).$$

Finally, using the Aspinwall-Morrison formula,

$$-\int_{[\overline{M}_1(\widehat{\pi}, d)]^{vir}} \lambda \cdot c_{top}(R^1\widehat{\pi}_*\widehat{\mathcal{L}}^*N_{X/S}) = -\frac{d^{-3+2}}{24} \int_S c_2(N_{X/S}).$$

For the second integral, we use the formula from [17] for genus 1 contributions,

$$\begin{aligned} \chi(F) \int_{[\overline{M}_1(\mathbb{P}^1, d)]^{vir}} c_{top}(R^1\widehat{\pi}_*\widehat{\mathcal{L}}^*N_{X/S}) &= \\ \chi(F) \int_{\overline{M}_1(\mathbb{P}^1, d)} c_{top}(R^1\widehat{\pi}_*\widehat{\mathcal{L}}^*(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1))) &= \frac{\chi(F)}{12d}. \end{aligned}$$

Summing the first and second integrals, we obtain, by (9),

$$\text{Cont}_F(N_{1,\beta}) = \frac{1}{24d} \left(- \int_S c_2(N_{X/S}) + 2\chi(F) \right)$$

Finally, using the Calabi-Yau condition and the geometry of \mathbb{P}^1 -bundles,

$$\begin{aligned} c_2(N_{X/S}) &= c_2(T_X) + c_1(T_S)^2 - c_2(T_S) \\ &= c_2(T_X) + c_2(T_S) \\ &= c_2(T_X) + 2\chi(F), \end{aligned}$$

concluding the Lemma. □

Modulo the corrections from nodal elements of F to be discussed in Step III, the derivation of the second term of (7) is complete since

$$n_{0,\beta}(c_2(T_X)) = \int_S c_2(T_X).$$

2.3 Step III

We now relax Condition 1 of Section 2.2, but keep Condition 2 in following stronger form. Let

$$\pi : S \rightarrow F \tag{11}$$

be the ideal family of embedded rational curves of class β . Let $\overline{M}_1(S, \widehat{\beta})$ be the moduli space of maps to S representing a π -vertical curve class

$$\widehat{\beta} \in H_2(S, \mathbb{Z}).$$

The morphism (11) is the blow-up of a \mathbb{P}^1 -bundle over finitely many points corresponding to the

$$\frac{1}{2} \sum_{\beta_1 + \beta_2 = \beta} m_{\beta_1, \beta_2}$$

nodal fibers. Since π is not \mathbb{P}^1 -bundle, $\widehat{\beta}$ need not be a multiple of the fiber class. As before let $\widehat{\pi}$ and $\widehat{\iota}$ denote the universal curve and map over the moduli space $\overline{M}_1(S, \widehat{\beta})$.

Condition 2'. $R^0 \widehat{\pi}_* \widehat{\iota}^*(N_{X/S})$ vanishes for every class $\widehat{\beta}$ satisfying

$$\widehat{\beta} - [\text{Fiber}(\pi)] > 0.$$

The inequality is required in Condition 2'. Ideally, the inequality is violated for connected curves only if $\widehat{\beta}$ equals a multiple of a single component of a reducible nodal fiber of π . Then,

$$R^0 \widehat{\pi}_* \widehat{\iota}^*(N_{X/S}) \neq 0.$$

We view F as not contributing at all to the Gromov-Witten invariants in classes violating the inequality (as these curves deform away from F).

With the vanishing of Condition 2',

$$\text{Cont}_F(N_{1, \widehat{\beta}}) = \int_{[\overline{M}_1(S, \widehat{\beta})]^{vir}} c_{top} (R^1 \widehat{\pi}_* \widehat{\iota}^* N_{X/S})$$

for classes $\widehat{\beta}$ satisfying the inequality.

Since the family F may contain nodal elements, Lemma 1 must be modified. We have

$$\text{Cont}_F \left(\sum_{\beta - [\text{Fiber}(\hat{\pi})] > 0} N_{1,\beta} q^\beta \right) = \frac{1}{24} n_{0,\beta}(c_2(T_X)) \log(1 - q^\beta) \quad (12)$$

$$+ \frac{1}{2} \sum_{d_1=1}^{\infty} \sum_{d_2=1}^{\infty} \sum_{\beta_1+\beta_2=\beta} c_{d_1,d_2} m_{\beta_1,\beta_2} q^{d_1\beta_1+d_2\beta_2}$$

for universal constants c_{d_1,d_2} . The first term on the right is the uncorrected answer of Lemma 1. The second term is the correction. The factor of 2 is included for the double counting induced by the ordering.

The universal form of the correction terms follows from the canonical local analytic geometry near the nodal fibers. Let

$$\pi^{-1}(p) = E_1 \cup E_2$$

be a nodal fiber. The local geometry of S near $\pi^{-1}(p)$ is the total space of the node smoothing deformation. The restriction of $N_{X/S}$ splits in the form $\mathcal{O}_S(E_1) \oplus \mathcal{O}_S(E_2)$. The universality of the correction terms then follows.

Lemma 2. *We have*

$$\frac{1}{2} \sum_{d_1=1}^{\infty} \sum_{d_2=1}^{\infty} c_{d_1,d_2} q_1^{d_1} q_2^{d_2} = -\frac{1}{24} \log(1 - q_1 q_2).$$

Lemma 2 concludes Step III and completes the justification of definition (7) of the invariants $n_{1,\beta}$.

2.3.1 Proof of Lemma 2

By universality, we can prove Lemma 2 by considering any exactly solved geometry that is sufficiently rich to yield all the constants c_{d_1,d_2} .

The simplest is the following local geometry. Let S be the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ at the point (∞, ∞) ,

$$S = \text{BL}_{(\infty,\infty)}(\mathbb{P}^1 \times \mathbb{P}^1) \xrightarrow{\nu} \mathbb{P}^1 \times \mathbb{P}^1.$$

Let L_1 and L_2 be line bundles on S ,

$$L_1 = \nu^*(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-1, -1)), \quad L_2 = \nu^*(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-1, -1))(E),$$

where E is the exceptional divisor. Let X be the Calabi-Yau total space

$$X = L_1 \oplus L_2 \rightarrow S.$$

Of course, X is non-compact.

The homology $H_2(X, \mathbb{Z})$ is freely spanned by

$$H_2(\mathbb{P}^1 \times \mathbb{P}^1, \mathbb{Z}) = \mathbb{Z}[C_1] \oplus \mathbb{Z}[C_2]$$

and $[E]$. Let

$$\beta = [C_1] \in H_2(X, \mathbb{Z}).$$

Certainly, $F_\beta \cong \mathbb{P}^1$ and the associated universal family is

$$\pi : S \rightarrow \mathbb{P}^1$$

obtained by composing ν with the projection onto the second factor.

The morphism π has a unique nodal fiber over $\infty \in F$ which splits as

$$\beta_1 = [C_1] - [E], \quad \beta_2 = [E].$$

Hence, the only nonzero meeting numbers for X are

$$m_{\beta_1, \beta_2} = m_{\beta_2, \beta_1} = 1.$$

Condition 2' is easily verified for the family F .

Proposition 1. *We have*

$$\text{Cont}_F(N_{1, d_1 \beta_1 + d_2 \beta_2}) = \frac{\delta_{d_1, d_2}}{12d_1}$$

for $d_1, d_2 > 0$.

Proof. Let $\mathbf{T}^2 = \mathbb{C}^* \times \mathbb{C}^*$ act on $\mathbb{P}^1 \times \mathbb{P}^1$ by

$$(\xi_1, \xi_2) \cdot ([x_0, x_1], [y_0, y_1]) = ([\xi_1 x_0, x_1], [\xi_2 y_0, y_1])$$

with fixed points

$$(0, 0), (0, \infty), (\infty, 0), (\infty, \infty). \quad (13)$$

The action of \mathbf{T}^2 lifts canonically to S . We calculate

$$\text{Cont}_F(N_{1, d_1 \beta_1 + d_2 \beta_2}) = \int_{[\overline{M}_1(S, d_1 \beta_1 + d_2 \beta_2)]^{vir}} c_{top} (R^1 \widehat{\pi}_* \widehat{t}^*(L_1 \oplus L_2)) \quad (14)$$

by \mathbf{T}^2 -localization. With the correct \mathbf{T}^2 -equivariant linearizations of L_1 and L_2 , the integral is possible evaluate explicitly.

Let s_1 and s_2 denote the weights of the two torus factors of \mathbf{T}^2 . The tangent weights of the \mathbf{T} -action on $\mathbb{P}^1 \times \mathbb{P}^1$ are

$$(-s_1, -s_2), (-s_1, s_2), (s_1, -s_2), (s_1, s_2)$$

at the respective fixed points (13).

(i) Let \mathbf{T}^2 act on $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-1, -1)$ with weights

$$s_1 + s_2, s_1, s_2, 0$$

at the respective fixed points (13). The choice induces a canonical \mathbf{T}^2 -linearization on L_1 .

(ii) Let \mathbf{T}^2 act on $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-1, -1)$ with weights

$$s_1, s_1 - s_2, 0, -s_2$$

at the fixed points (13). Together with the canonical linearization on $\mathcal{O}_S(E)$, the choice induces a canonical \mathbf{T}^2 -linearization on L_2 .

The \mathbf{T}^2 -localization contributions of the integral (14) over $0 \in F$ must first be calculated. The contribution over $0 \in F$ certainly vanishes unless $d_1 = d_2$. An unravelling of the formulas shows

$$\begin{aligned} \text{Cont}_{0 \in F}(N_{1, d\beta_1 + d\beta_2}) = \\ \int_{\overline{M}_1(\mathbb{P}^1, d)} \left(\frac{-s_1 - \lambda}{s_1} \right) c_{top} \left(-s_1 \otimes (R^1 \widehat{\pi}_* \widehat{t}^* (\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1))) \right). \end{aligned}$$

Then, by a straightforward expansion similar to the proof of Lemma 1, we obtain the vanishing

$$\text{Cont}_{0 \in F}(N_{1, d\beta_1 + d\beta_2}) = 0.$$

By the vanishing over $0 \in F$, the contribution over $\infty \in F$ must be a constant⁶,

$$\text{Cont}_{\infty \in F}(N_{1, d_1\beta_1 + d_2\beta_2}) \in \mathbb{Q}.$$

⁶By definition, the contribution over ∞ is a rational function in s_1 and s_2 .

The \mathbf{T}^2 -action on S has 3 fixed points

$$p_0, p_\infty, p'_\infty$$

over $\infty \in F$. Here, p_0 is the fixed point lying over $(0, \infty)$, p_∞ is the node of $\pi^{-1}(\infty)$, and p'_∞ is the remaining fixed point. With the linearizations (i) and (ii), L_1 has weight 0 over p_∞, p'_∞ , and L_2 has weight 0 over p_0, p_∞ .

Since the \mathbf{T}^2 -weight of L_1 at p_∞ and p'_∞ is 0, each node of the fixed map over these produces a \mathbf{T}^2 -trivial factor of $R^1\widehat{\pi}_*\widehat{\ell}^*(L_1)$ by the normalization sequence. Each \mathbf{T}^2 -fixed component mapping to β_2 produces a cancelling \mathbf{T}^2 -trivial factor of $R^1\widehat{\pi}_*\widehat{\ell}^*(L_1)$. Similarly for L_2 .

The only localization graphs⁷ which survive the \mathbf{T}^2 -trivial factors from the 0 weights of L_1 and L_2 are *double combs*. A double comb is a connected graph with a single vertex v_0 over p_0 , a single vertex v'_∞ over p'_∞ , and a single path

$$v_0 - v_\infty - v'_\infty.$$

connecting p_0 to p'_∞ through p_∞ :

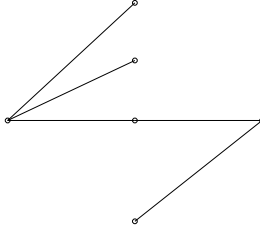


Figure 1: A double comb

Since a double comb has no loops, one of the vertices must have genus 1. The localization contribution of the double comb is understood to include all possible genus 1 vertex assignments.

The final part of the analysis requires taking the nonequivariant limit

$$\text{Lim}_{s_1 \rightarrow 0} \text{Cont}_{\infty \in F}(N_{1, d_1\beta_1 + d_2\beta_2}). \quad (15)$$

Since the contribution on the right is a constant, no information is lost.

Nonequivariant limits are often difficult to study, but for double combs the analysis is simple. A factor of s_1 in the denominator of the localization

⁷We follow [17] for the graphical terminology for the virtual localization formula

contribution of double comb can only occur if the two edges of the unique path connecting p_0 to p'_∞ have equal degrees

$$v_0 \stackrel{e}{=} v_\infty \stackrel{e}{=} v'_\infty.$$

The s_1 factor occurs here from the node smoothing deformation at v_∞ . Even then, the s_1 factor in the denominator is cancelled by s_1 factors in the numerator if either v_0 or v'_∞ has valence greater than 1.

In case $d_1 \neq d_2$, the latter valence condition *must* be satisfied, and the nonequivariant limit (15) can be taken for each double comb. In fact, each nonequivariant limit is easily seen to vanish, proving the Proposition in the unequal case.

If $d_1 = d_2$, there is unique double comb which does not satisfy the valence condition,

$$v_0 \stackrel{d_1}{=} v_\infty \stackrel{d_2}{=} v'_\infty. \quad (16)$$

However, since the nonequivariant limit $\text{Lim}_{s_1 \rightarrow 0}$ exists for *all* other double combs, the limit must exist as well for (16). As in the unequal case, the nonequivariant limit vanishes for all double combs except (16). The limit for (16) is explicitly calculated to equal

$$\frac{1}{12d_1}$$

in the equal case. □

To complete the proof of Lemma 2, we expand (12) for the local Calabi-Yau X . Since $c_2(T_X) = 0$,

$$\begin{aligned} \text{Cont}_F(N_{1,d_1\beta_1+d_2\beta_2}) &= \frac{1}{2} \sum_{d_1=1}^{\infty} \sum_{d_2=1}^{\infty} \sum_{\beta_1+\beta_2=\beta} c_{d_1,d_2} m_{\beta_1,\beta_2} q^{d_1\beta_1+d_2\beta_2} \\ &= c_{d_1,d_2}. \end{aligned}$$

Hence,

$$c_{d_1,d_2} = \frac{\delta_{d_1,d_2}}{12d}$$

by Lemma 1. □

The justification for definition (7) of the invariants $n_{1,\beta}$ is based on ideal geometry. Since the ideal hypotheses are typically false in algebraic geometry, Conjectures 0 and 1 are *not* proven. In fact, one may be suspicious of their validity. In the remaining Sections, we will compute many examples and find the Conjectures to always be valid.

3 Local examples I

3.1 Solutions

Proposition 1 already describes an exactly solved local Calabi-Yau 4-fold geometry. However, a complete solution is not given by Proposition 1 since only special curve classes of $\mathrm{BL}_{(\infty, \infty)}(\mathbb{P}^1 \times \mathbb{P}^1)$ are considered.

The two simplest nontrivial local Calabi-Yau 4-folds are studied here. The examples may be viewed as the analogues of the

$$\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \rightarrow \mathbb{P}^1 \tag{17}$$

local Calabi-Yau 3-fold. As in (17), we find closed form solutions for all curve classes.

3.2 Local \mathbb{P}^2

Let Y be the local Calabi-Yau determined by the total space of the rank 2 bundle

$$\mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(-2) \rightarrow \mathbb{P}^2.$$

Let P denote the point class on \mathbb{P}^2 .

Proposition 2. *We have*

$$N_{0,d}(P) = \frac{(-1)^d}{2d^2} \binom{2d}{d},$$

$$\begin{aligned} \sum_{d>0} N_{1,d} q^d &= \frac{1}{12} \log \left(\sum_{d \geq 0} (-1)^d \binom{2d}{d} q^d \right) \\ &= -\frac{1}{24} \log(1 + 4q). \end{aligned}$$

The Proposition is proven by localization. Let \mathbf{T}^2 act on \mathbb{P}^2 with fixed points

$$[1, 0, 0], [0, 1, 0], [0, 0, 1] \tag{18}$$

and respective tangent weights

$$(s_1, s_2), (-s_2, s_1 - s_2), (s_2 - s_1, -s_1).$$

Let P be the equivariant class of the fixed point $[1, 0, 0]$. Let \mathbf{T}^2 act on $\mathcal{O}_{\mathbb{P}^1}(-1)$ with weights

$$0, s_2, s_1$$

at the respective fixed points (18). Similarly, let \mathbf{T}^2 act on $\mathcal{O}_{\mathbb{P}^1}(-2)$ with weights

$$-s_1 - s_2, s_2 - s_1, s_1 - s_2.$$

The above choices kill the localization contributions to $N_{0,d}(P)$ and $N_{1,d}$ of all graphs with either a node over $[1, 0, 0]$ or an edge connecting $[0, 1, 0]$ and $[0, 0, 1]$. The sum over remaining comb graphs is not difficult and left to the reader.

The integral invariants $n_{0,d}(P)$ and $n_{1,d}$ can be easily calculated from the Gromov-Witten invariants by the defining formulas (2) and (3).

	1	2	3	4	5	6	7	8	9	10
$n_{0,d}(P)$	-1	1	-1	2	-5	13	-35	100	-300	925
$n_{1,d}$	0	0	-1	2	-8	27	-90	314	-1140	4158

The underlying moduli space of maps (with the point condition imposed) for the invariants $n_{0,1}(P)$ and $n_{0,2}(P)$ are projective spaces of dimension 1 and 4 respectively. It appears when the underlying moduli space is \mathbb{P}^k , the invariant is $(-1)^k$ reminiscent of Seiberg-Witten theory for surfaces.

The genus 1 invariants vanish in case there are no embedded genus 1 curves. The underlying moduli space for $n_{1,3}$ is \mathbb{P}^9 .

3.3 Local $\mathbb{P}^1 \times \mathbb{P}^1$

Let Z be the local Calabi-Yau determined by the total space of the rank 2 bundle

$$\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-1, -1) \oplus \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-1, -1) \rightarrow \mathbb{P}^1 \times \mathbb{P}^1.$$

Appropriate localization formulas⁸ for Y in genus 0 and 1 yield

$$N_{0,(d_1,d_2)}(P) = \sum_{m \in P(d_1)} \sum_{n \in P(d_2)} \frac{(-1)^{d_1+d_2}}{\mathfrak{z}(m)\mathfrak{z}(n)} \cdot \int_{\overline{M}_{0,\ell(m)+\ell(n)+1}} \frac{1}{\prod_{i=1}^{\ell(m)} (1 + m_i \psi_i) \prod_{j=1}^{\ell(n)} (1 - n_j \psi_{\ell(m)+j})}$$

and

$$N_{1,(d_1,d_2)} = \sum_{m \in P(d_1)} \sum_{n \in P(d_2)} \frac{(-1)^{d_1+d_2}}{\mathfrak{z}(m)\mathfrak{z}(n)} \cdot \int_{\overline{M}_{1,\ell(m)+\ell(n)}} \frac{1}{\prod_{i=1}^{\ell(m)} (1 + m_i \psi_i) \prod_{j=1}^{\ell(n)} (1 - n_j \psi_{\ell(m)+j})}.$$

Here, P is the point class on $\mathbb{P}^1 \times \mathbb{P}^1$, and $P(d)$ denotes the set of partitions of d . For $p \in P(d)$, the length is denoted by $\ell(p)$. The function

$$\mathfrak{z}(p) = |\text{Aut}(p)| \cdot \prod_{i=1}^{\ell(p)} p_i$$

is the usual factor.

By evaluating the above localization sums, we obtain the following exact solutions.

Proposition 3. *We have,*

$$N_{0,(d_1,d_2)}(P) = \frac{1}{(d_1 + d_2)^2} \binom{d_1 + d_2}{d_1}^2.$$

$$\sum_{(d_1,d_2) \neq (0,0)} N_{1,(d_1,d_2)} q_1^{d_1} q_2^{d_2} = \frac{1}{12} \log \left(\sum_{d_1 \geq 0} \sum_{d_2 \geq 0} \binom{d_1 + d_2}{d_1}^2 q_1^{d_1} q_2^{d_2} \right).$$

The integral invariants $n_{0,(d_1,d_2)}(P)$ and $n_{1,(d_1,d_2)}$ can be easily calculated from the Gromov-Witten invariants by the defining formulas (2) and (3).

⁸We now leave the optimal weight choice for the reader to discover.

$n_{0,(d_1,d_2)}(P)$	0	1	2	3	4	5	6
0	*	1	0	0	0	0	0
1	1	1	1	1	1	1	1
2	0	1	2	4	6	9	12
3	0	1	4	11	25	49	87
4	0	1	6	25	76	196	440
5	0	1	9	49	196	635	1764
6	0	1	12	87	440	1764	5926

$n_{1,(d_1,d_2)}$	0	1	2	3	4	5	6
0	*	0	0	0	0	0	0
1	0	0	0	0	0	0	0
2	0	0	1	2	5	8	14
3	0	0	2	10	28	68	144
4	0	0	5	28	112	350	922
5	0	0	8	68	350	1370	4426
6	0	0	14	144	922	4426	17220

For $n_{0,(1,d)}(P)$ the underlying moduli space is \mathbb{P}^{2d} . The elliptic invariants vanish in classes in which there are no embedded elliptic curves. For $n_{1,(2,2)}$ the moduli space is \mathbb{P}^8 .

4 1-Loop amplitude and Ray-Singer torsion

Let X be a nonsingular Calabi-Yau n -fold. The string amplitude which contains information about the genus 1 Gromov-Witten theory of X is the twisted 1-loop amplitude

$$F_1 = \frac{1}{2} \int_{\mathcal{F}} \frac{d^2\tau}{\text{Im}\tau} \text{Tr}_{\mathcal{H}} [(-1)^F F_L F_R Q^{H_L} \bar{Q}^{H_R}] . \quad (19)$$

Here, the integral is over the fundamental domain \mathcal{F} of the mapping class group of the world-sheet torus with respect to the $\text{SL}(2, \mathbb{Z})$ -invariant measure.

The trace is over the Ramond sector \mathcal{H} of the twisted non-linear σ -model on X . The operators F_L and F_R are the left and right fermion number operators,

$$F = F_L + F_R,$$

and H_L and H_R are the left and right moving Hamilton operators. The parameter τ is the complex modulus of the world-sheet torus, and

$$Q = \exp(2\pi i\tau).$$

The object F_1 is an index which depends either only on the complexified Kähler structure moduli of X in the A -model or only on the complex structure moduli of \hat{X} in the B -model. The dependence on the moduli is via the spectra of H_L and H_R .

We will use the B -model analysis to evaluate F_1 on the mirror \hat{X} of X . Predictions for the genus 1 invariants of X are then made by the mirror map. By the world-sheet analysis of [7, 2], F_1 satisfies the holomorphic anomaly equation

$$\partial_i \bar{\partial}_{\bar{j}} F_1 = \frac{1}{2} \text{Tr}_{\mathcal{H}} [(-)^F C_i \bar{C}_{\bar{j}}] - \frac{1}{24} \text{Tr}_{\mathcal{H}} (-)^F G_{i\bar{j}}. \quad (20)$$

Here $G_{i\bar{j}}$ is the Zamolodchikov metric and the derivatives are with respect to the $N = 2$ moduli. For $N = 2$ σ -models on Calabi-Yau n -folds, we can specialize to the complex moduli on \hat{X} . Then, $\text{Tr}_{\mathcal{H}} (-)^F$ becomes the Euler number χ of the \hat{X} and $G_{i\bar{j}}$ becomes the Weil-Petersen metric on the complex structure moduli space of \hat{X} . The C_i are genus 0, 3-point functions in the A -model. In the B -model on \hat{X} , the C_i can be calculated from the Picard-Fuchs equation for periods of the holomorphic $(n, 0)$ form on \hat{X} . The indices i, \bar{j} run from 1 to $h_{n-1,1}(\hat{X})$.

There two methods to integrate the equation (20). One can use the integrability conditions of special geometry for Calabi-Yau n -folds or, somewhat more generally, the tt^* -equations. The latter apply to any $N = 2$ conformal world-sheet theory. If the central charge satisfies

$$3c = n \in \mathbb{Z},$$

then the tt^* -equations imply the special geometry relations for Calabi-Yau n -folds. The tt^* equations are used in [7, 2] to obtain

$$F_1 = \frac{1}{2} \sum_{p,q} (-1)^{p+q} \binom{p+q}{2} \text{Tr}_{p,q} [\log(g)] - \frac{\chi}{24} K + \log |f|^2. \quad (21)$$

The sum here is over the Ramond-Ramond degenerate lowest energy states labeled by p, q which range for the σ -model case in the left and the right moving sector as follows

$$-\frac{n}{2}, -\frac{n}{2} + 1, \dots, \frac{n}{2}.$$

By the usual argument [42], the states are identified in the A -model with harmonic

$$(k, l) = \left(p + \frac{n}{2}, q + \frac{n}{2}\right)$$

forms. For 4-folds the $(2, 2)$ forms correspond to $(p, q) = (0, 0)$ and decouple from the sum in (21). Finally, g is the tt^* metric, K is the Kähler potential for Weil-Peterson metric, and f is the holomorphic ambiguity.

The metric g is related to the Weil-Peterson metric by

$$G_{i\bar{j}} = \frac{g_{i\bar{j}}}{g_{0\bar{0}}} = \partial_i \bar{\partial}_{\bar{j}} K, \quad g_{0\bar{0}} = e^K. \quad (22)$$

The Kähler potential K is given by

$$e^{-K} = \int_{\hat{X}} \Omega \wedge \bar{\Omega}, \quad (23)$$

where Ω is the holomorphic $(n, 0)$ -form on \hat{X} — e^{-K} can be calculated from the periods on \hat{X} .

In summary, specializing to 4-folds⁹ with $h_{21} = 0$, we evaluate (21) to

$$F_1 = \left(2 + h_{11}(X) - \frac{\chi(X)}{24}\right) K - \log \det G + \log |f|^2. \quad (24)$$

For 3-folds, in our normalization¹⁰,

$$F_1^{(3)} = \frac{1}{2} \left(3 + h_{11}(X) - \frac{\chi(X)}{12}\right) K - \frac{1}{2} \log \det G + \log |f^{(3)}|^2. \quad (25)$$

The Gromov-Witten invariants are extracted in the holomorphic limit of (24) in the large volume of X corresponding to the point P of maximal

⁹All of our compact examples will satisfy $h_{21} = 0$.

¹⁰This is, up to the normalization factor $\frac{1}{2}$, the result in [2].

unipotent monodromy of \hat{X} . Taking the holomorphic limit is very similar for all dimensions. We introduce the flat coordinates near P

$$t_i = \frac{X^i(z)}{X^0(z)}, \quad (26)$$

which are identified with the complexified Kähler parameters of X . As the coordinates z are the complex structure moduli of \hat{X} , equation (26) defines the mirror map between the complex structure on \hat{X} and the complexified Kähler structure on X . The function X^0 is the unique holomorphic period at P , which we chose to lie at $z_i = 0$. The functions

$$X^i = \frac{1}{2\pi i} (X^0 \log(z) + \text{holomorphic})$$

are the $h_{11}(X) = h_{n-1,1}(\hat{X})$ single logarithmic periods. The existence of X^0 and X^i satisfying the above conditions is part of the defining property of P . Using further the structure of the periods in an integer symplectic basis at P , we conclude

$$\begin{aligned} \lim_{\bar{t} \rightarrow i\infty} K &= -\log(X^0), \\ \lim_{\bar{t} \rightarrow i\infty} G_{i\bar{j}} &= \frac{\partial t_i}{\partial z_j} \delta_j^i. \end{aligned} \quad (27)$$

After substitution in (24), we obtain the holomorphic limit of F_1 at P

$$\mathcal{F}_1 = \left(\frac{\chi}{24} - h_{11} - 2 \right) \log(X^0) + \log \det \left(\frac{\partial z}{\partial t} \right) + \log |f|^2. \quad (28)$$

Here, $\partial z / \partial t$ is the Jacobian of the inverse mirror map, and $f(z)$ is the holomorphic ambiguity at genus 1. The latter is restricted by the space time modular invariance of $F_1(t, \bar{t})$. The first two terms in (24) can be shown to be modular invariant. Therefore $f(z)$ must be modular invariant as well. The modular constraints together with the large volume behavior, which in physical terms comes from a zero mode analysis,

$$\mathcal{F}_1 \rightarrow \frac{(-1)^{n+1}}{24} \sum_{i=1}^{h_{n-1,1}} t_i \int_X c_{n-1}(T) \wedge H_i, \quad t \rightarrow \infty, \quad (29)$$

and the expected universal local behavior at other singular limits in the complex structure moduli space fix the holomorphic ambiguity $f(z)$.

As explained in [3], the the genus 1 free energy F_1 is related to the Ray-Singer torsion [35]. The latter describes aspects of the spectrum of the Laplacians of $\Delta_{V,q} = \bar{\partial}_V \bar{\partial}_V^\dagger + \bar{\partial}_V^\dagger \bar{\partial}_V$ of a del-bar operator

$$\bar{\partial}_V : \wedge^q \bar{T}^* \otimes V \rightarrow \wedge^{q+1} \bar{T}^* \otimes V$$

coupled to a holomorphic vector bundle V over M . More precisely, with a regularized determinant over the non-zero mode spectrum of $\Delta_{V,q}$, one defines¹¹[35]

$$I^{RS}(V) = \prod_{q=0}^n (\det' \Delta_{V,q})^{\frac{q}{2}(-1)^{q+1}}. \quad (30)$$

The case $V = \wedge^p T^*$ with $\Delta_{p,q} = \Delta_{\wedge^p T^*,q}$ leads to the definition of a family index

$$F_1 = \frac{1}{2} \log \prod_{p=0}^n \prod_{q=0}^n (\det' \Delta_{pq})^{(-1)^{p+q} pq} \quad (31)$$

depending only on the complex structure of \hat{X} .

5 Local examples II

We now consider the local Calabi-Yau geometry

$$\mathcal{O}(-n) \rightarrow \mathbb{P}^{n-1}.$$

Since the space is toric, Batyrev's reflexive cone construction produces the mirror geometry: a compact Calabi-Yau $(n-1)$ -fold together with a meromorphic $(n-1,0)$ -form λ . The latter can be obtained as a reduction of the holomorphic $(n,0)$ -form to the Calabi-Yau $(n-1)$ -fold and has a non-vanishing residuum¹². The n periods of λ fulfill the Picard-Fuchs equation

$$\mathcal{L}X = \left[\theta^{n-1} - (-1)^n n z \prod_{k=1}^{n-1} (n\theta + k) \right] \theta X = 0, \quad (32)$$

¹¹[34] reviews these facts and relates the Ray-Singer torsion to Hitchin's generalized 3-form action at one loop.

¹²One can also consider the elliptic fibration over \mathbb{P}^n given by the hypersurface of degree $6n$ in the weighted projective space $\mathbb{P}^{n+1}(1^n, 2n, 3n)$, apply Batyrev's reflexive polyhedral mirror construction, and take the large fiber limit on both sides.

where $\theta = z \frac{d}{dz}$. The discriminant of the Picard-Fuchs equation is

$$\tilde{\Delta} = (1 - (-n)^n z).$$

Equation (32) has a constant solution corresponding to the residuum of λ , a general property of non-compact Calabi-Yau manifolds. We normalize constant period to $X^0 = 1$. The system (32) has 3 regular singular points:

- (i) the point $z = 0$ of maximal unipotent monodromy,
- (ii) the point $\tilde{\Delta} = 0$ corresponding to a nodal singularity (called the conifold point),
- (iii) the point $z \rightarrow \infty$ (a \mathbb{Z}_n orbifold point).

Because of (iii), a single cover variable ψ is sometimes more convenient. It is customary to introduce the latter as

$$z = \frac{(-1)^n}{(n\psi)^n} \quad (33)$$

so the conifold is at $\psi^n = 1$. More precisely we define the conifold divisor as

$$\Delta = (1 - \psi^n) . \quad (34)$$

Solutions to (32) can be obtained as

$$X^k = \left(\frac{\partial}{2\pi i \partial \rho} \right)^k X^0(z, \rho) \Big|_{\rho=0} \quad (35)$$

where we define

$$X^0(z, \rho) := \sum_{k=0}^{\infty} \frac{z^{k+\rho}}{\Gamma(-n(k+\rho)+1)\Gamma(k+\rho+1)^n} . \quad (36)$$

Specializing to $n = 4$, we find the compact part of the mirror geometry is related to the K3 given by the quartic in \mathbb{P}^3 obtained by setting $n = 4$ in the above equations. The meromorphic differential is given by

$$\lambda = \frac{1}{2\pi i} \int_{\gamma_0} \frac{d\Sigma}{p}$$

where the contour γ_0 is around $p = 0$ and $d\Sigma$ is the canonical measure on \mathbb{P}^3 . The single logarithmic solution is

$$X^1 = \frac{1}{2\pi i} (\log(z) + 24z + 1260z^2 + 123200z^3 + \mathcal{O}(z^4)) . \quad (37)$$

We define $q = \exp(2\pi it)$ and with $t = X^1/X^0 = X^1$ we obtain by inverting the mirror map the series

$$z = q - 24q^2 - 396q^3 - 39104q^4 + \mathcal{O}(q^5) . \quad (38)$$

The first term of (28) vanishes in the local case as $X^0 = 1$. The holomorphic limit of the Kähler potential term is trivial. We must determine the holomorphic ambiguity. As f is a modular invariant, f can be expressed in terms of ψ^4 . As there is a non-degenerate conformal field theory description at $\psi = 0$ given by the σ -model on the orbifold $\mathbb{C}^n/\mathbb{Z}_n$, F_1 cannot be singular at this point. On the other hand the CFT degenerates at $\psi^n = 1$ and at $\psi = \infty$, and F_1 is expected to be logarithmically divergent at the conifold and at the point of maximal unipotent monodromy. The former behavior can be argued by comparison with the 3-fold case while the latter follows directly from (29) and the leading behavior of (37). Therefore, we are left with the ansatz $f = x \log(\Delta)$, where x is unknown. We obtain

$$\mathcal{F}_1 = \log\left(\frac{\partial\psi}{\partial t}\right) - \frac{1}{24} \log(\Delta) . \quad (39)$$

The first term comes from the holomorphic limit of the Weil-Peterson metric. The x coefficient of the last term is matched to the first term in the localization calculation that can be done in the local case. The leading behavior at boundary divisors in the moduli space will only depend on the type of the singularities. We expect therefore that the leading $-\frac{1}{24} \log(\Delta)$ behavior will be universal at every conifold in 4-folds.

The series \mathcal{F}_1 generates the genus 1 Gromov-Witten invariants of the local Calabi-Yau $\mathcal{O}(-4) \rightarrow \mathbb{P}^4$,

$$\mathcal{F}_1 = C + \sum_{d>0} N_{1,d} q^d$$

where C is an integration constant. Our result for \mathcal{F}_1 agrees up to degree 6 with the calculation of Mayr [30]. Mayr uses the localization and Hodge integral formulas of [9, 17] to calculate up to degree 6.

The associated invariants $n_{0,d}(L)$ and $n_{1,d}$ are given in Table 1. We have checked integrality of the invariants up to degree $d = 100$.

d	g=0	g=1
1	-20	0
2	-820	0
3	-68060	11200
4	-7486440	3747900
5	-965038900	963762432
6	-137569841980	225851278400
7	-21025364147340	50819375678400
8	-3381701440136400	11209456846594400
9	-565563880222390140	2447078892879536000
10	-97547208266548098900	531302247998293196352
11	-17249904137787210605980	115033243754049262028000
12	-3113965536138337597215480	24874518281284024213236000

Table 1: Integer invariants $n_{0,d}(L)$ and $n_{1,d}$ for $\mathcal{O}(-4) \rightarrow \mathbb{P}^3$

6 Compact Calabi-Yau 4-folds

The holomorphic anomaly equation will now be used to verify the integrality conjectures for several compact Calabi-Yau 4-folds. The compact cases have much more interesting geometry than the local models previously considered.

6.1 The sextic 4-fold

Batyrev's mirror construction gives the 1-parameter complex mirror family for degree n hypersurfaces in \mathbb{P}^{n-1} as

$$p = \sum_{k=1}^n x_k^n - n\psi \prod_{k=1}^n x_k = 0 \quad (40)$$

in¹³ $\mathbb{P}^{n-1}/\mathbb{Z}_n^{n-2}$. The holomorphic $(n, 0)$ -form can be written as

$$\Omega = \frac{1}{2\pi i} \int_{\gamma_0} \frac{\psi d\Sigma}{p}$$

where the contour γ_0 is around $p = 0$. We obtain the Picard-Fuchs operator for the period integrals $\int_{\Gamma(4)} \Omega$ in this family parameterized by the variable

¹³The orbifold is essentially irrelevant for the B-model period calculation. It only changes the normalization of the periods by a factor $\frac{1}{n^{n-2}}$.

$z = (-n\psi)^{-n}$ as

$$\mathcal{L} = \theta^{n-1} - n z \prod_{k=1}^{n-1} (n\theta + k) . \quad (41)$$

The discriminant is $\tilde{\Delta} = 1 - n^n z$, and $z = 0$ is the point of maximal unipotent monodromy. Solutions as in (35) are obtained from

$$X^0(z, \rho) := \sum_{k=0}^{\infty} \frac{\Gamma(n(k + \rho) + 1)}{\Gamma(k + \rho + 1)^n} z^{k+\rho} . \quad (42)$$

Here, there is a non-trivial holomorphic solution

$$X^0 = \sum_{k=0}^{\infty} \frac{(nk)!}{(k!)^n} z^k . \quad (43)$$

Let $X_6 \subset \mathbb{P}^5$ be the sextic 4-fold. For X_6 , the first few terms of the inverse mirror map are

$$z = q - 6264 q^2 - 8627796 q^3 - 237290958144 q^4 + \mathcal{O}(q^5) . \quad (44)$$

The nonvanishing Hodge numbers of X_6 up to the symmetries of the Hodge diamond are

$$h_{00} = h_{4,0} = 1, \quad h_{11} = 1, \quad h_{31} = 426, \quad h_{22} = 1752.$$

Further one has

$$\chi = \int_X c_4 = 2610, \quad c_2 = 15H^2, \quad c_3 = -70H^3, \quad \int_X H^4 = 6 \quad (45)$$

where H is the hyperplane class.

The holomorphic ambiguity can be fixed as follows. A simple analytic continuation argument shows that $X^0 \sim \psi$ at the orbifold point $\psi = 0$. As there is no singularity in \mathcal{F}_1 at this point, only the combination $\frac{X^0}{\psi}$ can appear in \mathcal{F}_1 . Furthermore, we use the universal behavior at the conifold

$$\Delta = (1 - \psi^6) = 0$$

and obtain, using (28),

$$\mathcal{F}_1 = \frac{423}{4} \log \left(\frac{X^0}{\psi} \right) + \log \left(\frac{\partial \psi}{\partial t} \right) - \frac{1}{24} \log(\Delta) . \quad (46)$$

d	g=0	g=1
1	60480	0
2	440884080	0
3	6255156277440	2734099200
4	117715791990353760	387176346729900
5	2591176156368821985600	26873294164654597632
6	63022367592536650014764880	1418722120880095142462400
7	1642558496795158117310144372160	65673027816957718149246220800
8	45038918271966862868230872208340160	2828627118403192025358734275898400

Table 2: Integer invariants $n_{0,d}(H^2)$ and $n_{1,d}$ for X_6

m_{d_1,d_2}	$d_2 = 1$	2	3
$d_1 = 1$	15245496000	111118033656000	1576410499948536000
2		809911567810170000	11490828530432030136000
3			163029083563567893374136000

Table 3: Meeting invariants for X_6

As a consistency check, equation (29) is fulfilled.

A further consistency check is obtained from classical methods for degrees $d \leq 4$. The vanishing of the integer invariants

$$n_{1,1} = n_{1,2} = 0$$

is expected from geometrical considerations. The invariants $n_{1,3}$ and $n_{1,4}$ are enumerative. The number of elliptic cubics on a general sextic $X_6 \subset \mathbb{P}^5$ can easily be computed by Schubert calculus¹⁴ to be 2734099200. The number of elliptic quartics on X_6 was computed to be 387176346729900 by Ellingsrud and Stromme in [8]. Finally, the integrality of $n_{1,d}$, which we have checked to $d = 100$, is highly non-trivial. The values for the first few $n_{1,d}$ are listed in Table 2.

We also report a few of the meeting invariants as they have an interesting interpretation as BPS bound states at threshold in Table 3.

6.2 Quintic fibrations over \mathbb{P}^1

Genus 0 Gromov-Witten invariants for multiparameter Calabi-Yau 4-folds have been calculated in [21, 29]. We determine the genus 1 Gromov-Witten invariants and test the integer expansion of F_1 for two such cases.

¹⁴We used the Maple package Schubert written by S. Katz and S. Stromme.

We first consider the quintic fibration over \mathbb{P}^1 realized as the resolution of the degree 10 orbifold hypersurface $X_{10} \subset \mathbb{P}^5(1, 1, 2, 2, 2, 2)$. The non-vanishing Hodge numbers are

$$h_{0,0} = h_{4,0} = 1, \quad h_{1,1} = 2, \quad h_{3,1} = 1452$$

up to symmetries of the Hodge diamond.

We introduce the divisor F associated to the linear system generated by monomials of degree 2. For example, a representative would be $x_3 = 0$ yielding a degree 10 hypersurface in $\mathbb{P}^5(1, 1, 2, 2, 2, 2)$. The dual curve to F lies as a degree 1 curve in the quintic fiber with size t_1 . Another divisor B is associated to the linear system generated by monomials of degree 1. Since B lies in a linear pencil of quintic fibers, $B^2 = 0$. The dual curve is the base \mathbb{P}^1 with size t_2 .

We calculate the classical intersection data by toric geometry as follows

$$\begin{aligned} F^4 &= 10, \quad F^3 B = 5, \quad \int_M c_2 \wedge F^2 = 110, \quad \int_M c_2 \wedge B F = 50, \\ \int_M c_4 &= 2160, \quad \int_M c_3 \wedge F = -200, \quad \int_M c_3 \wedge B = -410. \end{aligned} \quad (47)$$

By Batyrev's construction the mirror is given also as an degree 10 hypersurface in $\mathbb{P}(1, 1, 2, 2, 2, 2)/(\mathbb{Z}_{10}\mathbb{Z}_5^4)$,

$$\sum_{k=1}^2 x_k^{2(n+1)} + \sum_{k=3}^{n+2} x_k^{(n+1)} - 2\phi \prod_{i=1}^2 x_i^{n+1} - \psi \prod_{k=1}^{n+2} x_k \quad (48)$$

with¹⁵ $n = 4$. We derive the Picard-Fuchs operators as

$$\begin{aligned} \mathcal{L}_1 &= \theta_1^{n-1} (2\theta_2 - \theta_1) - (n+2) \left[\prod_{k=1}^{n+1} ((n+2)\theta_1 - k) \right] z_1 \\ \mathcal{L}_2 &= \theta_2 - (2\theta_2 - \theta_1 - 1)(2\theta_2 - \theta_1 - 2)z_2, \end{aligned} \quad (49)$$

where $\theta_i = z_i \frac{d}{dz_i}$ with $z_1 = \frac{\phi}{(-(n+2)\psi)^{n+2}}$ and $z_2 = \frac{1}{(2\phi)^2}$. The system has one conifold discriminant Δ_{con} and a 'strong coupling' discriminant Δ_s at

$$\Delta_{con} = 1 - (\psi^{n+2} - \phi)^2, \quad \Delta_s = 1 - \phi^2. \quad (50)$$

Let us now turn to the calculation of $n_{0,\beta}(S_i)$ and $n_{0,\beta}(c_2)$. We denote by $A_1^{(1)} = J_F$ and $A_2^{(1)} = J_B$ the harmonic $(1, 1)$ -forms dual to F and B . We

¹⁵These formulas apply to n -dimensional degree $2(n+1)$ hypersurfaces in $\mathbb{P}(1, 1, 2^n)$.

chose further a basis $A_1^{(2)} = J_F^2$ and $A_2^{(2)} = J_B J_F$ for the vertical subspace of $H^{2,2}(M)$. Toric geometry implies that the latter can be obtained from the leading θ -polynomials $\hat{\mathcal{L}}_i$ of the Picard-Fuchs operators. More precisely the subspace is spanned by the degree two elements of the graded multiplicative ring

$$\mathcal{R} = \mathbb{C}[A_1^{(1)}, \dots, A_{h^{1,1}}^{(1)}] / \text{Id}(\hat{\mathcal{L}}_i|_{\theta_i \rightarrow A_i^{(1)}}), \quad (51)$$

where the $\hat{\mathcal{L}}_i$ are the formal limits $\hat{\mathcal{L}}_i = \lim_{z_i \rightarrow 0} \mathcal{L}_i$ of the Picard-Fuchs operators.

Following [19, 21, 29], we calculate the genus 0 quantum cohomology intersection

$$C_{ij\alpha}^{(1)} = \int_M A_i^{(1)} \wedge A_j^{(1)} \wedge A_\alpha^{(2)} + \text{instanton corrections.} \quad (52)$$

in the B model as follows. Using the flat coordinates

$$t_i = \frac{X^i}{X^0} = \frac{1}{2\pi i} \log(z_i) + \mathcal{O}(z)$$

for the identification of the B -model structure at the point of maximal unipotent monodromy with the A -model structure [19], we find

$$C_{ij\alpha}^{(1)} = \partial_{t_i} \partial_{t_j} \frac{\Pi_\alpha^{(2)}}{X^0}, \quad (53)$$

where given an $A_\alpha^{(2)}$ the dual period $\Pi_\alpha^{(2)}$ is specified by the leading quadratic behavior in the logarithms as¹⁶

$$\frac{\Pi_\alpha^{(2)}}{X^0} = \frac{1}{2} \sum_{ij} \int_M A_i^{(1)} \wedge A_j^{(1)} \wedge A_\alpha^{(2)} \times \frac{\log(z_i) \log(z_j)}{(2\pi i)^2} + \mathcal{O}(z). \quad (54)$$

For example, using (47), the period Π_{c_2} , whose expansion in q_i yields the $n_{0,\beta}(c_2)$, is specified by the leading logarithmic behavior

$$\frac{\Pi_{c_2}}{X^0} = \frac{1}{(2\pi i)^2} (55 \log(z_1)^2 + 50 \log(z_1) \log(z_2)).$$

¹⁶Note that admixtures of periods with lower leading logarithmic behavior does not affect $C_{ij\alpha}^{(1)}$ due to the derivatives in (52).

$n_{0,\beta}^1$	$d_2 = 0$	1	2	3	4
$d_1 = 0$	*	0	0	0	0
1	12250	12250	0	0	0
2	6462250	35338750	6462250	0	0
3	5718284750	85125750000	85125750000	5718284750	0
4	6349209995000	192339896968750	507648446407500	192339896968750	6349209995000
$n_{0,\beta}^2$	$d_2 = 0$	1	2	3	4
$d_1 = 0$	*	5	0	0	0
1	2875	9375	0	0	0
2	1218500	17669375	5243750	0	0
3	951619125	34150175000	50975575000	4766665625	0
4	969870120000	66623314796875	253824223203750	125716582171875	5379339875000
$n_{1,\beta}$	$d_2 = 0$	1	2	3	4
$d_1 = 0$	*	0	0	0	0
1	0	0	0	0	0
2	0	0	0	0	0
3	-2768250	7297250	7297250	-2768250	0
4	-17325370250	90447173500	699252105750	90447173500	-17325370250

Table 4: Integer invariants for the resolution of $X_{10} \subset \mathbb{P}^5(1, 1, 2, 2, 2)$.

With this information, we calculate the invariants

$$n_{0,\beta}^i = n_{0,\beta}(A_i^{(2)})$$

as well as $n_{0,\beta}(c_2)$.

Finally, for the genus 1 Gromov-Witten invariants, we obtain

$$\mathcal{F}_1 = 86 \log \left(\frac{X^0}{\psi} \right) + \log \det \left(\frac{\partial(\psi, \phi)}{\partial(t_1, t_2)} \right) - \frac{1}{24} \log(\Delta_{con}) - \frac{7}{24} \log(\Delta_s) . \quad (55)$$

The only difference from the calculation for the sextic is that the behavior at Δ_s must be determined. The latter determination is made by imposing (29) with $\int_M c_3 \wedge B = -410$. The second condition in (29) is a check. At $\Delta_s = 0$ we have divisor collapsing, which is an \mathbb{P}^1 fibration over the degree 5 hypersurface in \mathbb{P}^3 . Integer as well as meeting invariants are listed in Tables 4 and 5.

It is interesting to compare the above quintic fibration with a different quintic fibration given by hypersurface of bidegree $(2, 5)$,

$$X_{2,5} \in \mathbb{P}^1 \times \mathbb{P}^4 .$$

The Hodge diamond of the second fibration is the same as the previous case. The divisors B and F correspond to the pull-backs of the hyperplane classes

m_{β_1, β_2}	(0, 1)	(0, 2)	(1, 0)	(1, 1)	(1, 2)	(2, 0)	(2, 1)
(0, 1)	-10	-10	6500	0	0	4025250	4025250
(0, 2)		10	0	0	0	0	0
(1, 0)			10781250	19237750	0	5310625000	43309206500
(1, 1)				10768250	0	10532668750	43309206500
(1, 2)					0	0	0
(2, 0)						2555792968750	22836787744000
(2, 1)							124882678630250

Table 5: Meeting invariants for the resolution of $X_{10} \subset \mathbb{P}^5(1, 1, 2, 2, 2)$.

on \mathbb{P}^1 and \mathbb{P}^4 respectively. We use the same basis for the vertical subspace of $H^{2,2}(M)$ as before $A_1^{(2)} = J_F^2$ and $A_2^{(2)} = J_B J_F$. Due to the different fibration structure, the topological data differ from the previous case:

$$\begin{aligned}
F^4 &= 2, & F^3 B &= 5, & \int_M c_2 \wedge F^2 &= 70, & \int_M c_2 \wedge BF &= 50, \\
\int_M c_4 &= 2160, & \int_M c_3 \wedge F &= -200, & \int_M c_3 \wedge B &= -330.
\end{aligned} \tag{56}$$

We derive the Picard-Fuchs equations in the standard large volume variables z_1 and z_2 [19] as

$$\begin{aligned}
\mathcal{L}_1 &= \theta_1^3 (5\theta_1 - 2\theta_2) - 5^5 z_1 \prod_{k=1}^4 (5\theta_1 + 2\theta_2 + k) + 4z_2 (5\theta_1 + 2\theta_2 + 1) \\
\mathcal{L}_2 &= \theta_2 - z_2 \prod_{k=1}^2 (5\theta_2 + 2\theta_1 + k).
\end{aligned} \tag{57}$$

The system has only one conifold discriminant

$$\Delta = (1 - x_1^2) - 5x_2(1 + 4x_1) + 10x_2^2(1 - x_1) - x_2^3(10 - 5x_2 + x_2^2). \tag{58}$$

We have introduced rescaled variables $x_1 = 5^5 z_1$ and $x_2 = 2^2 z_2$. Here, we know no further regularity conditions in the interior of the moduli space. Therefore, we simply impose (29) with

$$\int_M c_3 \wedge B = -330 \text{ and } \int_M c_3 \wedge F = -200.$$

That fixes the coefficients of the $\log(z_1)$ and $\log(z_2)$ terms in the most general

$n_{0,\beta}^1$	$d_2 = 0$	1	2	3
$d_1 = 0$	*	0	0	0
1	9950	171750	609500	609500
2	5487450	533197250	9651689750	63917722000
3	4956989450	1342522028500	64483365881000	1152361680367750
4	5573313899000	3120681190272750	301443864603401500	10812807897775185750
$n_{0,\beta}^2$	$d_2 = 0$	1	2	3
$d_1 = 0$	*	125	0	0
1	2875	195875	1248250	1799250
2	1218500	369229625	10980854250	101591346500
3	951619125	713334157250	53873269172000	1308427978728875
4	969870120000	1390949237651750	205222409245164750	9819953566670512000
$n_{1,\beta}$	$d_2 = 0$	1	2	3
$d_1 = 0$	*	0	0	0
1	0	0	0	0
2	0	0	0	0
3	-2768250	218986250	82508848750	2759605738750
4	-17325370250	2510820252500	1468762788741875	94873159058300000

Table 6: Integer invariants for $X_{2,5} \subset \mathbb{P}^1 \times \mathbb{P}^4$.

ansatz of the ambiguity

$$\mathcal{F}_1 = 86 \log(X^0) + \log \det \left(\frac{\partial(z_1, z_2)}{\partial(t_1, t_2)} \right) - \frac{1}{24} \log(\Delta) + \frac{51}{4} \log(z_1) + \frac{22}{3} \log(z_2). \quad (59)$$

The integer invariants listed in Table 6 are compatible with the previous quintic fibration — we get the same invariants in the fiber direction, as expected.

6.3 Elliptically fibered Calabi-Yau 4-folds

A simple elliptic fibration over \mathbb{P}^3 compactifies the local model in Section 5. Consider the resolution of the degree 24 orbifold hypersurface

$$X_{24} \subset \mathbb{P}^5(1, 1, 1, 1, 8, 12)$$

in weighted projective space. The genus 0 invariants have been calculated in [21]. The resolution has the following non-vanishing Hodge numbers

$$h_{0,0} = h_{4,0} = 1, \quad h_{1,1} = 2, \quad h_{3,1} = 3878, \quad h_{2,2} = 15564.$$

up to symmetries.

We introduce the linear system B generated by linear polynomials in the four degree 1 variables. The linear system maps X_{24} to \mathbb{P}^3 with fibers given by elliptic curves. The second linear system E is generated by polynomials of degree 4. The curve dual to E is a curve extending over the fiber E with size denoted by t_1 . The curve dual to B is a degree one curve in \mathbb{P}^3 with size denoted by t_2 . The intersections of the divisors are

$$E^4 = 64, \quad E^3 B = 16, \quad E^2 B^2 = 4, \quad EB^3 = 1, \quad B^4 = 0. \quad (60)$$

Further topological data are

$$\begin{aligned} \int_M c_4 &= 23328, & \int_M c_3 \wedge B &= -960, & \int_M c_3 \wedge E &= -3860, \\ \int_M c_2 \wedge B^2 &= 48, & \int_M c_2 \wedge BE &= 182, & \int_M c_2 \wedge E^2 &= 728. \end{aligned} \quad (61)$$

The mirror family is likewise given by an hypersurface of degree 24 in $\mathbb{P}(1, 1, 1, 1, 8, 12)/(\mathbb{Z}_{24}^3)$

$$\sum_{k=1}^n x_1^{6n} + x_{n+1}^2 + x_{n+2}^3 - n\phi \prod_{k=1}^n x_i^{2n} - 6n\psi \prod_{i=1}^{n+2} x_i \quad (62)$$

with¹⁷ $n = 4$. We derive the Picard-Fuchs operators as

$$\begin{aligned} \mathcal{L}_1 &= \theta_1(\theta_1 - n\theta_2) - 12(6\theta_1 - 5)(6\theta_1 - 1)z_1 \\ \mathcal{L}_2 &= \theta_2^n - \prod_{k=1}^n (n\theta_2 - \theta_1 - k)z_2, \end{aligned} \quad (63)$$

where $\theta_i = z_i \frac{d}{dz_i}$ with $z_1 = \frac{n\phi}{(n\psi)^6}$ and $z_2 = \frac{(-1)^n}{(n\phi)^n}$. The system has two conifold discriminants

$$\Delta_1 = 1 - \phi^n, \quad \Delta_2 = 1 - \tilde{\phi}^n, \quad (64)$$

where we defined $\tilde{\phi} = \psi^6 - \phi$. The solutions to the Picard-Fuchs equations can be obtained similarly as in Section 6.1 using the methods outlined in [21]. For example the holomorphic solution at the point of maximal unipotent monodromy is given by

$$X^0 = \sum_{k_1=0, k_2=0}^{\infty} \frac{(6k_1)!(nk_2)!}{(2k_1)!(3k_1)!k_1!(k_2!)^n} z_1^{k_1} z_2^{k_2}. \quad (65)$$

¹⁷These formulas apply to n -dimensional degree $6n$ hypersurfaces in $\mathbb{P}(1^n, 2n, 3n)$.

$n_{0,\beta}^1$	$d_2 = 0$	1	2	3	4
$d_1 = 0$	0	0	0	0	0
1	960	5760	181440	13791360	1458000000
2	1920	-1817280	-98640000	-10715760000	-1476352644480
3	2880	421685760	29972448000	4447212981120	783432258136320
4	3840	2555202430080	-6353500619520	-1273702762398720	-285239128072550400
$n_{0,\beta}^2$	$d_2 = 0$	1	2	3	4
$d_1 = 0$	0	-20	-820	-68060	-7486440
1	0	7680	491520	56256000	7943424000
2	0	-1800000	-159801600	-24602371200	-4394584496640
3	0	278394880	35703398400	7380433205760	1662353371955200
4	0	623056099920	-6039828417600	-1683081588149760	-478655396625235200
$n_{1,\beta}$	$d_2 = 0$	1	2	3	4
$d_1 = 0$	0	0	0	11200	3747900
1	-20	-120	-3780	-7852120	-3536410200
2	0	45720	2245680	2858334000	1724679193440
3	0	-10662240	-719326800	-719497580160	-573686979645680
4	0	1638152760	160844654520	140278855296640	145314212874711600

Table 7: Integer invariants for the resolution of X_{24} .

The considerations, which lead to the expression of F_1 in the holomorphic limit, are very similar to those of Section 6.1,

$$\mathcal{F}_1 = 928 \log\left(\frac{X^0}{\psi}\right) + \log \det\left(\frac{\partial(\psi, \phi)}{\partial(t_1, t_2)}\right) + 3 \log(\psi) - \frac{1}{24} \sum_{i=1}^2 \log(\Delta_i). \quad (66)$$

A new feature here is

$$\lim_{\psi \rightarrow 0} \log \det\left(\frac{\partial(\psi, \phi)}{\partial(t_1, t_2)}\right) \sim \psi^{-3},$$

as is shown by simple analytic continuation of X^0 and the two logarithmic solutions to $\psi = 0$. To maintain the expected regularity at $\psi = 0$, we have to add the explicit $3 \log(\psi)$ term to the holomorphic ambiguity. As a check of the result (66), we note again that (29) with (61) is fulfilled.

We chose further a basis $A_1^{(2)} = \frac{1}{17}(4J_E^2 + J_E J_B)$ and $A_2^{(2)} = J_B^2$ and calculate as before the genus 0 and genus 1 invariants. As a consistency check we note that scaling the size of the elliptic fiber t_1 to infinity leaves us precisely with the $\mathcal{O}(-4) \rightarrow \mathbb{P}^3$ geometry. The corresponding invariants are listed in Table 7.

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