THE κ RING OF THE MODULI OF CURVES OF COMPACT TYPE

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ABSTRACT. The subalgebra of the tautological ring of the moduli of curves of compact type generated by the κ classes is studied in all genera. Relations, constructed via the virtual geometry of the moduli of stable quotients, are used to obtain minimal sets of generators. Bases and Betti numbers of the κ rings are computed. A universality property relating the higher genus κ rings to the genus 0 rings is proven using the virtual geometry of the moduli space of stable maps. The λ_g-formula for Hodge integrals arises as the simplest consequence.

CONTENTS

1. INTRODUCTION

1.1. **Curves of compact type.** Let C be a reduced and connected curve over $\mathbb{C}$ with at worst nodal singularities. The associated *dual*

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graph Γ_C has vertices corresponding to the irreducible components of C and edges corresponding to the nodes. The curve C is of *compact type* if Γ_C is a tree. Alternatively, C is of compact type if the Picard variety of line bundles of fixed multidegree on C is compact.

Standard marked points $p_1, \ldots, p_n$ on C must be distinct and lie in the nonsingular locus. The pointed curve $(C, p_1, \ldots, p_n)$ is *stable* if the line bundle $\omega_C(p_1 + \ldots + p_n)$ is ample. Stability implies the condition $2g - 2 + n > 0$ holds. Let

$$M^c_{g,n} \subset \overline{M}_{g,n}$$

denote the open subset of genus g, n-pointed stable curves of compact type. The complement

$$\overline{M}_{g,n} \setminus M^c_{g,n} = \delta_0$$

is the irreducible divisor of stable curves with a non-disconnecting node.

Since every nonsingular curve is of compact type, the inclusion

$$M_{g,n} \subset M^c_{g,n}$$

is obtained. While the Torelli map

$$M_{g,n} \to A_g$$

from the moduli of nonsingular curves to the moduli of principally polarized Abelian varieties does not extend to $\overline{M}_{g,n}$, the extension

$$M_{g,n} \subset M^c_{g,n} \to A_g$$

is easily defined.

1.2. κ **classes.** The κ classes in the Chow ring[1] $A^*(\overline{M}_{g,n})$ are defined by the following construction. Let

$$\epsilon : \overline{M}_{g,n+1} \to \overline{M}_{g,n}$$

be the universal curve viewed as the $(n+1)$-pointed space, let

$$\mathbb{L}_{n+1} \to \overline{M}_{g,n+1}$$

be the line bundle obtained from the cotangent space of the last marking, and let

$$\psi_{n+1} = c_1(\mathbb{L}_{n+1}) \ \in A^1(\overline{M}_{g,n+1})$$

[1]Since the moduli spaces here are Deligne-Mumford stacks, we will always take Chow rings with $\mathbb{Q}$ coefficients.

be the Chern class. The κ classes, first defined by Mumford, are

$$\kappa_i = \epsilon_*(\psi_{n+1}^{i+1}) \ \in A^i(\overline{M}_{g,n}), \quad i \geq 0 \ .$$

The simplest is κ_0 which equals $2g-2+n$ times the unit in $A^0(\overline{M}_{g,n})$. The convention

$$\kappa_{-1} = \epsilon_*(\psi_{n+1}^0) = 0$$

is often convenient.

The κ classes on $M_{g,n}$ and $M_{g,n}^c$ are defined via restriction from $\overline{M}_{g,n}$. Define the κ rings

$$\begin{aligned} \kappa^*(M_{g,n}) &\subset A^*(M_{g,n}), \\ \kappa^*(M_{g,n}^c) &\subset A^*(M_{g,n}^c), \\ \kappa^*(\overline{M}_{g,n}) &\subset A^*(\overline{M}_{g,n}), \end{aligned}$$

to be the $\mathbb{Q}$-subalgebras generated by the κ classes. Of course, the κ rings are graded by degree.

Since κ_i is a tautological class[2], the κ rings are subalgebras of the corresponding tautological rings. For unpointed nonsingular curves, the κ ring equals the tautological ring by definition [21],

$$\kappa^*(M_g) = R^*(M_g) \ .$$

The topic of the paper is the compact type case where the inclusion

$$\kappa^*(M_{g,n}^c) \subset R^*(M_{g,n}^c)$$

is proper even for divisor classes.

1.3. **Results.** We present here several results about the rings $\kappa^*(M_{g,n}^c)$. The first two yield a minimal set of generators in the $n > 0$ case.

Theorem 1. *$\kappa^*(M_{g,n}^c)$ is generated over $\mathbb{Q}$ by the classes*

$$\kappa_1, \kappa_2, \ldots, \kappa_{g-1+\lfloor \frac{n}{2} \rfloor}.$$

Theorem 2. *If $n > 0$, there are no relations among*

$$\kappa_1, \ldots, \kappa_{g-1+\lfloor \frac{n}{2} \rfloor} \in \kappa^*(M_{g,n}^c)$$

in degrees $\leq g-1+\lfloor \frac{n}{2} \rfloor$.

[2] A discussion of tautological classes is presented in Section 5.1.

Since $\kappa^*(M^c_{g,n}) \subset R^*(M^c_{g,n})$, the socle and vanishing results for the tautological ring [7, 13] imply

$$\kappa^{2g-3+n}(M^c_{g,n}) = \mathbb{Q}, \qquad \kappa^{>2g-3+n}(M^c_{g,n}) = 0 \ . \tag{1}$$

By Theorem 2, all the interesting relations among the κ classes lie in degrees $g + \lfloor \frac{n}{2} \rfloor$ to $2g-3+n$.

By Theorem 1, the classes $\kappa_1, \ldots, \kappa_{g-1}$ generate $\kappa^*(M^c_g)$. Since M^c_g is excluded in Theorem 2, the possibility of a relation among the κ classes in degree $g-1$ is left open. However, no lower relations exist.

Proposition 1. *There are no relations among $\kappa_1, \ldots, \kappa_{g-1} \in \kappa^*(M^c_g)$ in degrees $\leq g-2$ and at most a single relation in degree $g-1$.*

The structure of $\kappa^*(M_g)$ has been studied for many years [21]. Faber [4] conjectured the classes $\kappa_1, \ldots, \kappa_{\lfloor \frac{g}{3} \rfloor}$ form a minimal set of generators for $\kappa^*(M_g)$. The result was proven in cohomology by Morita [20], and a second proof, via admissible covers and valid in Chow, was given by Ionel [15]. A uniform view of M_g, M^c_g, and $\overline{M}_g$ was proposed in [6], but very few results in the latter two cases have been obtained.

1.4. **Relations.** Theorem 1 is proven by finding sufficiently many geometric relations among the κ classes. The method uses the virtual geometry of the moduli space of stable quotients introduced in [19] and reviewed in Section 2. Nonstandard moduli spaces of pointed curves, arising naturally as subloci of the moduli space of stable quotients, are required for the construction.

Following the notation of [19], let $\overline{M}_{g,n|d}$ be the moduli space of genus g stable curves with markings

$$\{p_1 \ldots, p_n\} \ \cup \ \{\widehat{p}_1, \ldots, \widehat{p}_d\} \in C$$

lying in the nonsingular locus and satisfying the conditions

(i) the points p_i are distinct,
(ii) the points $\widehat{p}_j$ are distinct from the points p_i,

with stability given by the ampleness of

$$\omega_C \left(\sum_{i=1}^{n} p_i + \epsilon \sum_{j=1}^{d} \widehat{p}_j \right)$$

for every strictly positive $\epsilon \in \mathbb{Q}$. The conditions allow the points $\widehat{p}_j$ and $\widehat{p}_{j'}$ to coincide. The moduli space $\overline{M}_{g,n|d}$ is a nonsingular, irreducible, Deligne-Mumford stack.[3]

Denote the open locus of curves of compact type by

$$M^c_{g,n|d} \subset \overline{M}_{g,n|d} \ .$$

Consider the universal curve

$$\pi : U \to M^c_{g,n|d}.$$

The morphism π has sections $\sigma_1, \ldots, \sigma_d$ corresponding to the markings $\widehat{p}_1, \ldots, \widehat{p}_d$. Let

$$\sigma \subset U$$

be the divisor obtained from the union of the d sections. The two rank d bundles on $M^c_{g,n|d}$,

$$\mathbb{A}_d = \pi_*(\mathcal{O}_\sigma), \qquad \mathbb{B}_d = \pi_*(\mathcal{O}_\sigma(\sigma)),$$

play important roles in the geometry.

The new relations studied here arise from the vanishing of the Chern classes of the virtual bundle $\mathbb{A}_d^* - \mathbb{B}_d$ on $M^c_{g,n|d}$ after push-forward via the proper forgetful map

$$\epsilon^c : M^c_{g,n|d} \to M^c_{g,n}.$$

Theorem 3. *For all $k > n$,*

$$\epsilon^c_* \left(c_{2g-2+k}(\mathbb{A}_d^* - \mathbb{B}_d) \right) = 0 \ \in A^*(M^c_{g,n}).$$

The proofs of Theorem 3 and richer variants are given in Section 3. The ϵ^c push-forwards are calculated by simple rules explained in Section 3.5. In particular, we will see Theorem 3 yields relations purely among the κ classes on the moduli space $M^c_{g,n}$.

Theorem 1 is proven for $M^c_{g,n}$ in Section 4 by examining the relations of Theorem 3. The coefficient of κ_i for $i > g - 1 + \lfloor \frac{n}{2} \rfloor$ is shown to be nonzero. The method yields an effective evaluation of the relations. Theorem 2 and Proposition 1 are proven in Section 5 by intersection calculations in the tautological ring.

[3]In fact, $\overline{M}_{g,n|d}$ is a special case of the moduli of pointed curves with weights studied by [14, 17]. Specifically, the points p_i are weighted with 1 and the points $\widehat{p}_j$ are weighted with ϵ.

1.5. **Genus 0.** The strategy of Theorem 3 does *not* generate all the relations in $\kappa^*(M_g^c)$. The first example of failure, occurring in genus 5, is discussed in Section 6.

Since all genus 0 curves are of compact type,

$$M_{0,n}^c = \overline{M}_{0,n}.$$

For emphasis here, we will use the notation $M_{0,n}^c$. The following universality property, motivated by the relations of Theorem 3, gives considerable weight to the genus 0 case.

Let $x_1, x_2, x_3, \ldots$ be variables with x_i of degree i. Let

$$f \in \mathbb{Q}[x_1, x_2, x_3, \ldots]$$

be *any* graded homogeneous polynomial.

Theorem 4. *If $f(\kappa_i) = 0 \in \kappa^*(M_{0,n}^c)$, then*

$$f(\kappa_i) = 0 \in \kappa^*(M_{g,n-2g}^c)$$

for all genera g for which $n - 2g \geq 0$.

Our proof of Theorem 4 uses relations constructed from the virtual geometry of the moduli spaces $\overline{M}_{g,n}(\mathbb{P}^1, d)$ of stable maps to $\mathbb{P}^1$. A crucial point is the calculation of the ranks of the vector spaces of relations. In fact, the stable maps relations will be shown to give *all* relations among κ classes in the ring $\kappa^*(M_{0,n}^c)$. The proof of Theorem 4 is given in Sections 7-9 of the paper.

1.6. λ_g**-formula.** The rank g Hodge bundle over the moduli space of curves

$$\mathbb{E} \to \overline{M}_{g,n}$$

has fiber $H^0(C, \omega_C)$ over $[C, p_1, \ldots, p_n]$. Let

$$\lambda_k = c_k(\mathbb{E})$$

be the Chern classes. Since λ_g vanishes when restricted to δ_0, we obtain a well-defined evaluation

$$\phi : A^*(M_{g,n}^c) \to \mathbb{Q}$$

given by integration

$$\phi(\gamma) = \int_{\overline{M}_{g,n}} \overline{\gamma} \cdot \lambda_g \ ,$$

where $\overline{\gamma}$ is any lift of $\gamma \in A^*(M_{g,n}^c)$ to $A^*(\overline{M}_{g,n})$.

A discussion of the evaluation ϕ and the associated Gorenstein conjecture for the tautological ring can be found in [7, 23]. For background on integrating the classes λ_i on the moduli space of curves, see [5].

The evaluation ϕ is determined on $R^*(M^c_{g,n})$ by the λ_g-formula for descendent integrals,

$$\int_{\overline{M}_{g,n}} \psi_1^{a_1} \cdots \psi_n^{a_n} \lambda_g = \binom{2g-3+n}{a_1, \ldots, a_n} \cdot \int_{\overline{M}_{g,1}} \psi_1^{2g-2} \lambda_g,$$

discovered in [10] and proven in [7]. Theorem 4 is much stronger. The λ_g-formula is a direct consequence of Theorem 4 in the special case where f has degree equal to

$$\dim_{\mathbb{C}}(M^c_{0,n}) = n - 3.$$

Theorem 4 may be viewed as an extension of the λ_g-formula from $\mathbb{Q}$ to cycle classes of all intermediate degrees.

1.7. **Bases and Betti numbers.** Let $P(d)$ be the set of partitions of d, and let

$$P(d,k) \subset P(d)$$

be the set of partitions of d into at most k parts. Let $|P(d,k)|$ be the cardinality. To a partition[4]

$$\mathbf{p} = (p_1, \ldots, p_\ell) \in P(d,k),$$

we associate a κ monomial by

$$\kappa_{\mathbf{p}} = \kappa_{p_1} \cdots \kappa_{p_\ell} \in \kappa^d(M^c_{0,n}) \ .$$

Theorem 5. *A $\mathbb{Q}$-basis of $\kappa^d(M^c_{0,n})$ is given by*

$$\{ \kappa_{\mathbf{p}} \mid \mathbf{p} \in P(d, n-2-d) \ \} \ .$$

For example, if $d \leq \lfloor \frac{n}{2} \rfloor$-1, then $n-2-d \geq d$ and

$$P(d, n-2-d) = P(d).$$

Hence, Theorem 5 agrees with Theorem 2. The Betti number calculation,

$$\dim_{\mathbb{Q}} \kappa^d(M^c_{0,n}) \ = \ |P(d, n-2-d)| \ ,$$

is implied by Theorem 5. The proof of Theorem 5 is given in Section 6.2.

[4]The parts of $\mathbf{p}$ are positive and satisfy $p_1 \geq \ldots \geq p_\ell$.

The relations of Theorem 3 and variants provide an indirect approach for multiplication in the canonical basis of $\kappa^*(M^c_{0,n})$ determined by Theorem 5.

Question 1. *Does there exist a direct calculus for multiplication in the canonical basis of* $\kappa^*(M^c_{0,n})$ *?*

1.8. **Universality.** The universality of Theorem 4 expresses the higher genus structures as canonical *ring* quotients,

$$\kappa^*(M^c_{0,2g+n}) \stackrel{\iota_{g,n}}{\to} \kappa^*(M^c_{g,n}) \to 0 \ .$$

Theorem 6. *If* $n > 0$*, then* $\iota_{g,n}$ *is an isomorphism.*

The rings $\kappa^*(M^c_{g,n})$ for $n > 0$ are determined by Theorem 6. For example,

$$\dim_{\mathbb{Q}} \kappa^d(M^c_{g,n}) \ = \ |P(d, 2g-2+n-d)|$$

by Theorem 5 for $n > 0$. The proof of Theorem 6 is presented in Section 10 via intersection calculations.

The quotient $\iota_{g,0}$ is not always an isomorphism. For example, a nontrivial kernel appears for $\iota_{5,0}$.

Question 2. *What is the kernel of* $\iota_{g,0}$ *?*

Universality appears to be special to the moduli of compact type curves. No similar phenomena have been found for M_g or $\overline{M}_g$.

1.9. **Gorenstein conjecture.** The tautological rings

$$R^*(M^c_{g,n}) \subset A^*(M^c_{g,n})$$

have been conjectured in [6, 23] to be Gorenstein algebras with socle in degree $2g-3+n$,

$$\phi : R^{2g-3+n}(M^c_{g,n}) \stackrel{\sim}{\to} \mathbb{Q} \ .$$

The following result, proven in Section 11, may be viewed as significant evidence for the Gorenstein conjecture for all $M^c_{g,n}$ with $n > 0$.

Theorem 7. *If* $n > 0$ *and* $\xi \in \kappa^d(M^c_{g,n}) \neq 0$*, the linear function*

$$L_\xi : R^{2g-3+n-d}(M^c_{g,n}) \to \mathbb{Q}$$

defined by the socle evaluation

$$L_\xi(\gamma) = \phi(\gamma \cdot \xi)$$

is non-trivial.

1.10. **Acknowledgments.** Theorem 3 was motivated by the study of stable quotients developed in [19]. Discussions with A. Marian and D. Oprea were very helpful. Easy exploration of the relations of Theorem 3 was made possible by code written by C. Faber. Conversations with C. Faber played an important role.

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2. Stable quotients

2.1. **Stability.** Our first set of relations in $\kappa(M^c_{g,n})$ will be obtained from the virtual geometry of the moduli space of stable quotients $\overline{Q}_{g,n}(\mathbb{P}^1, d)$. We start by reviewing basic definitions and results of [19].

Let C be a curve[5] with distinct markings $p_1, \ldots, p_n$ in the nonsingular locus C^{ns}. Let q be a quotient of the rank N trivial bundle C,

$$\mathbb{C}^N \otimes \mathcal{O}_C \xrightarrow{q} Q \to 0.$$

If the quotient sheaf Q is locally free at the nodes and markings of C, then q is a *quasi-stable quotient*. Quasi-stability of q implies the associated kernel,

$$0 \to S \to \mathbb{C}^N \otimes \mathcal{O}_C \xrightarrow{q} Q \to 0,$$

is a locally free sheaf on C. Let r denote the rank of S.

Let $(C, p_1, \ldots, p_n)$ be a pointed curve equipped with a quasi-stable quotient q. The data $(C, p_1, \ldots, p_n, q)$ determine a *stable quotient* if the $\mathbb{Q}$-line bundle

$$\omega_C(p_1 + \ldots + p_n) \otimes (\wedge^r S^*)^{\otimes \epsilon} \tag{2}$$

is ample on C for every strictly positive $\epsilon \in \mathbb{Q}$. Quotient stability implies $2g - 2 + n \geq 0$.

Viewed in concrete terms, no amount of positivity of S^* can stabilize a genus 0 component

$$\mathbb{P}^1 \stackrel{\sim}{=} P \subset C$$

unless P contains at least 2 nodes or markings. If P contains exactly 2 nodes or markings, then S^* *must* have positive degree.

[5] All curves here are reduced and connected with at worst nodal singularities.

A stable quotient $(C, p_1, \ldots, p_n, q)$ yields a rational map from the underlying curve C to the Grassmannian $\mathbb{G}(r, N)$. We will only require the $\mathbb{G}(1,2) = \mathbb{P}^1$ case for the proof Theorem 3.

2.2. **Isomorphism.** Let $(C, p_1, \ldots, p_n)$ be a pointed curve. Two quasi-stable quotients

$$\mathbb{C}^N \otimes \mathcal{O}_C \xrightarrow{q} Q \to 0, \quad \mathbb{C}^N \otimes \mathcal{O}_C \xrightarrow{q'} Q' \to 0 \tag{3}$$

on C are *strongly isomorphic* if the associated kernels

$$S, S' \subset \mathbb{C}^N \otimes \mathcal{O}_C$$

are equal.

An *isomorphism* of quasi-stable quotients

$$\phi : (C, p_1, \ldots, p_n, q) \to (C', p'_1, \ldots, p'_n, q')$$

is an isomorphism of curves

$$\phi : C \xrightarrow{\sim} C'$$

satisfying

(i) $\phi(p_i) = p'_i$ for $1 \leq i \leq n$,
(ii) the quotients q and $\phi^*(q')$ are strongly isomorphic.

Quasi-stable quotients (3) on the same curve C may be isomorphic without being strongly isomorphic.

The following result is proven in [19] by Quot scheme methods from the perspective of geometry relative to a divisor.

Theorem 8. *The moduli space of stable quotients $\overline{Q}_{g,n}(\mathbb{G}(r, N), d)$ parameterizing the data*

$$(C,\ p_1, \ldots, p_n,\ 0 \to S \to \mathbb{C}^N \otimes \mathcal{O}_C \xrightarrow{q} Q \to 0),$$

with $\mathrm{rank}(S) = r$ *and* $\deg(S) = -d$*, is a separated and proper Deligne-Mumford stack of finite type over* $\mathbb{C}$.

2.3. **Structures.** Over the moduli space of stable quotients, there is a universal curve

$$\pi : U \to \overline{Q}_{g,n}(\mathbb{G}(r, N), d) \tag{4}$$

with n sections and a universal quotient

$$0 \to S_U \to \mathbb{C}^N \otimes \mathcal{O}_U \xrightarrow{q_U} Q_U \to 0.$$

The subsheaf S_U is locally free on U because of the stability condition.

The moduli space $\overline{Q}_{g,n}(\mathbb{G}(r,N),d)$ is equipped with two basic types of maps. If $2g-2+n>0$, then the stabilization of $(C,p_1,\ldots,p_m)$ determines a map

$$\nu : \overline{Q}_{g,n}(\mathbb{G}(r,N),d) \to \overline{M}_{g,n}$$

by forgetting the quotient. For each marking p_i, the quotient is locally free over p_i, and hence determines an evaluation map

$$\mathrm{ev}_i : \overline{Q}_{g,n}(\mathbb{G}(r,N),d) \to \mathbb{G}(r,N).$$

The general linear group $\mathbf{GL}_N(\mathbb{C})$ acts on $\overline{Q}_{g,n}(\mathbb{G}(r,N),d)$ via the standard action on $\mathbb{C}^N \otimes \mathcal{O}_C$. The structures π, q_U, ν and the evaluations maps are all $\mathbf{GL}_N(\mathbb{C})$-equivariant.

2.4. **Obstruction theory.** The moduli of stable quotients maps to the Artin stack of pointed domain curves

$$\nu^A : \overline{Q}_{g,n}(\mathbb{G}(r,N),d) \to \mathcal{M}_{g,n}.$$

The moduli of stable quotients with fixed underlying curve

$$(C,p_1,\ldots,p_n) \in \mathcal{M}_{g,n}$$

is simply an open set of the Quot scheme. The following result of Section 3.2 of [19] is obtained from the standard deformation theory of the Quot scheme.

Theorem 9. *The deformation theory of the Quot scheme determines a 2-term obstruction theory on $\overline{Q}_{g,n}(\mathbb{G}(r,N),d)$ relative to ν^A given by $RHom(S,Q)$.*

More concretely, for the stable quotient,

$$0 \to S \to \mathbb{C}^N \otimes \mathcal{O}_C \xrightarrow{q} Q \to 0,$$

the deformation and obstruction spaces relative to ν^A are $\mathrm{Hom}(S,Q)$ and $\mathrm{Ext}^1(S,Q)$ respectively. Since S is locally free, the higher obstructions

$$\mathrm{Ext}^k(S,Q) = H^k(C, S^* \otimes Q) = 0, \quad k>1$$

vanish since C is a curve. An absolute 2-term obstruction theory on $\overline{Q}_{g,n}(\mathbb{G}(r,N),d)$ is obtained from Theorem 9 and the smoothness of $\mathcal{M}_{g,n}$, see [3, 11]. The analogue of Theorem 9 for the Quot scheme of a *fixed* nonsingular curve was observed in [18].

The $\mathbf{GL}_N(\mathbb{C})$-action lifts to the obstruction theory, and the resulting virtual class is defined in $\mathbf{GL}_N(\mathbb{C})$-equivariant cycle theory,

$$[\overline{Q}_{g,n}(\mathbb{G}(r,N),d)]^{vir} \in A_*^{\mathbf{GL}_N(\mathbb{C})}(\overline{Q}_{g,n}(\mathbb{G}(r,N),d)).$$

3. Relations via stable quotients

3.1. **$\mathbb{C}^*$-equivariant geometry.** Let $\mathbb{C}^*$ act on $\mathbb{C}^2$ with weights $[0,1]$ on the respective basis elements. Let

$$\mathbb{P}^1 = \mathbb{P}(\mathbb{C}^2),$$

and let $0, \infty \in \mathbb{P}^1$ be the $\mathbb{C}^*$-fixed points corresponding to the eigenspaces of weight 0 and 1 respectively.

There is an induced $\mathbb{C}^*$-action on $\overline{Q}_{g,n}(\mathbb{P}^1,d)$. Since the virtual dimension of $\overline{Q}_{g,n}(\mathbb{P}^1,d)$ is $2g-2+2d+n$,

$$[\overline{Q}_{g,n}(\mathbb{P}^1,d)]^{vir} \in A^{\mathbb{C}^*}_{2g-2+2d+n}(\overline{Q}_{g,n}(\mathbb{P}^1,d)),$$

see [19]. The $\mathbb{C}^*$-action lifts canonically[6] to the universal curve

$$\pi : U \to \overline{Q}_{g,n}(\mathbb{P}^1,d).$$

and to the universal subsheaf S_U. The higher direct image $R^1\pi_*(S_U)$ is a vector bundle of rank $g+d-1$ with top Chern class

$$\mathsf{e}(R^1\pi_*(S_U)) \in A^{g+d-1}_{\mathbb{C}^*}(\overline{Q}_{g,n}(\mathbb{P}^1,d)).$$

3.2. **Relations.** The relations of Theorem 3 will be obtained by studying the class

$$\Phi_{g,n,d} = \left(\mathsf{e}(R^1\pi_*(S_U)) \ \cup \ \prod_{i=1}^{n} \mathrm{ev}_i^*([\infty]) \right) \ \cap \ [\overline{Q}_{g,n}(\mathbb{P}^1,d)]^{vir} \ .$$

on the moduli space of stable quotients. A dimension calculation shows

$$\Phi_{g,n,d} \in A^{\mathbb{C}^*}_{g-1+d}(\overline{Q}_{g,n}(\mathbb{P}^1,d)) \ .$$

Let $2g-2+n>0$, and consider the proper morphism

$$\nu : \overline{Q}_{g,n}(\mathbb{P}^1,d) \to \overline{M}_{g,n}.$$

Let $[1]$ denote the trivial bundle with $\mathbb{C}^*$-weight 1, and let $\mathsf{e}([1])$ be the $\mathbb{C}^*$-equivariant first Chern class. Since the non-equivariant limit of $\mathsf{e}([1])$ is 0, the class

$$\nu_*\left(\Phi_{g,n,d}\ \mathsf{e}([1])^k\right) \in A_{g-1+d-k}(\overline{M}_{g,n}) \tag{5}$$

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[6] The particular $\mathbb{C}^*$-lift to S_U plays an important role in the calculation.

certainly vanishes in the non-equivariant limit for $k > 0$.

We will calculate the push-forward (5) via $\mathbb{C}^*$-localization to find relations. Theorem 3 will be obtained after restriction to the moduli space

$$M^c_{g,n} \subset \overline{M}_{g,n}$$

of curves of compact type.

3.3. $\mathbb{C}^*$**-fixed loci.** Since $\Phi_{g,n,d}\ \mathsf{e}([1])^k$ is a $\mathbb{C}^*$-equivariant class, we may calculate the non-equivariant limit of the push-forward (5) by the virtual localization formula [11] as applied in [19]. We will be interested in the restriction of $\nu_*\left(\Phi_{g,n,d}\ \mathsf{e}([1])^k\right)$ to $M^c_{g,n}$.

The first step is to determine the $\mathbb{C}^*$-fixed loci of $\overline{Q}_{g,n}(\mathbb{P}^1,d)$. The full list of $\mathbb{C}^*$-fixed loci is indexed by decorated graphs described in [19]. However, we will see most loci do not contribute to the localization calculation of

$$\nu_*\left(\Phi_{g,n,d}\ \mathsf{e}([1])^k\right)|_{M^c_{g,n}}$$

by our specific choices of $\mathbb{C}^*$-lifts.

The *principal* component of the $\mathbb{C}^*$-fixed point locus

$$\overline{Q}_{g,n}(\mathbb{P}^1,d)^{\mathbb{C}^*} \subset \overline{Q}_{g,n}(\mathbb{P}^1,d)$$

is defined as follows. Consider the quotient

$$\overline{M}_{g,n|d}\ /S_d \tag{6}$$

where the symmetric group acts by permutation of the d nonstandard markings. Given an element

$$[C,p_1,\ldots,p_n,\widehat{p}_1,\ldots,\widehat{p}_d] \in \overline{M}_{g,n|d}\ ,$$

there is a canonically associated sequence

$$0 \to \mathcal{O}_C(-\sum_{j=1}^d \widehat{p}_j) \to \mathcal{O}_C \to \widetilde{Q} \to 0. \tag{7}$$

By including $\mathcal{O}_C$ above the *second* factor of $\mathbb{C}^2 \otimes \mathcal{O}_C$, we obtain a stable quotient from (7),

$$0 \to \mathcal{O}_C(-\sum_{j=1}^d \widehat{p}_j) \to \mathcal{O}_C \oplus \mathcal{O}_C \to \mathcal{O}_C \oplus \widetilde{Q} \to 0.$$

The corresponding S_d-invariant morphism

$$\iota : \overline{M}_{g,n|d} \to \overline{Q}_{g,n}(\mathbb{P}^1,d)$$

surjects onto the principal component of $\overline{Q}_{g,n}(\mathbb{P}^1,d)^{\mathbb{C}^*}$.

Let $F \subset \overline{Q}_{g,n}(\mathbb{P}^1,d)^{\mathbb{C}^*}$ be a component of the $\mathbb{C}^*$-fixed locus, and let $[C,p_1,\ldots,p_n,q] \in F$ be a generic element of F:

(i) If an irreducible component of C lying over $0 \in \mathbb{P}^1$ has genus $h>0$, then $\mathsf{e}(R^1\pi_*(S_U))$ yields the class λ_h by the contribution formulas of [19]. Since
$$\lambda_h|_{M^c_{h,*}} = 0$$
by [24], such loci F have vanishing contribution to
$$\nu_*\left(\Phi_{g,n,d}\ \mathsf{e}([1])^k\right)|_{M^c_{g,n}} .$$

(ii) If an irreducible component of C lying over $0 \in \mathbb{P}^1$ is incident to more than a single irreducible component dominating $\mathbb{P}^1$, then $\mathsf{e}(R^1\pi_*(S_U))$ vanishes on F by the 0 weight space in $\mathbb{C}^2$ associated to $0 \in \mathbb{P}^1$.

(iii) If $p_i \in C$ lies over $0 \in \mathbb{P}^1$, then $\mathrm{ev}_i^*([\infty])$ vanishes on F.

By the vanishings (i-iii) together with the stability conditions, we conclude the principal locus (6) is the *only* $\mathbb{C}^*$-fixed component of $\overline{Q}_{g,n}(\mathbb{P}^1,d)$ which contributes to $\nu_*\left(\Phi_{g,n,d}\ \mathsf{e}([1])^k\right)|_{M^c_{g,n}}$.

3.4. **Proof of Theorem 3.** The contribution of the principal component of $\overline{Q}_{g,n}(\mathbb{P}^1,d)$ to the push-forward $\nu_*\left(\Phi_{g,n,d}\ \mathsf{e}([1])^k\right)|_{M^c_{g,n}}$ is obtained from the localization formulas of [19] together with an analysis of $\mathsf{e}(R^1\pi_*(S_U))$.

For $[C,p_1,\ldots,p_n,\widehat{p}_1,\ldots,\widehat{p}_d] \in \overline{M}_{g,n|d}$, the long exact sequence associated to (7) yields
$$0 \to H^0(C,\mathcal{O}_C) \to H^0(C,\mathcal{O}_{\widehat{p}_1+\ldots+\widehat{p}_d}) \to H^1(C,S) \to H^1(C,\mathcal{O}_C) \to 0 .$$

We conclude
$$\mathsf{e}(R^1\pi_*(S_U)) = \frac{\mathsf{e}(\mathbb{E}^*\otimes[1])\ \mathsf{e}(\mathbb{A}_d\otimes[1])}{\mathsf{e}([1])}$$
on the principal component. The evaluation
$$\prod_{i=1}^n \mathrm{ev}_i^*([\infty])\cdot \mathsf{e}([1])^k = \mathsf{e}([-1])^n\cdot \mathsf{e}([1])^k$$
is immediate.

By [19], the full localization contribution of the principal component is therefore

$$\frac{\mathsf{e}(\mathbb{E}^* \otimes [1])\ \mathsf{e}(\mathbb{A}_d \otimes [1])}{\mathsf{e}([1])} \mathsf{e}([-1])^n \mathsf{e}([1])^k \cdot \frac{\mathsf{e}(\mathbb{E}^* \otimes [-1])}{\mathsf{e}([-1])} \frac{1}{\mathsf{e}(\mathbb{B}_d \otimes [-1])} \ .$$

Using the Mumford relation $c(\mathbb{E}) \cdot c(\mathbb{E}^*) = 1$, we conclude, in the non-equivariant limit,

$$\nu_* \left(\Phi_{g,n,d}\ \mathsf{e}([1])^k\right) |_{M^c_{g,n}} = (-1)^{3g-3+d+k}\ \epsilon^c_* \left(c_{2g-2+n+k}(\mathbb{A}^*_d - \mathbb{B}_d)\right) \ .$$

Since the non-equivariant limit of $\nu_* \left(\Phi_{g,n,d}\ \mathsf{e}([1])^k\right) |_{M^c_{g,n}}$ vanishes, the proof of Theorem 3 is complete. $\square$

3.5. **Evaluation rules.**

3.5.1. *Chern classes.* Associated to each nonstandard marking $\widehat{p}_j$, there is cotangent line bundle

$$\widehat{\mathbb{L}}_j \to M^c_{g,n|d} \ .$$

Let $\widehat{\psi}_j = c_1(\widehat{\mathbb{L}}_j)$ be the first Chern class.

The nonstandard markings are allowed by the stability conditions to be coincident. The *diagonal*

$$D_{ij} \subset M^c_{g,n|d}$$

is defined to be the locus where $\widehat{p}_i = \widehat{p}_j$. Let

$$S_{ij} = \{\ \ell \mid \ell \neq i, j\ \} \ \cup\ \{\star\} \ .$$

The basic isomorphism

$$D_{ij} \stackrel{\sim}{=} M^c_{g,n|S_{ij}} \ .$$

gives the diagonal geometry a recursive structure compatible with the cotangent line classes,

$$\widehat{\psi}_\ell|_{D_{ij}} = \widehat{\psi}_\ell \ ,$$

$$\widehat{\psi}_i|_{D_{ij}} = \widehat{\psi}_j|_{D_{ij}} = \widehat{\psi}_\star \ .$$

The intersection of distinct diagonals leads to smaller diagonals

$$D_{ij} \cap D_{jk} = D_{ijk}$$

in the obvious sense. The self-intersection is determined by

$$[D_{ij}]^2 = -\widehat{\psi}_\star|_{D_{ij}} \ . \tag{8}$$

For convenience, let

$$\Delta_i = D_{1,i} + D_{2,i} + \ldots + D_{i-1,i}$$

with the convention $\Delta_1 = 0$.

The Chern classes of $\mathbb{A}_d$ and $\mathbb{B}_d$ are easily obtained inductively from the sequences

$$0 \to \mathcal{O}_{\sigma_1+\ldots+\sigma_{d-1}}(-\sigma_d) \to \mathcal{O}_\sigma \to \mathcal{O}_{\sigma_d} \to 0,$$

$$0 \to \mathcal{O}_{\sigma_1+\ldots+\sigma_{d-1}}(\sigma_1 + \ldots + \sigma_{d-1}) \to \mathcal{O}_\sigma(\sigma) \to \mathcal{O}_{\sigma_d}(\sigma) \to 0$$

on the universal curve U over $M^c_{g,n}$. We find

$$\begin{aligned} c(\mathbb{A}_d) &= \prod_{j=1}^{d}(1-\Delta_j), \\ c(\mathbb{B}_d) &= \prod_{j=1}^{d}(1-\widehat{\psi}_j+\Delta_j)\ , \end{aligned} \tag{9}$$

see [19] for similar calculations.

3.5.2. *Push-forward.* From the Chern class formulas (9) and the diagonal intersection rules of Section 3.5.1,

$$\epsilon^c_*(c_{2g-2+k}(\mathbb{A}^*_d - \mathbb{B}_d)) \in A^*(M^c_{g,n})$$

is canonically a sum of push-forwards of the type

$$\epsilon^c_*\left(\widehat{\psi}_1^{j_1+1}\cdots\widehat{\psi}_s^{j_s+1}\right) \in A^*(M^c_{g,n})$$

along the forgetful maps

$$\epsilon^c_* : M^c_{g,n|s} \to M^c_{g,n}$$

associated to the various diagonals.

Lemma 1. $\epsilon_*\left(\widehat{\psi}_1^{j_1+1}\cdots\widehat{\psi}_s^{j_s+1}\right) = \kappa_{j_1}\cdots\kappa_{j_d}$ *in* $A^*(M^c_{g,n})$.

Proof. There are forgetful maps

$$\gamma_j : M^c_{g,n|s} \to M^c_{g,n|1} = M^c_{g,n+1},$$

associated to each nonstandard marking where the isomorphism on the right follows from the definition of stability. Taking the fiber product over $M^c_{g,n}$ of all the γ_j yields a birational morphism

$$\gamma : M^c_{g,n|s} \to M^c_{g,n+1} \times_{M^c_{g,n}} M^c_{g,n+1} \times_{M^c_{g,n}} \cdots \times_{M^c_{g,n}} M^c_{g,n+1}\ .$$

The morphism γ is a small resolution. The exceptional loci are at most codimension 2 in $M^c_{g,n|s}$. Hence,

$$\mu^*(\psi_j) = \widehat{\psi}_j$$

for each nonstandard marking. We see

$$\mu_*\left(\widehat{\psi}_1^{j_1+1}\cdots\widehat{\psi}_s^{j_s+1}\right) = \psi_1^{j_1+1}\cdots\psi_s^{j_s+1}.$$

The result then follows after push-forward to $M^c_{g,n}$ by the definition of the κ classes. □

By Lemma 1, the relations of Theorem 3 are purely among the κ classes in $A^*(M^c_{g,n})$.

3.5.3. *Example.* The $d=1$ case of Theorem 3 immediately yields the relations

$$\forall\ k>n, \quad \kappa_{2g-2+k} = 0\ \in A^*(M^c_{g,n})$$

implied also by the vanishing results (1).

More interesting relations occur for $d=2$. By the Chern class calculation (9),

$$c(\mathbb{A}_2^* - \mathbb{B}_2) = \frac{1+\Delta_1}{1-\widehat{\psi}_1+\Delta_1}\cdot\frac{1+\Delta_2}{1-\widehat{\psi}_2+\Delta_2}\ .$$

Using the series expansion

$$\frac{1+x}{1-y+x} = 1+\sum_{r\geq 0} y(y-x)^r \tag{10}$$

and the diagonal intersection rules, we obtain

$$c(\mathbb{A}_2^* - \mathbb{B}_2) = \left(1+\sum_{r\geq 0}\widehat{\psi}_1^{r+1}\right)\cdot\left(1+\sum_{r\geq 0}\widehat{\psi}_2(\widehat{\psi}_2^r-(2^r-1)\widehat{\psi}_2^{r-1}\Delta_2)\right)$$

In genus 3 with $n=0$, the $k=1$ case of Theorem 3 concerns

$$c_5(\mathbb{A}_2^* - \mathbb{B}_2) = \sum_{r_1+r_2=5}\widehat{\psi}_1^{r_1}\widehat{\psi}_2^{r_2} - \sum_{r=1}^{4}(2^r-1)\widehat{\psi}_\star^4\Delta_2\ .$$

The push-forward is easily evaluated

$$\begin{aligned}\epsilon^c_*(c_5(\mathbb{A}_2^* - \mathbb{B}_2)) &= 4\kappa_3+\kappa_1\kappa_2+\kappa_2\kappa_1+4\kappa_3-(1+3+7+15)\kappa_3\\ &= -18\kappa_3+2\kappa_1\kappa_2\ .\end{aligned}$$

We obtain the nontrivial relation

$$-18\kappa_3 + 2\kappa_1\kappa_2 = 0 \quad \in A^*(M_3^c) \ .$$

3.6. **Richer relations.** The proof of Theorem 3 naturally yields a richer set of relations among the κ classes. The universal curve

$$\pi : U \to M^c_{g,n|d}$$

carries the basic divisor classes

$$s = c_1(S_U^*), \qquad \omega = c_1(\omega_\pi(p_1 + \ldots + p_n))$$

obtained from the universal subsheaf S_U and the π-relative log dualizing sheaf.

Proposition 2. *For all $a_i \geq 0$, $b_i \geq 0$ and $k > n$,*

$$\epsilon_* \left(\prod_{i=1}^{m} \pi_*(s^{a_i}\omega^{b_i}) \cdot c_{2g-2+k}(\mathbb{A}_d^* - \mathbb{B}_d) \right) = 0 \ \in A^*(M^c_{g,n}).$$

The proof of Proposition 2 exactly follows the proof of Theorem 3. We leave the details to the reader. By the rules of Section 3.5, the relations of Proposition 2 are also purely among the κ classes.

4. Evaluation of the stable quotient relations

4.1. **Overview.** Our goal here is to explicitly evaluate the relations of Theorem 3 as polynomials in the κ classes. By examining the coefficients, we will obtain a proof of Theorem 1.

4.2. **Term counts.** Consider the total Chern class

$$c(\mathbb{A}_d^* - \mathbb{B}_d) = \prod_{i=1}^{d} \frac{1+\Delta_i}{1-\widehat{\psi}_i + \Delta_i} \ . \tag{11}$$

After substituting

$$\Delta_i = D_{1,i} + \ldots + D_{i-1,i},$$

we may expand the right side of (11) fully. The resulting expression is a formal series in the $d + \binom{d}{2}$ variables[7]

$$\widehat{\psi}_1, \ldots, \widehat{\psi}_d, -D_{12}, -D_{13}, \ldots, -D_{d-1,d} \ .$$

[7] The sign on the diagonal variables is chosen because of the self-intersection formula (8).

Let M_r^d denote the coefficient in degree r,

$$c(\mathbb{A}_d^* - \mathbb{B}_d) = \sum_{r=0}^{\infty} M_r^d(\widehat{\psi}_i, -D_{ij}).$$

Lemma 2. *After setting all the variables to 1,*

$$\sum_{r=0}^{\infty} M_r^d(\widehat{\psi}_i = 1, -D_{ij} = 1)\ t^r \ = \ \frac{1}{1-dt}.$$

Proof. After setting the variables to 1 in (11), we find

$$c_t(\mathbb{A}_d^* - \mathbb{B}_d) = \prod_{i=1}^{d} \frac{1-(i-1)t}{1-it},$$

which is a telescoping product. □

Lemma 2 may be viewed counting the number of terms in the expansion of (11),

$$M_r^d(\widehat{\psi}_i = 1, -D_{ij} = 1) \ = \ d^r \ .$$

The simple answer will play a crucial role in the analysis.

4.3. **Connected counts.** A monomial in the diagonal variables

$$D_{12}, D_{13}, \ldots, D_{d-1,d} \tag{12}$$

determines a set partition of $\{1, \ldots, d\}$ by the diagonal associations. For example, the monomial $3D_{12}^2 D_{13} D_{56}^3$ determines the set partition

$$\{1,2,3\} \ \cup \ \{4\} \ \cup \ \{5,6\}$$

in the $d = 6$ case. A monomial in the variables (12) is *connected* if the corresponding set partition consists of a single part with d elements.

A monomial in the variables

$$\widehat{\psi}_1, \ldots, \widehat{\psi}_d, -D_{12}, -D_{13}, \ldots, -D_{d-1,d}$$

is connected if the corresponding monomial in the diagonal variables obtained by setting all $\widehat{\psi}_i = 1$ is connected. Let C_r^d be the summand of $M_r^d(\widehat{\psi}_i = 1, -D_{ij} = 1)$ consisting of the contributions of only the connected monomials.

Lemma 3. *We have*

$$\sum_{d=1}^{\infty} \sum_{r=0}^{\infty} C_r^d \ t^r \frac{z^d}{d!} = \log\left(1 + \sum_{d=1}^{\infty} \sum_{r=0}^{\infty} d^r t^r \frac{z^d}{d!}\right) \ .$$

Proof. By a standard application of Wick, the connected and disconnected counts are related by exponentiation,

$$\exp(\sum_{d=1}^{\infty}\sum_{r=0}^{\infty} C_r^d\ t^r \frac{z^d}{d!}) = 1 + \sum_{d=1}^{\infty}\sum_{r=0}^{\infty} M_r^d(\widehat{\psi}_i = 1, -D_{ij} = 1)\ t^r \frac{z^d}{d!}\ .$$

The right side is then evaluated by Lemma 2 □

4.4. C_r^d **for** $r \leq d$**.** We may write the series inside the logarithm in Lemma 3 in the following form,

$$F(t,z) = 1 + \sum_{d=1}^{\infty}\sum_{r=0}^{\infty} d^r t^r \frac{z^d}{d!} = \frac{1}{1 - tz\frac{d}{dz}}\ e^z\ .$$

Expanding the exponential of the differential operator by order in t yields,

$$F(t,z) = e^z + tze^z + t^2(z^2+z)e^z + \\ t^3(z^3+3z^2+z)e^z + t^4(z^4+6z^3+7z^2+z)e^z + \ldots\ .$$

We have proven the following result.

Lemma 4. *$F(t,z) = e^z \cdot \sum_{r=0}^{\infty} t^r p_r(z)$ where*

$$p_r(z) = \sum_{s=0}^{r} c_{r,s} z^{r-s}$$

is a degree r polynomial.

By Lemma 4 and the coefficient evaluation $c_{r,0} = 1$, we see

$$\log(F(t,z)) = z + \log(\frac{1}{1-tz}) + \ldots$$

where the dots stand for terms of the form $t^r z^d$ with $r > d$. We obtain the following result.

Proposition 3. *The only nonvanishing C_r^d for $r \leq d$ are $C_0^1 = 1$ and*

$$\sum_{r=1}^{\infty} C_r^r\ t^r \frac{z^r}{r!} = -\log(1 - tz).$$

4.5. **Evaluation.** Let g and n be fixed. We are interested in calculating

$$R_{g,n}(t,z) = \sum_{d=1}^{\infty} \epsilon_*^c \left(c_t(\mathbb{A}_d^* - \mathbb{B}_d)\right) \frac{z^d}{d!} \ .$$

By the straightforward application of the evaluation rules of Section 3.5, we find

$$R_{g,n}(t,z) = \exp\left(\sum_{d=1}^{\infty}\sum_{r\geq d}^{\infty}(-1)^{d-1}C_r^d\kappa_{r-d}\ t^r z^d\right) \ . \tag{13}$$

We rewrite (13) after separating out the $r=d$ terms using Proposition 3 and the evaluation $\kappa_0 = 2g-2+n$,

$$\begin{aligned} R_{g,n}(t,-z) &= \exp\left(-\sum_{d=1}^{\infty}\sum_{r\geq d}^{\infty} C_r^d\kappa_{r-d}\ t^r z^d\right) \\ &= (1-tz)^{2g-2+n}\exp\left(-\sum_{d=1}^{\infty}\sum_{r>d}^{\infty} C_r^d\kappa_{r-d}\ t^r z^d\right) \ . \end{aligned}$$

The $t^r z^d$ coefficient of $R_{g,n}$ is a valid relation in $A^*(M_{g,n}^c)$ if

$$r > 2g-2+n.$$

The above formula, taken together with Lemma 3, provides a very effective approach to the relations of Theorem 3.

4.6. **Proof of Theorem 1.** The generating series for the coefficients of the singleton $\kappa_{\ell>0}$ in the $t^{d+\ell}z^d$ terms of $R_{g,n}(t,-z)$ is

$$R_{g,n}^{\ell}(t,-z) \ = \ -(1-tz)^{2g-2+n}\sum_{d=1}^{\infty} C_{d+\ell}^d\ (tz)^d t^\ell \ . \tag{14}$$

In order to analyze the right side of (14), we will use Lemma 4. For $\ell \geq 0$, let

$$G_\ell(t,z) = \sum_{d=1}^{\infty} c_{d+\ell,\ell}\ (tz)^d \ . \tag{15}$$

By Lemma 4 and Proposition 3,

$$\begin{aligned}
\sum_{\ell\geq 0}\sum_{d=1}^{\infty} C^d_{d+\ell}\ (tz)^d t^\ell &= \log\left(\sum_{\ell\geq 0} G_\ell(t,z)\ t^\ell\right)\\
&= \log\left(\frac{1}{1-tz}+\sum_{\ell\geq 1} G_\ell(t,z)\ t^\ell\right)\\
&= \log\left(\frac{1}{1-tz}\right)+\log\left(1+(1-zt)\sum_{\ell\geq 1} G_\ell(t,z)\ t^\ell\right) .
\end{aligned}$$

So for $\ell > 0$,

$$\sum_{d=1}^{\infty} C^d_{d+\ell}\ (tz)^d = \text{Coeff}_\ell\left(\log\left(1+(1-zt)\sum_{\ell\geq 1} G_\ell(t,z)\ t^\ell\right)\right) . \tag{16}$$

Here, Coeff_ℓ extracts all the terms of the form $t^{*+\ell}z^*$ and divides by t^ℓ.

The behavior of the coefficients $c_{r,s}$ is easily determined by induction on s.

Lemma 5. *For $r \geq s$, $c_{r,s} = f_s(r)$ where $f_s(r)$ is a polynomial of degree $2s$ with leading term*

$$f_s(r) = \frac{1}{2^s s!} r^{2s} + \dots .$$

For example, $f_0(r) = 1$ and

$$f_1(r) = \frac{1}{2}r^2 + \frac{1}{2}r .$$

We leave the elementary proof of Lemma 5 to the reader

From (15) and Lemma 5, we conclude for $\ell > 0$,

$$G_\ell(t,z) = \frac{1}{2^\ell \ell!}\frac{(2\ell)!}{(1-tz)^{2\ell+1}} + \sum_{i=0}^{2\ell}\frac{\tilde{c}_{i,\ell}}{(1-tz)^i}$$

for $\tilde{c}_{i,\ell} \in \mathbb{Q}$. Then by (16),

$$\sum_{d=1}^{\infty} C^d_{d+\ell}\ (tz)^d = \text{Coeff}_\ell\left(\log\left(1+\sum_{\ell\geq 1}\frac{(2\ell-1)!!}{(1-tz)^{2\ell}}t^\ell\right)\right)\dots, \tag{17}$$

where the dots stand for finitely many terms of the form $(1-tz)^{-j}$ where $j < 2\ell$. By Proposition 4 proven in Section 4.7 below,

$$\sum_{d=1}^{\infty} C^d_{d+\ell}\ (tz)^d = \frac{\alpha_\ell}{(1-zt)^{2\ell}} + \dots \tag{18}$$

with $\alpha_\ell \neq 0$.

We now return to the coefficients of the singleton $\kappa_{\ell>0}$ in the $t^{d+\ell}z^d$ terms of $R_{g,n}(t,-z)$. By (14),

$$R^\ell_{g,n}(t,-z) \;=\; -\alpha_\ell(1-tz)^{2g-2+n-2\ell}t^\ell + \ldots \tag{19}$$

where the dots stand for finitely many terms of the form $(1-tz)^j t^\ell$ where $j > 2g-2+n-2\ell$. If

$$2g-2+n-2\ell < 0, \tag{20}$$

then the coefficient of $(tz)^d t^\ell$ in $R^\ell_{g,n}$ will be nonzero for all large d. Once

$$d+\ell > 2g-2+n,$$

the corresponding κ relation is valid by Theorem 3. If (20) is satisfied, κ_ℓ lies in the subring of $\kappa^*(M^c_{g,n})$ generated by $\kappa_1,\ldots,\kappa_{\ell-1}$. □

4.7. **Series analysis.** Define the double factorial by

$$(2\ell-1)!! = \frac{(2\ell)!}{2^\ell \ell!} = (2\ell-1)\cdot(2\ell-3)\cdots 1$$

and let

$$\phi(x) = 1 + \sum_{\ell\geq 1}(2\ell-1)!!\; x^\ell = 1 + x + 3x^2 + 15x^3 + \ldots$$

be the generating series. Define $\alpha_\ell \in \mathbb{Q}$ for $\ell > 0$ by

$$\log(\phi) = \sum_{\ell\geq 1} \alpha_\ell x^\ell \ .$$

Series expansion yields the first terms

$$\log(\phi(x)) = x + \frac{5}{2}x^2 + \frac{37}{3}x^3 + \frac{353}{4}x^4 + \ldots \ .$$

To complete the proof of Theorem 3, we must prove the following result.

Proposition 4. *$\alpha_\ell \neq 0$ for all $\ell > 0$.*

Let $x = y^2$. Then $\phi(x(y))$ satisfies the differential equation

$$y^2 \frac{d}{dy}(y\phi) = \phi - 1 \ .$$

Equivalently,

$$y^3 \frac{d}{dy}\log(\phi) + y^2 - 1 = -\frac{1}{\phi}$$

Changing variables back to x, we find

$$2x^2 \frac{d}{dx} \log(\phi) + x - 1 = -\frac{1}{\phi} \tag{21}$$

Let β_ℓ denote the coefficients of the inverse series,

$$\begin{aligned} \phi(x)^{-1} &= 1 + \sum_{\ell \geq 0} \beta_\ell x^\ell \\ &= 1 - x - 2\alpha_1 x^2 - 4\alpha_2 x^3 - 6\alpha_3 x^4 - \ldots \ , \end{aligned}$$

where the second equality is obtained from (21).

Lemma 6. *$\beta_\ell \neq 0$ for all $\ell > 0$.*

Proof. Series expansion yields

$$\phi(x)^{-1} = 1 - x - 2x^2 - 10x^3 - 74x^4 - \ldots \ .$$

We will establish the following two properties for $\ell > 0$ by joint induction:

(i) $\beta_\ell < 0$,
(ii) $|\beta_\ell| \leq (2\ell - 1)!!$.

By inspection, the conditions hold in the base case $\ell = 1$.

Let $\ell > 1$ and assume conditions (i)-(ii) hold for all $\ell' < \ell$. Since $\phi \cdot \phi^{-1} = 1$,

$$\begin{aligned} (2\ell - 1)!! + \beta_\ell &= -\sum_{k=1}^{\ell-1} (2k-1)!! \cdot \beta_{\ell-k} \\ &\leq \sum_{k=1}^{\ell-1} (2k-1)!! \cdot (2\ell - 2k - 1)!!, \end{aligned} \tag{22}$$

where the second line uses (ii). For $\frac{\ell}{2} \leq k \leq \ell - 1$,

$$\begin{aligned} (2k-1)!! \cdot (2\ell - 2k - 1)!! &= (2\ell - 1)!! \frac{1}{2\ell - 1} \frac{3}{2\ell - 3} \cdots \frac{2\ell - 2k - 1}{2k + 1} \\ &\leq (2\ell - 1)!! \frac{1}{2\ell - 1}. \end{aligned}$$

By putting the two above inequalities together, we obtain

$$(2\ell - 1)!! + \beta_\ell \leq (\ell - 1) \cdot (2\ell - 1)!! \frac{1}{2\ell - 1} < (2\ell - 1)!! \ .$$

Hence, $\beta_\ell < 0$. Since also

$$(2\ell - 1)!! + \beta_\ell > 0$$

by the first equality of (22) and (i), we see $|\beta_\ell| < (2\ell-1)!!$. □

Lemma 6 and the relation

$$-2\ell\alpha_\ell = \beta_{\ell+1}$$

together complete the proof of Proposition 4.

5. Independence

5.1. **Tautological classes.** The moduli space $M^c_{g,n}$ has an algebraic stratification by topological type. The push-forward of the κ and ψ classes from the strata generate the *tautological ring*

$$R^*(M^c_{g,n}) \subset A^*(M^c_{g,n}) \ ,$$

see [12]. Following the Gorenstein philosophy explained in [6], we will study the independence of

$$\kappa_1, \ldots, \kappa_{g-1+\lfloor \frac{n}{2} \rfloor} \in R^*(M^c_{g,n})$$

through degree $g-1+\lfloor \frac{n}{2} \rfloor$ by pairing with strata classes.

5.2. **Case $n=1$.** We first prove Theorem 2 for $M^c_{g,1}$. By stability, $g \geq 1$. To each partition $\mathbf{p} \in P(d)$, we associate a κ monomial,

$$\kappa_{\mathbf{p}} = \kappa_{p_1}\kappa_{p_2}\cdots\kappa_{p_\ell} \ \in R^*(M^c_{g,1}) \ .$$

Theorem 2 is equivalent to the independence of the $|P(g-1)|$ monomials

$$\{ \ \kappa_{\mathbf{p}} \mid \mathbf{p} \in P(g-1) \ \}$$

in $R^*(M^c_{g,1})$.

To each partition $\mathbf{p} \in P(g-1)$ of length ℓ, we associate a codimension $g-1$ stratum $S_{\mathbf{p}} \subset M^c_{g,1}$ by the following construction. Start with a chain of elliptic curves E_i of length $\ell+1$ with the marking on the first,

$$E_1^* - E_2 - E_3 - \ldots - E_\ell - E_{\ell+1} \ . \tag{23}$$

The asterisk indicates the marking. Since $\ell \leq g-1$, such a chain does not exceed genus g. Next, we add elliptic tails[8] to the first ℓ elliptic components. To the curve E_i, we add $p_i - 1$ elliptic tails. Let C be the resulting curve. The total genus of C is

$$\ell + 1 + (g-1) - \ell = g \ .$$

[8]An elliptic tail is an unmarked elliptic curve meeting the rest of the curve in exactly 1 point.

The number of nodes of C is

$$\ell + (g-1) - \ell = g-1 \ .$$

Hence, C determines a codimension $g-1$ stratum $S_{\mathbf{p}} \subset M^c_{g,1}$.

The moduli in $S_{\mathbf{p}}$ is found mainly on the first ℓ components of the original chain (23). Each such E_i has $p_i + 1$ moduli parameters. All other components (including $E_{\ell+1}$) are elliptic tails with 1 moduli parameter each.

The λ_g-evaluation on $R^*(M^c_{g,1})$ discussed in Section 1.6 yields a pairing on partitions $\mathbf{p}, \mathbf{q} \in P(g-1)$,

$$\mu_g(\mathbf{p},\mathbf{q}) = \int_{\overline{M}_{g,1}} \kappa_{\mathbf{p}} \cdot [S_{\mathbf{q}}] \cdot \lambda_g \ \in \mathbb{Q} \ .$$

Lemma 7. *For all $g \geq 1$, the matrix μ_g is nonsingular.*

Proof. To evaluate the pairing, we first restrict λ_g to $S_{\mathbf{q}}$ by distributing a λ_1 to each elliptic component. To pair $\kappa_{\mathbf{p}}$ with the class $[S_{\mathbf{q}}] \cdot \lambda_g$, we must distribute the factors κ_{p_i} to the components E_j of $S_{\mathbf{q}}$ in all possible ways. By the dimension constraints imposed by the moduli parameters of the components of $S_{\mathbf{q}}$, we immediately conclude

$$\mu_g(\mathbf{p},\mathbf{q}) = 0$$

unless $\ell(\mathbf{p}) \geq \ell(\mathbf{q})$. Moreover, if $\ell(\mathbf{p}) = \ell(\mathbf{q})$, the pairing vanishes unless $\mathbf{p} = \mathbf{q}$.

We have already shown μ_g to be upper-triangular with respect to the length partial ordering on $P(g-1)$. To establish the nonsingularity of μ_g, we must show the diagonal entries $\mu_g(\mathbf{p},\mathbf{p})$ do not vanish. Since $\mu_g(\mathbf{p},\mathbf{p})$ is a product of factors of the form

$$\int_{\overline{M}_{1,p+1}} \kappa_p \lambda_1 = \frac{1}{24},$$

the required nonvanishing holds. $\square$

By Lemma 7, the κ monomials of degree $g-1$ are independent. The proof of Theorem 2 for $M^c_{g,1}$ is complete.

5.3. **Case $n=2$.** We now consider Theorem 2 for $M^c_{g,2}$. By stability, $g \geq 1$. We must prove the independence of the $|P(g)|$ monomials

$$\{ \ \kappa_{\mathbf{p}} \mid \mathbf{p} \in P(g) \ \}$$

in $R^*(M^c_{g,2})$.

To each partition $\mathbf{p} \in P(g)$ of length ℓ, we associate a codimension $g-1$ stratum $T_{\mathbf{p}} \subset M^c_{g,1}$ by the following construction. Start with a chain of elliptic curves E_i of length ℓ with the markings on the first and last,

$$E_1^* - E_2 - E_3 - \ldots - E_\ell^* \ . \tag{24}$$

Since $\ell \leq g$, such a chain does not exceed genus g. Next, we add elliptic tails to the ℓ elliptic components of (24). To the curve E_i, we add $p_i - 1$ elliptic tails. Let C be the resulting curve. The total genus of C is

$$\ell + g - \ell = g$$

The number of nodes of C is

$$\ell - 1 + g - \ell = g - 1 \ .$$

Hence, C determines a codimension $g-1$ stratum $T_{\mathbf{p}} \subset M^c_{g,1}$.

As before, the λ_g-evaluation on $R^*(M^c_{g,2})$ yields a pairing on partitions $\mathbf{p}, \mathbf{q} \in P(g)$,

$$\nu_g(\mathbf{p}, \mathbf{q}) = \int_{\overline{M}_{g,2}} \kappa_{\mathbf{p}} \cdot [T_{\mathbf{q}}] \cdot \lambda_g \ \in \mathbb{Q} \ .$$

Lemma 8. *For all $g \geq 1$, the matrix ν_g is nonsingular.*

The proof is identical to the proof of Lemma 7. We leave the details to the reader. The proof of Theorem 2 for $M^c_{g,2}$ is complete.

5.4. **Proof of Theorem 2.** To complete the proof of Theorem 2, we must consider the case $n \geq 3$ and prove the independence of the monomials

$$\{ \ \kappa_{\mathbf{p}} \mid \mathbf{p} \in P(g - 1 + \lfloor \frac{n}{2} \rfloor) \ \}$$

in $R^*(M^c_{g,n})$.

We will relate the question to the established cases with 1 and 2 markings. Let

$$\widehat{g} = g + \lfloor \frac{n-1}{2} \rfloor, \quad \widehat{n} = n - 2 \lfloor \frac{n-1}{2} \rfloor \ .$$

If n is odd, then $\widehat{n} = 1$. If n is even, $\widehat{n} = 2$. Note

$$\widehat{g} - 1 + \lfloor \frac{\widehat{n}}{2} \rfloor = g - 1 + \lfloor \frac{n}{2} \rfloor \ .$$

To start, assume $\widehat{n} = 1$. We have constructed strata classes in $M^c_{\widehat{g},1}$ which show the independence of the monomials

$$\{ \kappa_{\mathbf{p}} \mid \mathbf{p} \in P(\widehat{g} - 1) \}$$

in $R^*(M^c_{\widehat{g},1})$. For each $\mathbf{q} \in P(\widehat{g} - 1)$, the stratum

$$S_{\mathbf{q}} \subset M^c_{\widehat{g},1}$$

consists of a configuration of $\widehat{g}$ elliptic curves. We construct a corresponding stratum

$$S'_{\mathbf{q}} \subset M^c_{g,n}$$

by the following method. Choose any subset[9] of $\lfloor \frac{n-1}{2} \rfloor$ elliptic components of $S_{\mathbf{q}}$. For each elliptic component E selected, replace E with a rational component carrying 2 additional markings.[10] The construction trades $\lfloor \frac{n-1}{2} \rfloor$ genus for $2\lfloor \frac{n-1}{2} \rfloor$ markings.

Theorem 2 is implied by the nonsingularity of the λ_g-pairing between the κ monomials of degree $\widehat{g} - 1$ and the strata classes $[S'_{\mathbf{q}}]$. The proof of the nonsingularity is identical to the proof of Lemma 7.

The $\widehat{n} = 2$ case proceeds by exactly the same method. Again, elliptic components of the strata

$$T_{\mathbf{q}} \subset M^c_{\widehat{g},2}$$

are traded for rational components with 2 additional markings. Theorem 2 is deduced by nonsingularity of the λ_g-pairing. $\square$

5.5. **Proof of Proposition 1.** Consider M^c_g for $g \geq 2$. Let

$$P^*(g-1) = P(g-1) \ \setminus \ \{(1, \ldots, 1)\}$$

be the subset excluding the longest partition. We will first prove the independence of the monomials

$$\{ \kappa_{\mathbf{p}} \mid \mathbf{p} \in P^*(g-1) \}$$

in $R^*(M^c_g)$. The result shows there can be at most a single κ relation in degree $g - 1$.

To each partition $\mathbf{p} \in P^*(g-1)$ of length $\ell \leq g - 2$, we associate a codimension $g - 2$ stratum $U_{\mathbf{p}} \subset M^c_g$ by the following construction. Start with a chain of curves of length $\ell + 1$,

$$X - E_2 - E_3 - \ldots - E_\ell - E_{\ell+1} \ ,$$

[9]The particular choice of subset is not important.

[10]The particular markings chosen are not important.

where X has genus 2 and all the E_i are elliptic curves. Since $\ell \leq g-2$, such a chain does not exceed genus g. Next, we add elliptic tails to the first ℓ components. Since p_1 is the greatest part of $\mathbf{p}$, $p_1 \geq 2$. To the curve X, we add $p_1 - 2$ elliptic tails. To the curve E_i, we add $p_i - 1$ elliptic tails for $2 \leq i \leq \ell$. Let C be the resulting curve. The total genus of C is

$$2 + \ell + (g-1) - \ell - 1 = g$$

The number of nodes of C is

$$\ell + (g-1) - \ell - 1 = g - 2 \ .$$

Hence, C determines a codimension $g-2$ stratum $U_{\mathbf{p}} \subset M_g^c$.

The λ_g-evaluation on $R^*(M_g^c)$ yields a pairing on $\mathbf{p}, \mathbf{q} \in P^*(g-1)$,

$$\omega_g(\mathbf{p}, \mathbf{q}) = \int_{\overline{M}_g} \kappa_{\mathbf{p}} \cdot [U_{\mathbf{q}}] \cdot \lambda_g \ \in \mathbb{Q} \ .$$

The argument of Lemma 7 yields the following result.

Lemma 9. *For all $g \geq 2$, the matrix ω_g is nonsingular.*

The independence of the κ monomials in degrees at most $g-2$ is easier and proven in a similar way. To each partition $\mathbf{p} \in P(g-2)$ of length ℓ, we associate a codimension $g-1$ stratum $U'_{\mathbf{p}} \subset M_g^c$ by the following construction. Start with a chain of elliptic curves of length $\ell + 2$,

$$E_0 - E_1 - E_2 - E_3 - \ldots - E_\ell - E_{\ell+1} \ .$$

Since $\ell \leq g-2$, such a chain does not exceed genus g. Next, we add elliptic tails to the components. To E_i, for $1 \leq i \leq \ell$, we add $p_i - 1$ elliptic tails. To E_0 and $E_{\ell+1}$, we add nothing. Let C be the resulting curve. The total genus of C is

$$\ell + 2 + (g-2) - \ell = g$$

The number of nodes of C is

$$\ell + 1 + (g-2) - \ell = g - 1 \ .$$

Hence, C determines a codimension $g-1$ stratum $U'_{\mathbf{p}} \subset M_g^c$.

The λ_g-evaluation on $R^*(M_g^c)$ yields a pairing on $\mathbf{p}, \mathbf{q} \in P(g-2)$,

$$\omega'_g(\mathbf{p}, \mathbf{q}) = \int_{\overline{M}_g} \kappa_{\mathbf{p}} \cdot [U'_{\mathbf{q}}] \cdot \lambda_g \ \in \mathbb{Q} \ .$$

Again, the argument of Lemma 7 yields the required result.

Lemma 10. *For all $g \geq 2$, the matrix ω'_g is nonsingular.*

Together, Lemmas 9 and 10 complete the proof of Proposition 1. $\square$

6. Universality of genus 0

6.1. **Genus 5.** Do the relations of Theorem 3 generate the entire ideal of relations in $\kappa^*(M_g^c)$? Since Proposition 2 contains the relations of Theorem 3, we may ask the same question of the richer system. The answer to these questions is no. The first example occurs in $\kappa^6(M_5^c)$.

There are 11 κ monomials of degree 6. By the evaluation rules of Section 3.5, the κ relations in codimension 6 generated by Proposition 2 are the *same* for all the rings

$$\kappa^*(M_5^c),\ \kappa^*(M_{4,2}^c),\ \kappa^*(M_{3,4}^c),\ \kappa^*(M_{2,6}^c),\ \kappa^*(M_{1,8}^c),\ \kappa^*(M_{0,10}^c)\ .$$

On $M_{0,10}^c$, there are 4 types[11] of boundary divisors determined by the point splittings

$$8+2,\quad 7+3,\quad 6+4,\quad 5+5.$$

The pairings of these divisors with the κ monomials

$$\kappa_6,\ \kappa_5\kappa_1,\ \kappa_4\kappa_2,\ \kappa_3^2$$

on $M_{0,10}^c$ are easily seen to determine a nonsingular 4×4 matrix. Hence, the number of independent κ relations in $\kappa^6(M_{0,10}^c)$ is at most 7. In fact, Proposition 2 generates 7 independent relations.

The number of divisor classes in $R^*(M_5^c)$ is 3 given by κ_1 and the 2 boundary divisors with genus splittings $4+1$ and $3+2$. The Gorenstein conjecture for M_5^c predicts $R^6(M_5^c)$ to have rank 3. The rank of $R^6(M_5^c)$ can be proven to be 3 via an application[12] of Getzler's relation [9, 22]. Therefore, there *must* be at least 8 relations among the κ monomials

[11]There are several actual divisors of each type depending on the marking distribution. We select one of each type.

[12]The crucial geometry here is the gluing map

$$\overline{M}_{1,4} \times \prod_{i=1}^{4} \overline{M}_{1,1} \to \overline{M}_5\ .$$

Getzler's codimension 2 relation on the factor $\overline{M}_{1,4}$ pushes-forward to a codimension 6 relation on $\overline{M}_5$ which may be restricted to M_5^c. The result is a nontrivial relation in $R^6(M_5^c)$ not implied by the 7 relations of Proposition 2. We thank C. Faber for pointing out the argument.

of degree 6 in M_5^c. We have proven the method of Proposition 2 does not yield all the κ relations in $R^6(M_5^c)$.

6.2. **Genus 0.** In Sections 7-9 below, a set of relations obtained from the virtual geometry of the moduli space of stable maps will be proven to generate all the κ relations in the rings $\kappa^*(M_{0,n}^c)$.

Question 3. *Does Proposition 2 generate all the κ relations in the rings $\kappa^*(M_{0,n}^c)$?*

The answer to Question 3 is affirmative at least for $n \leq 12$. We list below the Betti polynomials $B_n(t)$ of $\kappa^*(M_{0,n}^c)$ for low n.

$$\begin{aligned}
B_3 &= 1 \\
B_4 &= 1+t \\
B_5 &= 1+t+t^2 \\
B_6 &= 1+t+2t^2+t^3 \\
B_7 &= 1+t+2t^2+2t^3+t^4 \\
B_8 &= 1+t+2t^2+3t^3+3t^4+t^5 \\
B_9 &= 1+t+2t^2+3t^3+4t^4+3t^5+t^6 \\
B_{10} &= 1+t+2t^2+3t^3+5t^4+5t^5+4t^6+t^7 \\
B_{11} &= 1+t+2t^2+3t^3+5t^4+6t^5+7t^6+4t^7+t^8 \\
B_{12} &= 1+t+2t^2+3t^3+5t^4+7t^5+9t^6+8t^7+5t^8+t^9
\end{aligned}$$

From the table of Betti numbers, a formula is easily guessed. Let

$$P(d,k) \subset P(d)$$

be the subset of partitions of d of length at most k, and let $|P(d,k)|$ be the order. We see

$$\dim_{\mathbb{Q}} \kappa^d(M_{0,n}^c) = |P(d,n-d-2)|$$

holds in all the above cases.

Theorem 5. *A $\mathbb{Q}$-basis of $\kappa^d(M_{0,n}^c)$ is given by*

$$\{\kappa_{\mathbf{p}} \mid \mathbf{p} \in P(d,n-2-d)\ \} \ .$$

Proof. In order for $P(d, n-d-2)$ to be nonempty, we must have

$$d \leq n-3.$$

We first prove the independence of the κ monomials associated to $P(d, n-d-2)$ by intersection with strata classes in $R^{n-3-d}(M^c_{0,n})$. To each partition

$$\mathbf{p} \in P(d, n-d-2),$$

we associate a codimension $n-3-d$ stratum $V_{\mathbf{p}} \subset M^c_{0,n}$ by the following construction. We write the parts of $\mathbf{p}$ as

$$(p_1, \ldots, p_\ell, p_{\ell+1}, \ldots, p_{n-d-2})$$

where $p_{\ell+\delta} = 0$ for $\delta > 0$. Start with a chain of rational curves of length $n-d-2$,

$$R_1 - R_2 - R_3 - \ldots - R_{n-d-2} \ .$$

Next, we add markings[13] to the components:

- $p_1 + 2$ markings to R_1,
- $p_i + 1$ markings to R_i for $2 \leq i \leq n-d-3$,
- $p_{n-d-2} + 2$ markings to R_{n-d-2},

Let C be the resulting curve. The total number of markings of C is

$$2 + d + n - d - 2 = n \ .$$

The number of nodes of C is $n-3-d$. Hence, C determines a codimension $n-3-d$ stratum $V_{\mathbf{p}} \subset M^c_{0,n}$.

A simple analysis following the strategy of the proof of Lemma 7 shows the paring on $P(d, n-d-2)$ given by

$$(\mathbf{p}, \mathbf{q}) \mapsto \int_{M^c_{0,n}} \kappa_{\mathbf{p}} \cdot [V]_{\mathbf{q}}$$

is upper-triangular and nonsingular. We conclude the κ monomials associated to $P(d, n-d-2)$ are linearly independent.

The strata of $M^c_{0,n}$ are indexed by marked trees. Given a marked tree Γ with $n-d-2$ vertices, the associated stratum

$$S_\Gamma \subset M^c_{0,n}$$

[13] The particular markings chosen are not important.

parameterizes curves C with marked dual graph Γ. In other words, C is a tree of marked rational components

$$R_1, \ldots, R_{n-2-d} \ .$$

To S_Γ, we associate a partition $\mathbf{q}(\Gamma) \in P(d, n-d-2)$ by the following construction. Let $\mathsf{m}(R_i)$ and $\mathsf{n}(R_i)$ denote the numbers of markings and nodes incident to R_i. Let

$$q_i = \mathsf{m}(R_i) + \mathsf{n}(R_i) - 3.$$

By stability, $q_i \geq 0$. After reordering by size,

$$\mathbf{q}(\Gamma) = (q_1, \ldots, q_{n-d-2}) \in P(d, n-d-2) \ .$$

Let $\mathbf{p} \in P(d)$. The intersection of $\kappa_{\mathbf{p}}$ with a stratum class S is obtained by distributing the factors κ_{p_i} to the components of S. We conclude

$$\int_{M^c_{0,n}} \kappa_{\mathbf{p}} \cdot S_\Gamma = \int_{M^c_{0,n}} \kappa_{\mathbf{p}} \cdot V_{\mathbf{q(\Gamma)}} \tag{25}$$

for all $\mathbf{p} \in P(d)$.

By Poincaré duality[14], the dimension of $\kappa^d(M^c_{0,n})$ is the rank of the intersection pairing

$$\kappa^d(M^c_{0,n}) \times A^{n-3-d}(M^c_{0,n}) \to \mathbb{Q}.$$

The classes of strata generate $A^{n-3-d}(M^c_{0,n})$. Moreover, only the special strata $V_{\mathbf{q}}$ need to be considered by (25). So,

$$\dim_{\mathbb{Q}} \kappa^d(M^c_{0,n}) \leq |P(d, n-d-2)| \ .$$

The independence property together with the above dimension estimate yields the basis result. $\square$

7. Strategy for Theorem 4

7.1. **Overview.** By Theorem 5, we know

$$\dim_{\mathbb{Q}} \kappa^d(M^c_{0,n}) = |P(d, n-2-d)| \ .$$

Hence, Theorem 4 is a consequence of the following result.

[14]For $M^c_{0,n}$, singular cohomology and Chow agree.

Proposition 5. *Let $\zeta > 0$ be fixed. The space of relations among κ monomials of degree d valid simultaneously in* all *the rings*

$$\{\kappa^*(M^c_{g,n}) \mid 2g-2+n=\zeta\ \}$$

is of rank at least $|P(d)| - |P(d,\zeta-d)|$.

Proposition 5 is proven in Sections 8 and 9 by constructing universal relations in $\kappa^*(M^c_{g,n})$ via the virtual geometry of the moduli space of stable maps. The interplay between stable quotients and stable maps is an interesting aspect of the study of $\kappa^*(M^c_{g,n})$.

7.2. ψ **classes.** Consider the cotangent line classes

$$\psi_{n+1},\ldots,\psi_{n+\ell} \in A^1(M^c_{g,n+\ell})$$

at the last ℓ marked points. Let

$$\epsilon^c : M^c_{g,n+\ell} \to M^c_{g,n}$$

be the proper forgetful map. For each partition $\mathbf{p} \in P(d)$ of length ℓ, we associate the class

$$\epsilon^c_*\left(\psi_{n+1}^{1+p_1}\cdots\psi_{n+\ell}^{1+p_\ell}\right) \in A^d(M^c_{g,n})\ .$$

The relation between the above push-forwards of ψ monomials and the κ classes is easily obtained. For $\mathbf{p}=(d)$, we have

$$\epsilon^c_*(\psi_{n+1}^{1+d}) = \kappa_d$$

by definition. The standard cotangent line comparison formulas yield the length 2 case,

$$\epsilon^c_*(\psi_{n+1}^{1+p_1}\psi_{n+2}^{1+p_2}) = \kappa_{p_1}\kappa_{p_2} + \kappa_{p_1+p_2}\ .$$

The formula for arbitrary $\mathbf{p}$, due to Faber, is

$$\epsilon^c_*\left(\psi_{n+1}^{1+p_1}\cdots\psi_{n+\ell}^{1+p_\ell}\right) = \sum_{\sigma\in S_\ell} \kappa_{\sigma(\mathbf{p})}\ , \tag{26}$$

where the sum is over the symmetric group S_ℓ. For $\sigma \in S_\ell$, let

$$\sigma = \gamma_1\ldots\gamma_r$$

be the canonical cycle decomposition (including the 1-cycles), and let $\sigma(\mathbf{p})_i$ be the sum of the parts of $\mathbf{p}$ with indices in the cycle γ_i. Then,

$$\kappa_{\sigma(\mathbf{p})} = \kappa_{\sigma(\mathbf{p})_1}\cdots\kappa_{\sigma(\mathbf{p})_r}\ .$$

A discussion of (26) can be found in [1], see equation (1.13) there.

Lemma 11. *The sets of classes in $A^d(M^c_{g,n})$ defined by*

$$\{\ \epsilon^c_*\left(\psi_{n+1}^{1+p_1}\cdots\psi_{n+\ell}^{1+p_\ell}\right)\ |\ \mathbf{p}\in P(d)\ \}\quad \textit{and}\quad \{\ \kappa_{\mathbf{p}}\ |\ \mathbf{p}\in P(d)\ \}$$

are related by an invertible linear transformation independent of g and n.

Proof. Formula (26) defines a universal transformation independent of g and n. Since the transformation is triangular in the partial ordering of $P(d)$ by length (with 1's on the diagonal), the invertibility is clear. □

7.3. **Bracket classes.** Let $\mathbf{p}\in P(d)$ be a partition of length ℓ. Let

$$\langle\mathbf{p}\rangle = \epsilon^c_*\left[\prod_{i=1}^{\ell}\frac{1}{1-p_i\psi_{n+i}}\right]^{\ell+d} \in A^d(M^c_{g,n})\ . \tag{27}$$

The superscript in the inhomogeneous expression $\left[\prod_{i=1}^{\ell}\frac{1}{1-p_i\psi_{n+i}}\right]^{\ell+d}$ indicates the summand in $A^{\ell+d}(M^c_{g,n+\ell})$.

We can easily expand definition (27) to express the class $\langle\mathbf{p}\rangle$ linearly in terms of the classes

$$\{\ \epsilon^c_*\left(\psi_{n+1}^{1+p_1}\cdots\psi_{n+\ell}^{1+p_\ell}\right)\ |\ \mathbf{p}\in P(d)\ \}\ .$$

Since the string and dilation equation must be used to remove the ψ^0_{n+i} and ψ^1_{n+i} factors, the transformation depends upon g and n only through $2g-2+n$.

Lemma 12. *The sets of classes in $A^d(M^c_{g,n})$ defined by*

$$\{\ \langle\mathbf{p}\rangle\ |\ \mathbf{p}\in P(d)\ \}\quad \textit{and}\quad \{\ \epsilon^c_*\left(\psi_{n+1}^{1+p_1}\cdots\psi_{n+\ell}^{1+p_\ell}\right)\ |\ \mathbf{p}\in P(d)\ \}$$

are related by an invertible linear transformation depending only upon $2g-2+n$.

Proof. Only the invertibility remains to be established. The result exactly follows from the proof of Proposition 3 in [7]. □

By Lemmas 11 and 12, the bracket classes lie in the κ ring,

$$\langle\mathbf{p}\rangle\in\kappa^d(M^c_{g,n})\ .$$

We will prove Proposition 5 in the following equivalent form.

Proposition 6. *Let $\zeta > 0$ be fixed. The space of relations among the classes*

$$\{ \langle \mathbf{p} \rangle \mid \mathbf{p} \in P(d) \}$$

valid in all *the rings*

$$\{\kappa^*(M^c_{g,n}) \mid 2g-2+n = \zeta \}$$

is of rank at least $|P(d)| - |P(d, \zeta - d)|$.

8. Relations via stable maps

8.1. **Moduli of stable maps.** Let $\overline{M}_{g,n+m}(\mathbb{P}^1, d)$ denote the moduli of stable maps[15] to $\mathbb{P}^1$ of degree d, and let

$$\nu : \overline{M}_{g,n+m}(\mathbb{P}^1, d) \to \overline{M}_{g,n}$$

be the morphism forgetting the map and the last m markings. The moduli space

$$M^c_{g,n+m}(\mathbb{P}^1, d) \subset \overline{M}_{g,n+m}(\mathbb{P}^1, d)$$

is defined by requiring the domain curve to be of compact type. The restriction

$$\nu^c : M^c_{g,n+m}(\mathbb{P}^1, d) \to M^c_{g,n}$$

is proper and equivariant with respect to the symmetries of $\mathbb{P}^1$.

We will find relations in $A^*(M^c_{g,n})$ by localizing ν^c push-forwards which vanish geometrically. A complete analysis in the socle $A^{2g-3}(M^c_g)$ was carried out in [7], but much more will be required for Theorem 4. While the relations in $A^*(M^c_{g,n})$ of Theorem 3 via stable quotients are more elegantly expressed, the ranks of the relations via stable maps appear easier to compute.

8.2. **Construction of relations.**

8.2.1. *Indexing.* Let $d \leq 2g-3+n$, and let

$$\delta = 2g - 3 + n - d \ .$$

We will construct a series of relations $I(g, d, \alpha)$ in $A^d(M^c_{g,n})$ where

$$\alpha = (\alpha_1, \ldots, \alpha_m)$$

is a (non-empty) vector of non-negative integers satisfying two conditions:

[15]Stable maps were defined in [16], see [8] for an introduction.

(i) $|\alpha| = \sum_{i=1}^{m} \alpha_i \leq d - 2 - \delta$,
(ii) $\alpha_i > 0$ for $i > 1$.

By condition (i), $d - 2 - \delta \geq 0$ so

$$d > g - 1 + \lfloor \frac{n}{2} \rfloor \ .$$

Condition (ii) implies α_1 is the only integer permitted to vanish. The relation $I(g, d, \alpha)$ will be a variant of the equations considered in [7].

8.2.2. *Formulas.* Let Γ denote the data type

$$(p_1, \ldots, p_m) \cup \{p_{m+1}, \ldots, p_\ell\}, \tag{28}$$

satisfying

$$p_i > 0, \quad \sum_{i=1}^{\ell} p_i = d.$$

The first part of Γ is an ordered m-tuple $(p_1, \ldots, p_m)$. The second part $\{p_{m+1}, \ldots, p_\ell\}$ is an unordered set. Let $\mathrm{Aut}(\{p_{m+1}, \ldots, p_\ell\})$ be the group which permutes equal parts. The group of automorphisms $\mathrm{Aut}(\Gamma)$ equals $\mathrm{Aut}(\{p_{m+1}, \ldots, p_\ell\})$.

Theorem 7. *For all α satisfying (i-ii),*

$$\sum_{\Gamma} \frac{1}{|\mathrm{Aut}(\Gamma)|} \prod_{i=1}^{m} p_i^{-\alpha_i} \prod_{i=m+1}^{\ell} (-p_i)^{-1} \prod_{j=1}^{\ell} \frac{p_i^{p_i}}{p_i!} \ \ \langle p_1, \ldots, p_\ell \rangle$$
$$= 0 \ \in A^d(M_{g,n}^c),$$

where the sum is over all Γ of type (28).

The bracket $\langle p_1, \ldots, p_\ell \rangle \in A^d(M_{g,n}^c)$ denotes the class associated to the partition defined by the union of all the parts p_i of Γ.

8.3. **Proof of Theorem 7.**

8.3.1. *Torus actions.* The first step is to define the appropriate torus actions. Let

$$\mathbb{P}^1 = \mathbb{P}(V)$$

where $V = \mathbb{C} \oplus \mathbb{C}$. Let $\mathbb{C}^*$ act diagonally on V:

$$\xi \cdot (v_1, v_2) = (v_1, \xi \cdot v_2). \tag{29}$$

Let $\mathsf{p}_1, \mathsf{p}_2$ be the fixed points $[1, 0], [0, 1]$ of the corresponding action on $\mathbb{P}(V)$. An equivariant lifting of $\mathbb{C}^*$ to a line bundle L over $\mathbb{P}(V)$ is uniquely determined by the weights $[l_1, l_2]$ of the fiber representations at the fixed points

$$L_1 = L|_{\mathsf{p}_1}, \quad L_2 = L|_{\mathsf{p}_2}.$$

The canonical lifting of $\mathbb{C}^*$ to the tangent bundle $T_{\mathbb{P}^1}$ has weights $[1, -1]$. We will utilize the equivariant liftings of $\mathbb{C}^*$ to $\mathcal{O}_{\mathbb{P}(V)}(1)$ and $\mathcal{O}_{\mathbb{P}(V)}(-1)$ with weights $[1, 0]$, $[0, 1]$ respectively.

Over the moduli space of stable maps $\overline{M}_{g,n+m}(\mathbb{P}(V), d)$, we have

$$\pi : U \to \overline{M}_{g,n+m}(\mathbb{P}(V), d), \quad \mu : U \to \mathbb{P}(V)$$

where U is the universal curve and μ is the universal map. The representation (29) canonically induces $\mathbb{C}^*$-actions on U and $\overline{M}_{g,n+m}(\mathbb{P}(V), d)$ compatible with the maps π and μ. The $\mathbb{C}^*$-equivariant virtual class

$$[\overline{M}_{g,n+m}(\mathbb{P}(V), d)]^{vir} \in A^{\mathbb{C}^*}_{2g+2d-2+n+m}(\overline{M}_{g,n+m}(\mathbb{P}(V), d))$$

will play an important role.

8.3.2. *Equivariant classes.* Three types of equivariant Chow classes on $\overline{M}_{g,n+m}(\mathbb{P}(V), d)$ will be considered here:

- The linearization $[0, 1]$ on $\mathcal{O}_{\mathbb{P}(V)}(-1)$ defines an $\mathbb{C}^*$-action on the rank $d + g - 1$ bundle
$$\mathbb{R} = R^1\pi_*(\mu^*\mathcal{O}_{\mathbb{P}(V)}(-1))$$
on $\overline{M}_{g,n+m}(\mathbb{P}(V), d)$. Let
$$c_{top}(\mathbb{R}) \in A^{g+d-1}_{\mathbb{C}^*}(\overline{M}_{g,n+m}(\mathbb{P}(V), d))$$
be the top Chern class.
- For each marking i, let $\psi_i \in A^1_{\mathbb{C}^*}(\overline{M}_{g,n+m}(\mathbb{P}(V), d)$ denote the first Chern class of the canonically linearized cotangent line corresponding to i.
- Denote the i^{th} evaluation morphism by
$$\text{ev}_i : \overline{M}_{g,n+m}(\mathbb{P}(V), d) \to \mathbb{P}(V).$$
With $\mathbb{C}^*$-linearization $[1, 0]$ on $\mathcal{O}_{\mathbb{P}(V)}(1)$, let
$$\rho_i = c_1(\text{ev}_i^*\mathcal{O}_{\mathbb{P}(V)}(1)) \in A^1_{\mathbb{C}^*}(\overline{M}_{g,n+m}(\mathbb{P}(V), d) \ .$$
With $\mathbb{C}^*$-linearization $[0, -1]$ on $\mathcal{O}_{\mathbb{P}(V)}(1)$, let
$$\widetilde{\rho}_i = c_1(\text{ev}_i^*\mathcal{O}_{\mathbb{P}(V)}(1)) \in A^1_{\mathbb{C}^*}(\overline{M}_{g,n+m}(\mathbb{P}(V), d) \ .$$

In the non-equivariant limit, $\rho_i^2 = 0$. Our notation here closely follows [7].

8.3.3. *Vanishing integrals.* The forgetful morphism

$$\nu : \overline{M}_{g,n+m}(\mathbb{P}(V), d) \to \overline{M}_{g,n}$$

is $\mathbb{C}^*$-equivariant with respect to the trivial action on $\overline{M}_{g,n}$. As in Section 8.2.1, let

$$d \leq 2g-3+n, \quad \delta = 2g-3+n-d,$$

and let $\alpha = (\alpha_1, \ldots, \alpha_m)$ satisfy

(i) $|\alpha| = \sum_{i=1}^m \alpha_i \leq d-2-\delta$,
(ii) $\alpha_i > 0$ for $i > 1$.

Let $I(g,d,\alpha)$ be the $\mathbb{C}^*$-equivariant push-forward

$$\nu_* \left(\rho_{n+1}^{d-1-\delta-|\alpha|} \prod_{i=1}^m \rho_{n+i} \psi_{n+i}^{\alpha_i} \prod_{j=1}^n \widetilde{\rho}_j \ c_{top}(\mathbb{R}) \ \cap \ [\overline{M}_{g,n+m}(\mathbb{P}(V), d)]^{vir} \right) .$$

The degree of the class

$$\rho_{n+1}^{d-1-\delta-|\alpha|} \prod_{i=1}^m \rho_{n+i} \psi_{n+i}^{\alpha_i} \prod_{j=1}^n \widetilde{\rho}_j \ c_{top}(\mathbb{R})$$

is easily computed to be

$$d-1-\delta-|\alpha|+m+|\alpha|+n+d+g-1 = \\ g+2d-2+n+m-\delta \ .$$

Since the cycle dimension of the virtual class is $2g+2d-2+n+m$, the push-forward $I(g,d,\alpha)$ has cycle dimension

$$\begin{aligned} 2g+2d-2+n+m-(g+2d-2+n+m-\delta) &= g+\delta \\ &= 3g-3+n-d \ . \end{aligned}$$

Equivalently, $I(g,d,\alpha) \in A^d_{\mathbb{C}^*}(\overline{M}_{g,n})$. Since the class ρ_{n+1} appears with exponent

$$d-\delta-|\alpha| \geq 2,$$

$I(g,d,\alpha)$ vanishes in the non-equivariant limit.

8.3.4. *Localization terms.* The virtual localization formula of [11] calculates $I(g,d,\alpha)$ in terms of tautological classes on the moduli space $\overline{M}_{g,n}$. To prove Theorem 7, we will calculate the restriction of the localization formula to $M^c_{g,n}$.

The localization formula expresses $I(g,d,\alpha)$ as a sum over connected decorated graphs Γ indexing the $\mathbb{C}^*$-fixed loci of $\overline{M}_{g,n+m}(\mathbb{P}(V),d)$. The vertices of the graphs lie over the fixed points $\mathsf{p}_1, \mathsf{p}_2 \in \mathbb{P}(V)$ and are labelled with genera (which sum over the graph to $g - h^1(\Gamma)$). The edges of the graphs lie over $\mathbb{P}^1$ and are labelled with degrees (which sum over the graph to d). Finally, the graphs carry $n+m$ markings on the vertices. The valence $\mathrm{val}(v)$ of a vertex $v \in \Gamma$ counts both the incident edges and markings. The edge valence of v counts only the incident edges.

Only a very restricted subset of graphs will yield non-vanishing contributions to $I(g,d,\alpha)$ in the non-equivariant limit. If a graph Γ contains a vertex lying over p_1 of edge valence greater than 1, then the contribution of Γ to $I(g,d,\alpha)$ vanishes by our choice of linearization on the bundle $\mathbb{R}$. A vertex over p_1 of edge valence greater than 1 yields a trivial Chern root of $\mathbb{R}$ (with trivial weight 0) in the numerator of the localization formula to force the vanishing.

By the above vanishing, only *comb* graphs Γ contribute to $I(g,d,\alpha)$. Comb graphs contain $\ell \leq d$ vertices lying over p_1 each connected by a distinct edge to a unique vertex lying over p_2.

If Γ contains a vertex over p_1 of positive genus, then the restriction to $M^c_{g,n}$ of the contribution of Γ to $I(g,d,\alpha)$ vanishes by the following argument. Let v be a genus $g(v) > 0$ vertex lying over p_1. The integrand term $c_{top}(\mathbb{R})$ yields a factor $c_{g(v)}(\mathbb{E}^*)$ with trivial $\mathbb{C}^*$-weight on the genus $g(v)$ moduli space corresponding to the vertex v. Since

$$\lambda_{g(v)}|_{M^c_{g(v),\mathrm{val}(v)}} = 0$$

by [24], the required vanishing holds.

The linearizations of the classes ρ_i and $\widetilde{\rho}_j$ place restrictions on the marking distribution. Since the class $\widetilde{\rho}_j$ is obtained from $\mathcal{O}_{\mathbb{P}(V)}(1)$ with linearization $[0,-1]$, the first n markings must lie on the unique vertex over over p_2. Since the class ρ_i is obtained from $\mathcal{O}_{\mathbb{P}(V)}(1)$ with linearization $[1,0]$, the last m markings must lie on vertices over p_1.

Finally, we claim the last m markings of Γ must lie on distinct vertices over p_1 for nonvanishing contribution to $I(g,d,\alpha)$. Let v be a vertex over p_1 (with $g(v)=0$). If v carries at least two markings, the fixed locus corresponding to Γ contains a product factor $\overline{M}_{0,r+1}$ where r is the number of markings incident to v. The classes $\psi_{n+i}^{\alpha_i}$ carry trivial $\mathbb{C}^*$-weight. Moreover, as each $\alpha_i > 0$ for $i > 1$, we see the sum of the α_i as i ranges over the set of markings incident to v is at least $r-1$. Since the sum exceeds the dimension of $\overline{M}_{0,r+1}$, the graph contribution to $I(g,d,\alpha)$ vanishes.

The proof of the main result about the localization terms for $I(g,d,\alpha)$ is now complete.

Proposition 7. *The restriction of $I(g,d,\alpha)$ to $M^c_{g,n}$ is expressed via the virtual localization formula as a sum over genus g, degree d, marked comb graphs Γ satisfying:*

(i) *all vertices over p_1 are of genus 0,*
(ii) *the unique vertex over p_2 carries all of the first n markings,*
(iii) *the last m markings all lie over p_1,*
(iv) *each vertex over p_1 carries at most 1 of the last m markings.*

8.3.5. *Formulas.* The precise contributions of allowable graphs Γ to the non-equivariant limit of $I(g,d,\alpha)$ are now calculated.

Let Γ be a genus g, degree d, comb graph with $n+m$ markings satisfying conditions (i-iv) of Proposition 7. By condition (iv), Γ must have $\ell \geq m$ edges. Γ may be described uniquely by the data

$$(p_1,\ldots,p_m) \cup \{p_{m+1},\ldots,p_\ell\}, \tag{30}$$

satisfying:

$$p_i > 0, \quad \sum_{i=1}^{\ell} p_i = d.$$

The elements of the ordered m-tuple $(p_1,\ldots,p_m)$ correspond to the degree assignments of the edges incident to the vertices marked by the last m markings. The elements of the unordered partition $\{p_{m+1},\ldots,p_\ell\}$ correspond to the degrees of edges incident to the unmarked vertices over p_1. The group of graph automorphisms is

$$\mathrm{Aut}(\Gamma) = \mathrm{Aut}(\{p_{m+1},\ldots,p_\ell\})\ .$$

By a direct application of the virtual localization formula of [11], we find the contribution of the graph (30) to the normalized[16] push-forward

$$(-1)^{g+1+|\alpha|+n+m} \cdot I(g,d,\alpha)$$

equals

$$\frac{1}{|\mathrm{Aut}(\Gamma)|} \prod_{i=1}^{m} p_i^{-\alpha_i} \prod_{i=m+1}^{\ell} (-p_i)^{-1} \prod_{i=1}^{\ell} \frac{p_i^{p_i}}{p_i!} \ \ \langle p_1,\ldots,p_\ell\rangle \ .$$

Hence, the vanishing of $I(g,d,\alpha)$ yields the relation

$$\sum_{\Gamma} \frac{1}{|\mathrm{Aut}(\Gamma)|} \prod_{i=1}^{m} p_i^{-\alpha_i} \prod_{i=m+1}^{\ell} (-p_i)^{-1} \prod_{i=1}^{\ell} \frac{p_i^{p_i}}{p_i!} \ \ \langle p_1,\ldots,p_\ell\rangle = 0 \ ,$$

where the sum is over all graphs (30). □

9. Proof of Theorem 4

9.1. **Matrix of relations.** Theorem 7 yields relations in $\kappa^d(M^c_{g,n})$, indexed by $\alpha=(\alpha_1,\ldots,\alpha_m)$ satisfying conditions (i-ii) of Section 8.2.1 with

$$\delta = 2g-3+n-d \geq 0.$$

We rewrite the relation obtained from the vanishing of $I(g,d,\alpha)$ as

$$\sum_{\mathbf{p}\in P(d)} \mathsf{C}^{\mathbf{p}}_{\alpha}\ \langle \mathbf{p}\rangle = 0 \ . \tag{31}$$

The coefficients are

$$\mathsf{C}^{\mathbf{p}}_{\alpha} = \frac{1}{|\mathrm{Aut}(\mathbf{p})|} \prod_{i=1}^{\ell} \frac{p_i^{p_i}}{p_i!} \ \sum_{\phi} \ \prod_{i=1}^{m} p_{\phi(i)}^{-\alpha_i} \prod_{j\in \mathrm{Im}(\phi)^c} (-p_j)^{-1} \ ,$$

where the sum is over all injections

$$\phi : \{1,\ldots,m\} \to \{1,\ldots,\ell\}$$

and

$$\mathrm{Im}(\phi)^c \subset \{1,\ldots,\ell\}$$

is the complement of the image of ϕ.

To prove Proposition 6, we will show the system (31) is of rank at least $|P(d)|-|P(d,\delta+1)|$. The claim is empty unless $0\leq \delta\leq d-2$.

[16]The parallel equation on page 106 of [7] has a sign error in the normalization. Instead of $(-1)^{g+1}I(g,d,\alpha)$ there, the normalization should be $(-1)^{g+1+|\alpha|+\ell(\alpha)}I(g,d,\alpha)$. The sign change makes no difference.

9.2. **Ordering.** For $0 \leq \delta \leq d-2$, define the subset $P_\delta(d) \subset P(d)$ by removing partitions of length at most $\delta+1$,

$$P_\delta(d) = P(d) \setminus P(d, \delta+1) \ .$$

We order $P_\delta(d)$ by the following rules

- longer partitions appear before shorter partitions,
- for partitions of the same length, we use the lexicographic ordering with larger parts[17] appearing before smaller parts.

For example, the ordered list of the 10 elements of $P_0(6)$ is

$$(1^6),\ (2,1^4),\ (3,1^3),\ (2^2,1^2),\ (4,1^2),\ (3,2,1),\ (2^3),\ (5,1),\ (4,2),\ (3,3) \ .$$

Given a partition $\mathbf{p} \in P(d)$, let $\widehat{\mathbf{p}}$ be the partition obtained removing all parts equal to 1. For example,

$$\widehat{(1^6)} = \emptyset, \ \ \widehat{(3,2,1)} = (3,2) \ .$$

Let $\mathbf{p}^-$ be the partition obtained by lowering all the parts of $\mathbf{p}$ by 1,

$$(1^6)^- = \emptyset, \ \ (3,2,1)^- = (2,1) \ .$$

If $\mathbf{p}$ has length ℓ, then

$$\mathbf{p}^- \in P(d-\ell).$$

To each partition $\mathbf{p} \in P_\delta(d)$, we associate data $\alpha[\mathbf{p}]$ satisfying conditions (i)-(ii) with respect to δ by the following rules. The special designation

$$\alpha[(1^d)] = (0)$$

is given. Otherwise

$$\alpha[\mathbf{p}] = \mathbf{p}^- \ .$$

We note condition (i) of Section 8.2.1,

$$|\alpha[\mathbf{p}]| \leq d-2-\delta \ ,$$

is satisfied in all cases.

Let $\mathsf{M}_\delta(d)$ be the square matrix indexed by the ordered set $P_\delta(d)$ with elements

$$M_\delta(d)[\mathbf{p},\mathbf{q}] = \mathsf{C}^{\mathbf{q}}_{\alpha[\mathbf{p}]} \ .$$

The rank of the system (31) is at least

$$|P_\delta(d)| = |P(d)| - |P(d,\delta+1)|$$

by the following nonsingularity result proven in Sections 9.3 - 9.6 below.

[17] Remember the parts of $\mathbf{p} = (p_1, \ldots, p_\ell)$ are ordered by $p_1 \geq \ldots \geq p_\ell$.

Proposition 8. *For $0 \le \delta \le d-2$, the matrix $\mathsf{M}_\delta(d)$ is nonsingular.*

Proposition 8 implies Proposition 6 and thus Theorem 4. Moreover, Proposition 8 provides a new approach to [7].

9.3. **Scaling.** Let $\mathsf{X}_\delta(d)$ be the square matrix indexed by the ordered set $P_\delta(d)$ with elements

$$\begin{aligned} \mathsf{X}_\delta(d)[(1^d),\mathbf{q}] &= (-1)^{\ell(\mathbf{q})-1} d \\ \mathsf{X}_\delta(d)[\mathbf{p}\neq(1^d),\mathbf{q}] &= \sum_\phi (-1)^{\ell(\mathbf{q})-\ell(\widehat{\mathbf{p}})} \prod_{i=1}^{\ell(\widehat{\mathbf{p}})} q_{\phi(i)}^{-\widehat{p}_i+2}\ , \end{aligned}$$

where the sum is over all injections

$$\phi : \{1,\ldots,\ell(\widehat{\mathbf{p}})\} \to \{1,\ldots,\ell(\mathbf{q})\}\ .$$

For example, $\mathsf{X}_0(6)$ is

$$\left(\begin{array}{cccccccccc}
-6 & 6 & -6 & -6 & 6 & 6 & 6 & -6 & -6 & -6 \\
-6 & 5 & -4 & -4 & 3 & 3 & 3 & -2 & -2 & -2 \\
-6 & \frac{9}{2} & -\frac{10}{3} & -3 & \frac{9}{4} & \frac{11}{6} & \frac{3}{2} & -\frac{6}{5} & -\frac{3}{4} & -\frac{2}{3} \\
30 & -20 & 12 & 12 & -6 & -6 & -6 & 2 & 2 & 2 \\
-6 & \frac{17}{4} & -\frac{28}{9} & -\frac{5}{2} & \frac{33}{16} & \frac{49}{36} & \frac{3}{4} & -\frac{26}{25} & -\frac{5}{16} & -\frac{2}{9} \\
30 & -18 & 10 & 9 & -\frac{9}{2} & -\frac{11}{3} & -3 & \frac{6}{5} & \frac{3}{4} & \frac{2}{3} \\
-120 & 60 & -24 & -24 & 6 & 6 & 6 & 0 & 0 & 0 \\
-6 & \frac{33}{8} & -\frac{82}{27} & -\frac{9}{4} & \frac{129}{64} & \frac{251}{216} & \frac{3}{8} & -\frac{126}{125} & -\frac{9}{64} & -\frac{2}{27} \\
30 & -17 & \frac{28}{3} & \frac{15}{2} & -\frac{33}{8} & -\frac{49}{18} & -\frac{3}{2} & \frac{26}{25} & \frac{5}{16} & \frac{2}{9} \\
30 & -16 & 8 & \frac{13}{2} & -3 & -2 & -\frac{3}{2} & \frac{2}{5} & \frac{1}{4} & \frac{2}{9}
\end{array}\right).$$

The matrix $\mathsf{X}_\delta(d)$ is obtained from $\mathsf{M}_\delta(d)$ by dividing each column corresponding to $\mathbf{q}$ by

$$\frac{1}{|\mathrm{Aut}(\mathbf{q})|} \prod_{i=1}^{\ell(\mathbf{q})} \frac{q_i^{q_i-1}}{q_i!}\ .$$

Hence, $\mathsf{X}_\delta(d)$ is nonsingular if and only if $\mathsf{M}_\delta(d)$ is nonsingular.

9.4. **Elimination.** Our strategy for proving Proposition 8 is to find an upper-triangular square matrix $\mathsf{Y}_0(d)$ for which the product

$$\mathsf{X}_0(d) \cdot \mathsf{Y}_0(d) \tag{32}$$

is lower-triangular with ± 1's on the diagonal. Since $\mathsf{X}_\delta(d)$ for

$$0 \leq \delta \leq d-2$$

occurs as an upper left minor of $\mathsf{X}_0(d)$, the lower-triangularity of the product (32) will establish Proposition 8 for the full range of δ values.

We define $\mathsf{Y}_0(d)$ to be the square matrix indexed by the ordered set $P_0(d)$ given by the following rules. The upper left corner is

$$\mathsf{Y}_0(d)[(1^d),(1^d)] = \frac{1}{d}$$

If at least one of $\{\mathbf{p},\mathbf{q}\}$ is not equal to (1^d), then the matrix elements are

$$\mathsf{Y}_0(d)[\mathbf{p},\mathbf{q}] = \frac{1}{|\mathrm{Aut}(\mathbf{p})|}\frac{1}{|\mathrm{Aut}(\widehat{\mathbf{q}})|}\sum_\theta \prod_{i=1}^{\ell(\mathbf{q})} \binom{q_i}{p_{i[1]},\ldots,p_{i[\ell_i]}} q_i^{\ell_i-2} \prod_{j=1}^{\ell_i} p_{ij}^{p_{ij}-1} \ ,$$

where the sum is over all functions

$$\theta : \{1,\ldots,\ell(\mathbf{p})\} \to \{1,\ldots,\ell(\mathbf{q})\}$$

with

$$\theta^{-1}(i) = \{i[1],\ldots,i[\ell_i]\}$$

satisfying

$$q_i = \sum_{j=1}^{\ell_i} p_{i[j]} \ .$$

For example, $\mathsf{Y}_0(6)$ is

$$\begin{pmatrix} \frac{1}{6} & 1 & 3 & \frac{1}{2} & 16 & 3 & \frac{1}{6} & 125 & 16 & \frac{9}{2} \\ 0 & 1 & 6 & 1 & 48 & 9 & \frac{1}{2} & 500 & 64 & 18 \\ 0 & 0 & 3 & 0 & 36 & 3 & 0 & 450 & 36 & 9 \\ 0 & 0 & 0 & \frac{1}{2} & 12 & 6 & \frac{1}{2} & 300 & 60 & 18 \\ 0 & 0 & 0 & 0 & 16 & 0 & 0 & 320 & 16 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 & 180 & 36 & 18 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & 0 & 12 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 125 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 16 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{9}{2} \end{pmatrix}.$$

By the conditions on θ in the definition, $\mathsf{Y}_0(d)$ is easily seen to be upper-triangular.

9.5. **Generating functions.** Let $\mathbb{Q}[t]$ denote the polynomial ring in infinitely many variables

$$t = \{t_1, t_2, t_3, \ldots\} \ .$$

Define a $\mathbb{Q}$-linear function

$$\langle\ \rangle : \mathbb{Q}[t] \to \mathbb{Q}$$

by the equations $\langle 1 \rangle = 1$ and

$$\langle t_{d_1} t_{d_2} \cdots t_{d_k} \rangle = (d_1 + d_2 + \ldots + d_k)^{k-3} \ .$$

We may extend $\langle\ \rangle$ uniquely to define a x-linear function:

$$\langle\ \rangle : \mathbb{Q}[t][[x]] \to \mathbb{Q}[[x]].$$

For each non-negative integer i, let

$$Z_i(t,x) = \sum_{j>0} x^j t_j \frac{j^{j-i}}{j!} \in \mathbb{Q}[t][[x]].$$

Applying the bracket, we define

$$\mathsf{F}_{\alpha_1,\ldots,\alpha_m} = \langle \exp(-Z_1) \cdot Z_{\alpha_1} \cdots Z_{\alpha_m} \rangle \ \in \mathbb{Q}[[x]].$$

Lemma 13. *Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a non-empty sequence of non-negative integers satisfying $\alpha_i > 0$ for $i > 1$. The series*

$$\mathsf{F}_{\alpha_1,\ldots,\alpha_m} \in \mathbb{Q}[[x]]$$

is a polynomial *of degree at most $1 + \sum_{i=1}^m \alpha_i$ in x.*

Lemma 14. *Let $\alpha_1 \geq 0$. Then,*

$$\mathsf{F}_{\alpha_1} = \frac{(-1)^{\alpha_1}}{(1+\alpha_1)(1+\alpha_1)!} x^{1+\alpha_1} + \ldots$$

where the dots stand for lower order terms.

Lemma 13 can be proven by various methods. A proof via localization on moduli space is given in [7] in Section 1.7. □

Lemma 14 is more interesting. The integral

$$J_{1+\alpha_1} = \int_{\overline{M}_{0,1}(\mathbb{P}^1, 1+\alpha_1)} \rho_1 \psi_1^{\alpha_1} \; c_{top}(\mathbb{R}) \tag{33}$$

can be evaluated by exactly following[18] the localization analysis of Section 8.3. We find

$$J_{1+\alpha_1} = (-1)^{\alpha_1} \sum_{\Gamma} \frac{1}{|\mathrm{Aut}(\Gamma)|} \; p_1^{-\alpha_1} \prod_{i=2}^{\ell} (-p_i)^{-1} \prod_{i=1}^{\ell} \frac{p_i^{p_i}}{p_i!} \;\; (1+\alpha_1)^{\ell-3}$$

where the sum is over all 1-pointed comb graphs (30) of total degree $1+\alpha_1$. We conclude $J_{1+\alpha_1}$ equals, up to the factor of $(-1)^{\alpha_1}$, the leading $x^{1+\alpha_1}$ coefficient of $\langle \exp(-Z_1) \cdot Z_{\alpha_1} \rangle$.

To calculate the integral (33), we use well-known equations in Gromov-Witten theory. Certainly

$$J_1 = 1 \; . \tag{34}$$

By two applications of the divisor equation,

$$k^2 J_k = \int_{\overline{M}_{0,3}(\mathbb{P}^1, k)} \rho_1 \psi_1^{k-1} \rho_2 \rho_3 \; c_{top}(\mathbb{R})$$

By the topological recursion relation [2] applied to the right side,

$$k^2 J_k = \int_{\overline{M}_{0,2}(\mathbb{P}^1, k-1)} \rho_1 \psi_1^{k-2} \rho_2 \; c_{top}(\mathbb{R}) \;\; \cdot \;\; \int_{\overline{M}_{0,3}(\mathbb{P}^1, 1)} \rho_1 \rho_2 \rho_3 \; c_{top}(\mathbb{R}) \; .$$

We obtain the recursion

$$\begin{aligned} k^2 J_k &= (k-1) J_{k-1} J_1 \\ &= (k-1) J_{k-1} \end{aligned}$$

which we can easily solve

$$J_k = \frac{1}{k \cdot k!}$$

[18]The equivariant lifts are taken just as in Section 8.3.2.

starting with the initial condition (34). □

The case where the α data is empty will arise naturally. We define

$$\mathsf{F}_\emptyset = \langle \exp(-Z_1) \rangle.$$

The following result is derived from Lemma 13 by the relation

$$x \frac{d}{dx} \mathsf{F}_\emptyset = -\mathsf{F}_0.$$

Lemma 15. $\mathsf{F}_\emptyset = 1 - x$.

9.6. **Product.** We will now prove the basic identity

$$\mathsf{X}_0(d) \cdot \mathsf{Y}_0(d) = \mathsf{L}_0(d) \tag{35}$$

where $\mathsf{L}_0(d)$ is lower triangular with diagonal entries all ± 1.

We first address the special upper left corner. The product on the left side of (35) is

$$\mathsf{L}_0(d)[(1^d),(1^d)] = (-1)^{d-1} d \cdot \frac{1}{d} = (-1)^{d-1} \ ,$$

a diagonal entry of the specified form.

Next assume $\mathbf{p} \neq (1^d)$. Then, the matrix elements are

$$\mathsf{L}_0(d)[\mathbf{p},\mathbf{q}] = \frac{1}{|\mathrm{Aut}(\widehat{\mathbf{q}})|} \sum_\gamma \prod_{i=1}^{\ell(\mathbf{q})} \mathrm{Coeff}(F_{\gamma^{-1}(i)}, x^{q_i}) \ q_i \ q_i! \ , \tag{36}$$

where the sum is over all functions

$$\gamma : \{1, \ldots, \ell(\widehat{\mathbf{p}})\} \to \{1, \ldots, \ell(\mathbf{q})\} \ .$$

In case $\gamma^{-1}(i) = \{i[1], \ldots, i[\ell_i]\}$ is nonempty, we define

$$\mathsf{F}_{\gamma^{-1}(i)} = \mathsf{F}_{\widehat{p}_{i[1]}-1, \ldots, \widehat{p}_{i[\ell_i]}-1} \ .$$

If $\gamma^{-1}(i) = \emptyset$, then

$$\mathsf{F}_\emptyset = \langle \exp(-Z_1) \rangle = 1 - x.$$

Equation (36) is obtained from a simple unravelling of the definitions.

If $q_i > 1$, $\mathrm{Coeff}(F_{\gamma^{-1}(i)}, x^{q_i})$ vanishes unless $\gamma^{-1}(i)$ is nonempty by Lemma 15 and unless

$$q_i \leq 1 - \ell_i + \sum_{j=1}^{\ell_i} \widehat{p}_{i[j]} \tag{37}$$

by Lemma 13. Inequality (37) for all parts $q_i > 1$ implies

$$\ell(\mathbf{q}) \geq \ell(\mathbf{p}) \ .$$

Moreover, if equality of length holds, then inequality (37) implies either $\mathbf{q}$ precedes $\mathbf{p}$ in the ordering of $P_0(d)$ or $\mathbf{q} = \mathbf{p}$.

We conclude the matrix $\mathsf{L}_0(d)$ is lower-triangular when the first coordinate $\mathbf{p}$ is not (1^d). The diagonal elements for $\mathbf{p} \neq (1^d)$ are

$$\mathsf{L}_0(d)[\mathbf{p}, \mathbf{p}] = \prod_{i=1}^{\ell(\widehat{\mathbf{p}})} (-1)^{\widehat{p}_i - 1} \cdot (-1)^{\ell(\mathbf{p}) - \ell(\widehat{\mathbf{p}})}$$

by Lemmas 14 and 15.

To complete the proof of the lower-triangularity of $\mathsf{L}_0(d)$, we must show the vanishing of $\mathsf{L}_0(d)[(1^d), \mathbf{q} \neq (1^d)]$. The matrix elements are

$$\mathsf{L}_0(d)[(1^d), \mathbf{q} \neq (1^d)] = \frac{1}{|\mathrm{Aut}(\widehat{\mathbf{q}})|} \sum_{\tilde{\gamma}} \prod_{i=1}^{\ell(\mathbf{q})} \mathrm{Coeff}(\widetilde{F}_{\tilde{\gamma}^{-1}(i)}, x^{q_i})\ q_i\ q_i!\ ,$$

where the sum is over all functions

$$\tilde{\gamma} : \{1\} \to \{1, \ldots, \ell(\mathbf{q})\}\ .$$

In case $\tilde{\gamma}^{-1}(i) = \{1\}$ is nonempty, we define

$$\widetilde{\mathsf{F}}_{\tilde{\gamma}^{-1}(i)} = \mathsf{F}_0\ .$$

If $\tilde{\gamma}^{-1}(i) = \emptyset$, then

$$\widetilde{\mathsf{F}}_{\emptyset} = \langle \exp(-Z_1) \rangle = 1 - x.$$

Let $q_1 > 1$ be the largest part of $\mathbf{q}$. Then

$$\mathrm{Coeff}(\widetilde{F}_{\tilde{\gamma}^{-1}(1)}, x^{q_1}) = 0$$

by Lemmas 13 and 15. Hence,

$$\mathsf{L}_0(d)[(1^d), \mathbf{q} \neq (1^d)] = 0,$$

and the lower-triangularity of $\mathsf{L}_0(d)$ is fully proven.

The proof of Proposition 8 is complete. Following the implications back, the proof of Theorem 4 is also complete. □

Since we know explicitly the diagonal elements of the triangular matrices $\mathsf{Y}_0(d)$ and $\mathsf{L}_0(d)$, the product

$$\mathsf{X}_0(d) \cdot \mathsf{Y}_0(d) = \mathsf{L}_0(d)$$

yields a simple formula for the determinant,

$$\det(\mathsf{X}_{0,d}) = (-1)^{d-1} \prod_{\mathbf{p} \in P_0(d) \setminus \{(1^d)\}} \left(\frac{|\mathrm{Aut}(\widehat{\mathbf{p}})|}{\prod_{i=1}^{\ell(\mathbf{p})} p_i^{p_i - 2}}\ (-1)^{\ell(\mathbf{p})} \prod_{i=1}^{\ell(\widehat{\mathbf{p}})} (-1)^{\widehat{p}_i} \right)\ .$$

10. Proof of Theorem 6

10.1. Bound. By Theorem 4, we have a surjection

$$\kappa^d(M^c_{0,2g+n}) \stackrel{\iota_{g,n}}{\to} \kappa^d(M^c_{g,n}) \to 0.$$

By Theorem 5, to prove $\iota_{g,n}$ is an isomorphism, we need only establish

$$\dim_{\mathbb{Q}} \kappa^d(M^c_{g,n}) \geq |P(d, 2g-2+n-d)|$$

for $n > 0$. We will obtain the bound by refining the argument for Theorem 2.

10.2. Dual graph types. A dual graph of type $A(g_1, \ldots, g_r)$ with $g_i \geq 1$ is a chain of r vertices of genera $g_1, \ldots, g_r$ with 2 markings on the ends. The corresponding curves are of the form:

$$C^*_{g_1} - C_{g_2} - \ldots - C^*_{g_r} \ .$$

If $r = 1$, the unique vertex carries both markings.

A dual graph of type $B(g_1, \ldots, g_r | h_1, \ldots, h_{r-1})$ with $g_i, h_j \geq 1$ is comb of $2r-1$ vertices with 1 marking. The corresponding curves are of the form:

$$\begin{array}{ccccccccc} C^*_{g_1} & - & C_{g_2} & - & \ldots & - & C_{g_{r-1}} & - & C_{g_r} \\ | & & | & & & & | & & \\ C_{h_1} & & C_{h_2} & & \ldots & & C_{h_{r-1}} \ . & & \end{array}$$

There are $r-1$ vertices of valence 3 and r vertices of valence 1. The markings are included in the valence count.

10.3. Case $n = 1$. Let $\mathbf{p} \in P(d)$ be a partition of length $\ell = a + b$ with parts[19]

$$(p_1, \ldots, p_a, p'_1, \ldots, p'_b),$$

where the p_i are odd and the p'_j are even. We see

$$d + \ell = b \mod 2 \ .$$

[19]All parts of $\mathbf{p}$ here are positive.

If $d+\ell$ is odd, then $b = 2r-1$ for $r>0$. Let $\Gamma_{\mathbf{p}}$ be the dual graph obtained by the following construction:

$$\begin{array}{rcl} \Gamma_{\mathbf{p}} & = & A\left(\dfrac{p_1+1}{2}, \ldots, \dfrac{p_a+1}{2}\right) \\ & & \mid \\ & & B\left(\dfrac{p'_1}{2}, \ldots, \dfrac{p'_{r-1}}{2}, \dfrac{p'_r}{2}+1 \mid \dfrac{p'_{r+1}}{2}+1, \ldots, \dfrac{p'_{2r-1}}{2}+1\right) \ , \end{array}$$

where the graphs are attached at the first marking of A and the unique marking of B. The graph $\Gamma_{\mathbf{p}}$ has a unique marking (obtained from the second marking of A). The genus of $\Gamma_{\mathbf{p}}$ is easily calculated,

$$2g(\Gamma_{\mathbf{p}}) - 1 = d + a + 2r - 1 = d + \ell \ . \tag{38}$$

If $a = 0$, then $\Gamma_{\mathbf{p}}$ consists just of B, but the genus and marking results are the same.

The dual graph $\Gamma_{\mathbf{p}}$ determines a stratum in $M^c_{g(\Gamma_{\mathbf{p}}),1}$ which is a product of the moduli spaces,

$$\prod_{v \in \mathrm{Vert}(\Gamma_{\mathbf{p}})} M^c_{g(v),\mathrm{val}(v)} \ \to \ M^c_{g(\Gamma_{\mathbf{p}}),1} \ .$$

The socle dimensions of $M^c_{g(v),\mathrm{val} v}$ for $v \in \mathrm{Vert}(\Gamma_{\mathbf{p}})$ are exactly the parts of d.

If $d+\ell$ is even, then b must be even. If $b>0$, then

$$b = 2r - 1 + 1$$

for $r > 0$. Let

$$\begin{array}{rcl} \Gamma_{\mathbf{p}} & = & A\left(\dfrac{p_1+1}{2}, \ldots, \dfrac{p_a+1}{2}\right) - C^*_{\frac{p'_{2r}}{2}} - E \\ & & \mid \\ & & B\left(\dfrac{p'_1}{2}, \ldots, \dfrac{p'_{r-1}}{2}, \dfrac{p'_r}{2}+1 \mid \dfrac{p'_{r+1}}{2}+1, \ldots, \dfrac{p'_{2r-1}}{2}+1\right) \ . \end{array}$$

where the graphs A and B are attached at the markings. The graph $\Gamma_{\mathbf{p}}$ has a unique marking (on $C^*_{\frac{p'_{2r}}{2}}$) and an elliptic tail E. The genus of $\Gamma_{\mathbf{p}}$ is

$$2g(\Gamma_{\mathbf{p}}) - 1 = d + a + 2r + 2 - 1 = d + \ell + 1 \ . \tag{39}$$

If $a = 0$, then A is empty, but the genus and marking results are the same. The socle dimensions of $M^c_{g(v),\mathrm{val}v}$ for $v \in \mathrm{Vert}(\Gamma_{\mathbf{p}})$ are exactly the parts of d together with 0 for the elliptic tail.

If $d + \ell$ is even and $b = 0$, let

$$\Gamma_{\mathbf{p}} \;\; = \;\; A\left(\frac{p_1+1}{2}, \ldots, \frac{p_a+1}{2}\right) - E \ .$$

The graph $\Gamma_{\mathbf{p}}$ has a unique marking (obtained from the first marking of A) and ends in the elliptic tail E. The genus of $\Gamma_{\mathbf{p}}$ is

$$2g(\Gamma_{\mathbf{p}}) - 1 = d + a + 2 - 1 = d + \ell + 1 \ . \tag{40}$$

The socle dimensions of $M^c_{g(v),\mathrm{val}v}$ for $v \in \mathrm{Vert}(\Gamma_{\mathbf{p}})$ are exactly the parts of d together with 0 for the elliptic tail.

We now turn to the proof of Theorem 6 in the $n = 1$ case. We will prove

$$\dim_{\mathbb{Q}} \kappa^d(M^c_{g,1}) \geq |P(d, 2g-1-d)| \tag{41}$$

by intersecting κ monomials with tautological classes.

Let $\mathbf{p} \in P(d, 2g-1-d)$ be a partition of length ℓ. Let $\Gamma_{\mathbf{p}}$ be the dual graph of genus $g(\Gamma_{\mathbf{p}})$ obtained by the above constructions. Since

$$2g - 1 \geq d + \ell \ ,$$

equations (38)-(40) imply

$$g - g(\Gamma_{\mathbf{p}}) = \delta \geq 0 \ .$$

We associate to $\mathbf{p}$ a class $w_{\mathbf{p}} \in R^{2g-2-d}(M^c_{g,1})$ by the following construction. Let $v^* \in \mathrm{Vert}(\Gamma_{\mathbf{p}})$ be the vertex which carries the marking. Increase the genus of v^* by δ. The resulting graph determines a stratum

$$W_{\mathbf{p}} \subset M^c_{g,1}$$

of codimension $2g(\Gamma_{\mathbf{p}}) - 2 - d$. Let

$$w_{\mathbf{p}} = \psi_1^{2\delta} \cdot [W_{\mathbf{p}}] \in R^{2g-2-d}(M^c_{g,1}) \ .$$

The pairing on $P(d, 2g-1-d)$ given by

$$(\mathbf{p}, \mathbf{q}) \mapsto \int_{\overline{M}_{g,1}} \kappa_{\mathbf{p}} \cdot w_{\mathbf{q}} \tag{42}$$

is upper-triangular. The diagonal elements are nonvanishing because

$$\int_{\overline{M}_h} \kappa_{2h-3}\lambda_h = \frac{2^{2h-1}-1}{2^{2h-1}} \frac{|B_{2h}|}{(2h)!} \neq 0 \ ,$$

$$\int_{\overline{M}_{h,1}} \psi_1^k \kappa_{2h-2-k}\lambda_h = \binom{2h-1}{k} \int_{\overline{M}_h} \kappa_{2h-3}\lambda_h \neq 0$$

by [7]. Here, B_{2h} is the Bernoulli number. Hence, the pairing (42) is nonsingular and the bound (41) is established.

10.4. **Case** $n = 2$. We will need an additional dual graph type. A dual graph of type $\widetilde{B}(g_1, \ldots, g_r | h_1, \ldots, h_{r-1})$ with $g_i, h_j \geq 1$ is comb of $2r-1$ vertices with 3 markings. The corresponding curves are of the form:

$$\begin{array}{ccccccccc} C^*_{g_1} & - & C_{g_2} & - & \ldots & - & C_{g_{r-1}} & - & C^{**}_{g_r} \\ | & & | & & & & | & & \\ C_{h_1} & & C_{h_2} & & \ldots & & C_{h_{r-1}} \ . & & \end{array}$$

There are r vertices of valence 3 and $r-1$ vertices of valence 1. The marking is included in the valence count.

As before, let $\mathbf{p} \in P(d)$ be a partition of length $\ell = a + b$ with parts

$$(p_1, \ldots, p_a, p'_1, \ldots, p'_b),$$

where the p_i are odd and the p'_j are even.

If $d + \ell$ is even, then b must be even. If $b > 0$, then

$$b = 2r - 1 + 1$$

for $r > 0$. Let

$$\begin{array}{rcl} \widetilde{\Gamma}_{\mathbf{p}} & = & A\left(\frac{p_1+1}{2}, \ldots, \frac{p_a+1}{2}\right) - C_{\frac{p'_{2r}}{2}+1} \\ & & | \\ & & \widetilde{B}\left(\frac{p'_1}{2}, \ldots, \frac{p'_{r-1}}{2}, \frac{p'_r}{2} \mid \frac{p'_{r+1}}{2}+1, \ldots, \frac{p'_{2r-1}}{2}+1\right) \ . \end{array}$$

where the graphs A and $\widetilde{B}$ are attached at the initial markings. The graph $\widetilde{\Gamma}_{\mathbf{p}}$ has two markings (on the extremal component of $\widetilde{B}$). The genus of $\widetilde{\Gamma}_{\mathbf{p}}$ is

$$2g(\widetilde{\Gamma}_{\mathbf{p}}) = d + a + 2r = d + \ell \ . \tag{43}$$

If $a = 0$, then A is empty, but the genus and marking results are the same. The socle dimensions of $M^c_{g(v),\mathrm{val}v}$ for $v \in \mathrm{Vert}(\widetilde{\Gamma}_{\mathbf{p}})$ are exactly the parts of d.

If $d+\ell$ is even and $b=0$, let

$$\widetilde{\Gamma}_{\mathbf{p}} \quad = \quad A\left(\frac{p_1+1}{2},\ldots,\frac{p_a+1}{2}\right) \ .$$

The graph $\widetilde{\Gamma}_{\mathbf{p}}$ has two markings. The genus of $\widetilde{\Gamma}_{\mathbf{p}}$ is

$$2g(\Gamma_{\mathbf{p}}) = d+a = d+\ell \ . \tag{44}$$

The socle dimensions of $M^c_{g(v),\mathrm{val}v}$ for $v \in \mathrm{Vert}(\widetilde{\Gamma}_{\mathbf{p}})$ are exactly the parts of d.

If $d+\ell$ is odd, then $b=2r-1$ for $r>0$. Let

$$\begin{array}{rcl} \widetilde{\Gamma}_{\mathbf{p}} & = & A\left(\frac{p_1+1}{2},\ldots,\frac{p_a+1}{2}\right) - E \\ & & | \\ & & \widetilde{B}\left(\frac{p'_1}{2},\ldots,\frac{p'_{r-1}}{2},\frac{p'_r}{2} \mid \frac{p'_{r+1}}{2}+1,\ldots,\frac{p'_{2r-1}}{2}+1\right) \ , \end{array}$$

where the graphs A and $\widetilde{B}$ are attached at the initial markings. The graph $\widetilde{\Gamma}_{\mathbf{p}}$ has two markings (on the extremal component of $\widetilde{B}$). The genus of $\widetilde{\Gamma}_{\mathbf{p}}$ is

$$2g(\widetilde{\Gamma}_{\mathbf{p}}) = d+a+2(r-1)+2 = d+\ell+1 \ . \tag{45}$$

If $a=0$, then A is empty, but the genus and marking results are the same. The socle dimensions of $M^c_{g(v),\mathrm{val}v}$ for $v \in \mathrm{Vert}(\Gamma_{\mathbf{p}})$ are exactly the parts of d together with 0 for the elliptic tail.

The proof of Theorem 6 now follows the $n=1$ case. Let

$$\mathbf{p} \in P(d, 2g-d)$$

be a partition of length ℓ. Let $\widetilde{\Gamma}_{\mathbf{p}}$ be the dual graph of genus $g(\widetilde{\Gamma}_{\mathbf{p}})$ obtained by the above constructions. Since

$$2g \geq d+\ell \ ,$$

we see $g - g(\widetilde{\Gamma}_{\mathbf{p}}) = \delta \geq 0$.

We associate to $\mathbf{p}$ a class $\widetilde{w}_{\mathbf{p}} \in R^{2g-1-d}(M^c_{g,2})$ by the following construction. Let $v^* \in \mathrm{Vert}(\widetilde{\Gamma}_{\mathbf{p}})$ be the vertex which carries the first marking. Increase the genus of v^* by δ. The resulting graph determines a stratum

$$\widetilde{W}_{\mathbf{p}} \subset M^c_{g,2}$$

of codimension $2g(\widetilde{\Gamma}_{\mathbf{p}}) - 1 - d$. Let

$$\widetilde{w}_{\mathbf{p}} = \psi_1^{2\delta} \cdot [\widetilde{W}_{\mathbf{p}}] \in R^{2g-1-d}(M_{g,2}^c) \ .$$

The pairing on $P(d, 2g-d)$ given by

$$(\mathbf{p}, \mathbf{q}) \mapsto \int_{\overline{M}_{g,2}} \kappa_{\mathbf{p}} \cdot \widetilde{w}_{\mathbf{q}}$$

is upper-triangular and nonsingular as before. Hence,

$$\dim_{\mathbb{Q}} \kappa^d(M_{g,2}^c) \geq |P(d, 2g-d)|,$$

which is the required bound.

10.5. **Case $n \geq 3$.** The higher pointed cases are easily reduced to the 1 or 2 pointed cases depending upon the parity of n. The trading of genera for markings follows the proof of Theorem 2 in Section 5.4. We leave the details to the reader. □

11. Gorenstein conjecture

11.1. **Proof of Theorem 7.** If $n > 0$, the pairing

$$\kappa^d(M_{g,n}^c) \times R^{2g-3+n-d}(M_{g,n}^c) \to \mathbb{Q}$$

is shown to have rank at least $|P(d, 2g-2+n-d)|$ in Section 10. Since

$$\dim_{\mathbb{Q}} \kappa^d(M_{g,n}^c) = |P(d, 2g-2+n-d)|$$

by Theorem 6, Theorem 7 follows. □

11.2. **Further directions.** Perhaps the universality of Theorem 4 extends to larger subrings of $R^*(M_{g,n}^c)$. A natural place to start is the ring

$$S^*(M_{g,n}^c) \subset R^*(M_{g,n}^c)$$

generated by all the κ and ψ classes.

Question 4. *Is $S^*(M_{g,n}^c)$ canonically a subring of $S^*(M_{0,2g+n}^c)$?*

At least the condition $n > 0$ must be imposed in Question 4. How to include the strata classes in a universality statement is not clear.

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