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Algebraic Geometry of Moduli Spaces

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Consider first the moduli space

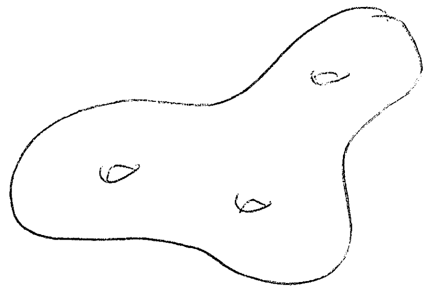
$\mathcal{M}_g$ of nonsingular complete

genus g curves $/ \mathbb{C}$

Such a curve can also be

viewed as complex Rieman surface

with g holes



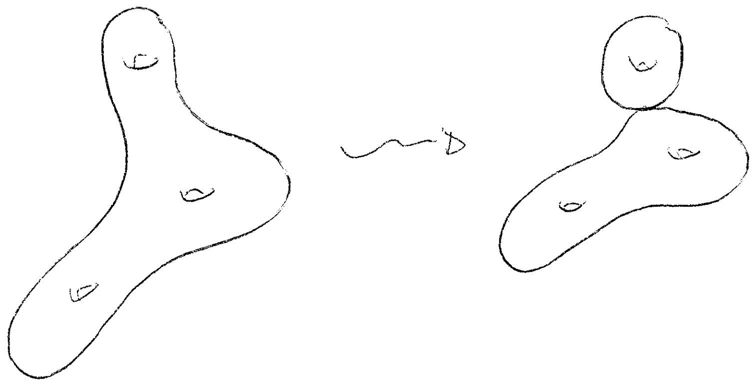
Already Rieman knew

$$\dim_{\mathbb{C}} \mathcal{M}_g = 3g-3 \qquad (g \geq 2)$$

Since 60's Deligne - Mumford

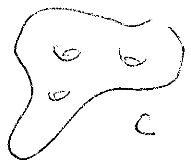
$\mathcal{M}_g \subset \overline{\mathcal{M}}_g$ compactification

by nodal curves



There is a basic rank g bundle on $\mathcal{M}_g$

$$\begin{array}{ccc} \mathbb{E}_g & \supset & H^0(C, \omega_C) \\ \downarrow & & \downarrow \\ \mathcal{M}_g & \ni & [\;C\;] \end{array}$$



$\leftarrow$ rank g space of holomorphic differentials

$$\begin{array}{c} \mathbb{E}_g \\ \downarrow \\ \overline{\mathcal{M}}_g \end{array}$$ extends

$$\lambda_1 = c_1(\mathbb{E}) \in H^2(\overline{\mathcal{M}}_g)$$

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Intrinsic questions about the structure of $H^*(\overline{\mathcal{M}}_g)$ guide much of what I will discuss today.

In the 90s, with Carel Faber, we developed methods to evaluate

$$\int_{\overline{\mathcal{M}}_g} p(\lambda_1, \lambda_2, \ldots, \lambda_g)$$

$$\lambda_i = c_i(\mathbb{E}_g)$$

Hodge integrals

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As an example

$$\int_{\overline{M}_g} \lambda_g \lambda_{g-1} \lambda_{g-2} = \frac{|B_{2g}|}{2g(2g-2)} \frac{|B_{2g-2}|}{(2g-2)!}$$

Harer - Zagier $|\chi_{top}(M_g)|$

[Faber - P 99]

Not a random integral.

Has geometric meaning in GW theory.

But today I want to

talk about a different

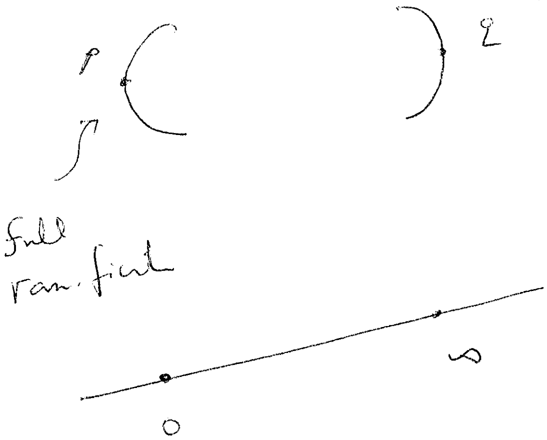
integral.

[Aside $\displaystyle\frac{t}{e^t - 1} = \sum_{m=0}^{\infty} B_m \frac{t^m}{m!}$]

Hurwitz Covers

$\mathcal{H}ur^{g}_{d} = \Big\{$ moduli space of coverings

$$\begin{array}{c} C_g \\ \text{degree } d \;\Big\downarrow f \\ \mathbb{P}^1 \end{array}$$



$\Big\}$

$$f^{-1}[0] = d\,[p]$$

$$f^{-1}[\infty] = d\,[q]$$

$$\dim_{\mathbb{C}} \mathcal{H}ur^{g}_{d} = 2g-1$$

$$\int_{\mathcal{H}ur^{g}_{d}} \lambda_g \lambda_{g-1}$$

arose in work with J. Bryan 03

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$$S_d = \sum_{g=1}^{\infty} u^{2g-1} \int_{\mathcal{H}ur_d^g} \lambda_g \lambda_{g-1}$$

$$= \frac{1}{2} \left(d \cot\left(\frac{du}{2}\right) - \cot\left(\frac{u}{2}\right) \right)$$

[Bryan-P]

Geometric argument uses many properties including relation between torsion points and 0 section of family of abelian varieties

$$-q = e^{iu}$$

$$S_d = \frac{1}{2} \left(d \, \frac{(-q)^d + 1}{(-q)^d - 1} - \frac{(-q) + 1}{(-q) - 1} \right)$$

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Next consider the simplest interesting Hilbert scheme.

$$\mathrm{Hilb}(\mathbb{C}^2,n)=\left\{ I\subset \mathbb{C}[x,y] \;\middle|\; \dim_{\mathbb{C}} \frac{\mathbb{C}[x,y]}{I}=n \right\}$$

given $I \subset \mathbb{C}[x,y]$

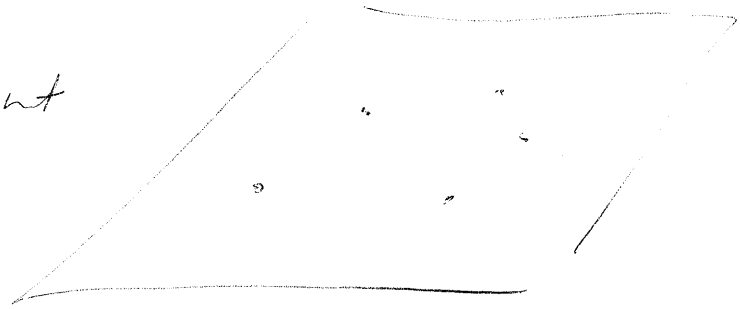
$\rightsquigarrow$ subscheme of $\mathbb{C}^2$

generic point of $\mathrm{Hilb}(\mathbb{C}^2,n)$

$\rightsquigarrow$

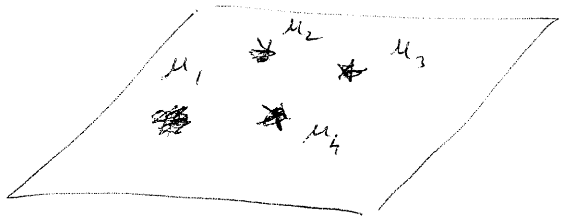
subscheme is

n distinct points



$\mathbb{C}^2$

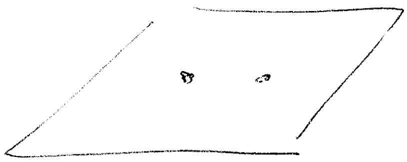
But the Hilbert Scheme also contains ideals which correspond to collisions



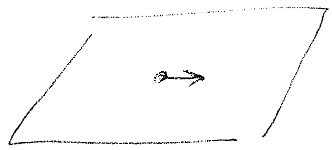
$\mu_1 + \mu_2 + \mu_3 + \mu_4 = n$

Shape

Most well-known of these



$t \to 0$



$$\mathbb{C}[x,y] \big/ (y,\ x^2 - t^2)$$

$$\mathbb{C}[x,y] \big/ (y,\ x^2)$$

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Let $\mu = (\mu_1, \dots, \mu_\ell)$

be a partition of n

$\left[\mu_i > 0 \,, \text{ unordered}, \sum_{i=1}^{\ell} \mu_i = n \right]$

We associate to μ

a cycle $\quad C_\mu \subset \mathrm{Hilb}(\mathbb{C}^2, n)$

$$C_\mu = \overline{\left\{ I \;\middle|\; \begin{array}{c} \mathbb{C}[x,y]/I \text{ has} \\ \text{shape } \mu \end{array} \right\}}$$

Example

$$C_{\underbrace{1,1,\dots,1}_{n}} = \mathrm{Hilb}(\mathbb{C}^2, n)$$