Algebraic Curves, Hurwitz Numbers, and Meromorphic Differentials

UIC Colloquium

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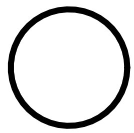
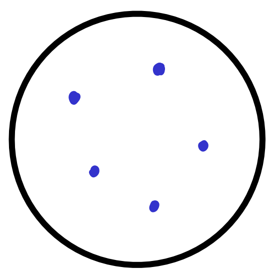
ETHZ

Hurwitz proposed (around 1900) the following problem:

(i) $\mathbb{P}^1 =$  is the Riemann Sphere

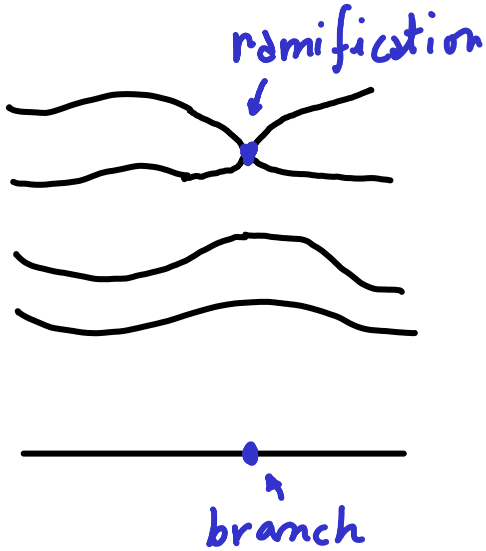
(ii) Choose b points of $\mathbb{P}^1$



(iii) How many Riemann Surfaces appear as degree d covers of $\mathbb{P}^1$ with simple branching over the b points?

$z \to z^2$

d sheets

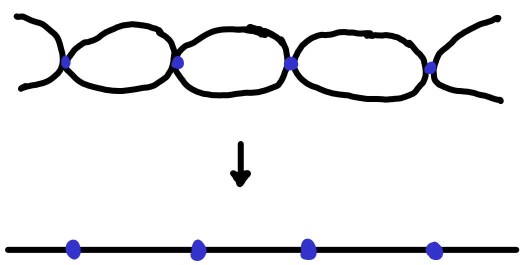


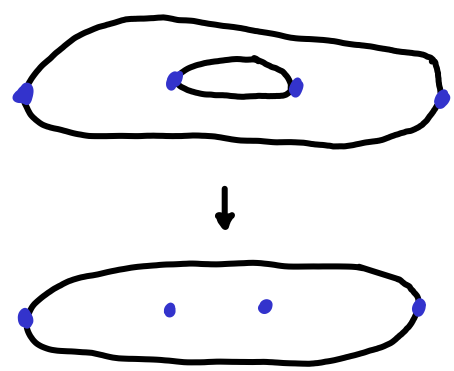
Nonsingular

$$\begin{array}{c} C \\ \downarrow \pi \\ \mathbb{P}^1 \end{array} \quad \left\{ \begin{array}{l} \text{algebraic geometry} \\ \text{Complex geometry} \\ \text{Topology} \end{array} \right.$$

Simple branching means 2 sheets come together

Example with $d=2$, $b=4$ (genus $=1$)





Riemann-Hurwitz formula :

$$2\cdot g(C) - 2 = -2d + b$$

Proof by Euler χ

count by $1/\mathrm{Aut}$

$H_{g,d}$ = Number of covers of $\mathbb{P}^1$ of degree d with $b = 2g+2d-2$ simple branch points

We do not assume cover is connected!

Related by inclusion/exclusion

$H^{\circ}_{g,d}$ = Count of connected covers

Ueber die Anzahl der Riemann'schen Flächen mit gegebenen Verzweigungspunkten.

Von

A. Hurwitz in Zürich.

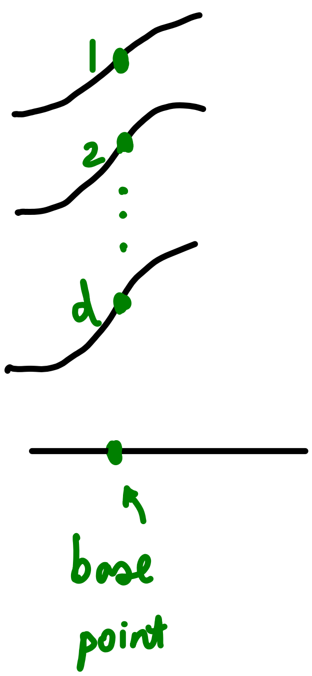
Math. Ann. 1901

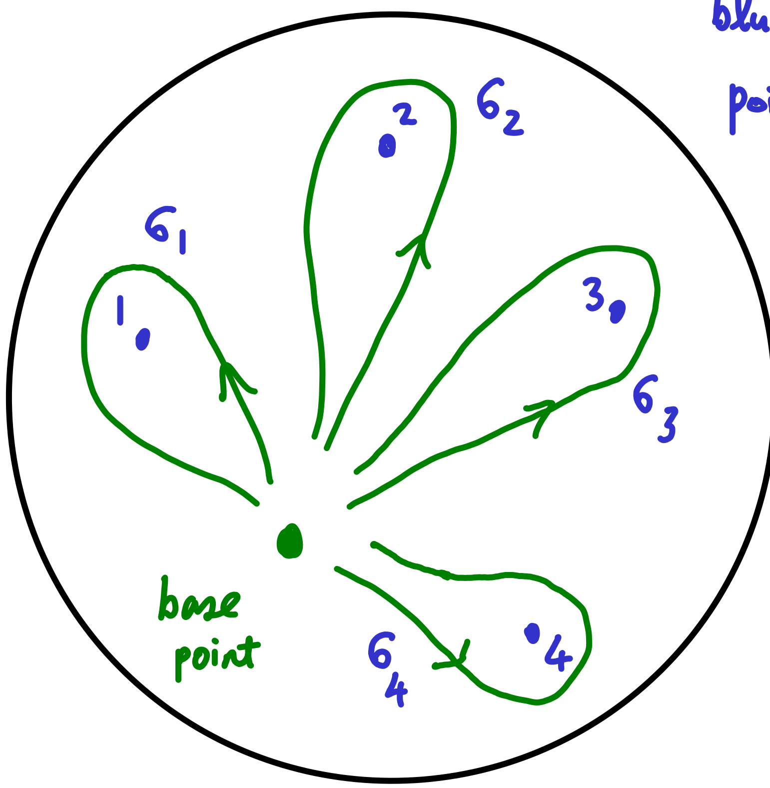
Hurwitz presents a solution

(with help from Chess World Champion Emanuel Lasker)

Solution via the Symmetric group Σ_d

Label preimages of a base point





blue branch points

σ_i monodromies

$$\sigma_1, \sigma_2, \sigma_3, \sigma_4 \in \Sigma_d$$

all are transpositions

Product is trivial: $\sigma_1 \cdot \sigma_2 \cdot \sigma_3 \cdot \sigma_4 = \mathrm{Id}$

Theorem (Hurwitz): $H_{g,d}$ equals

$\frac{1}{d!}$ times the number of solution of

$$\sigma_1 \cdot \sigma_2 \cdots \sigma_b = Id \in \Sigma_d$$

where $b = 2g+2d-2$ and all σ_i are transpositions

We can compute $H_{g,d}$ using the group algebra and the characters of the symmetric group:

$$H_{g,d} = \sum_{|\lambda|=d} \left(\frac{\dim\lambda}{d!}\right)^2 \left(|C_2| \frac{\chi_2^\lambda}{\dim\lambda}\right)^{2g+2d-2}$$

Burnside

- λ irrep of Σ_d
- χ_2^λ char
- C_2 conj class

1913

ETH Archives

I. Toda equations

Gromov-Witten theory concerns

$$\int$$ ← theory of integration

Topological string

Moduli of maps

$C \to X$

genus g curves

algebraic variety / symplectic manifold

I was working on various aspects of the theory (in Chicago 94-98).

Eguchi and S-K Yang had conjectured that $GW(\mathbb{P}^1)$ was governed by the Toda equations.

hep-th/9407134

I knew part of $GW(\mathbb{P}^1)$ contained the Hurwitz count.

Connected Covers

degree parameter

$$H(\lambda, y) = \sum_{g \geq 0} \sum_{d > 0} \lambda^{2g-2} e^{dy} \frac{H^{\circ}_{g,d}}{(2g+2d-2)!}$$

genus parameter

branch number

I could write a Toda equation for Hurwitz :

$$\boxed{\exp\Big(H(\lambda, y+\lambda) + H(\lambda, y-\lambda) - 2H \Big) = \lambda^2 e^{-y} \frac{\partial^2 H}{\partial y^2}}$$

which uniquely determines $H^{\circ}_{g,d}$ with no further mathematical input.

Can the Toda equation be proven from the symmetric group formula for the $H_{g,d}$?

Answer : Yes! (Okounkov)

Using the connection found here between the Hurwitz numbers and Toda as a starting point, Okounkov and I were able to solve the entire $GW(\mathbb{P}^1)$.

- P, The Toda equation and GW theory of $\mathbb{P}^1$, Lett. Math. Phys. (2000)
- Okounkov, Toda equation for Hurwitz numbers, Math. Res. Lett. (2000)
- OP, GW theory, Hurwitz Numbers, and Completed Cycles, Ann. of Math. (2006)

Brief remarks for the curious:

(i) $$V = \bigoplus_{k \in \mathbb{Z} + \frac{1}{2}} \mathbb{Z}\,\underline{k}$$

(ii) Consider $\Lambda^{\frac{\infty}{2}} V$ Fock space v_ϕ , $\langle , \rangle$

(iii) $\tau = \exp(\mathcal{H})$ generating series for $H^{\circ}_{g,d}$

generating series for $H_{g,d}$

Energy 2-cycle

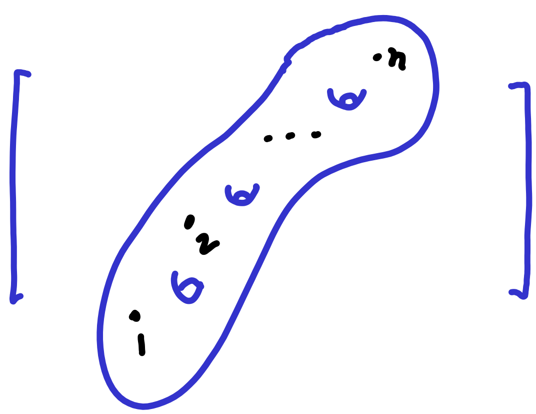
(iv) $$\tau = \langle \Gamma_+ e^{yE} e^{\lambda F_2} \Gamma_- v_\phi , v_\phi \rangle$$

Vertex operators

(v) Such matrix products are known to produce τ-functions for Toda

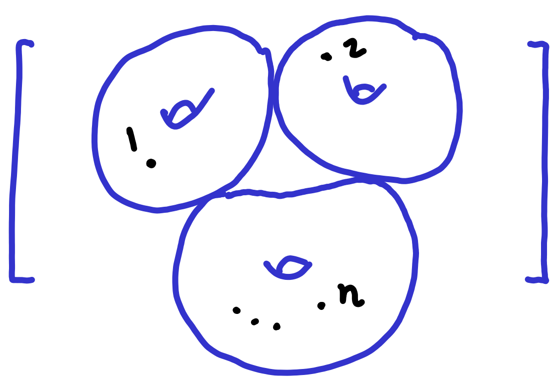
II. KdV

Let $\overline{\mathcal{M}}_{g,n}$ be the moduli space of stable curves.



$\in \mathcal{M}_{g,n}$

$\cap$ Deligne Mumford



$\overline{\mathcal{M}}_{g,n}$

$\psi_i \in H^2(\overline{\mathcal{M}}_{g,n})$ cotangent line class

$$\psi_i = c_1(\mathbb{L}_i) \qquad \begin{array}{ccc} \mathbb{L}_i & \ni & T^*_{C,p_i} \\ \downarrow & & \downarrow \\ \overline{\mathcal{M}}_{g,n} & \ni & [C, p_1, \ldots, p_n] \end{array}$$

Descendent integrals

$$\langle \tau_{k_1} \cdots \tau_{k_n} \rangle_g = \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{k_1} \cdots \psi_n^{k_n} \in \mathbb{Q}$$

Define free energy (following Witten 1990)

$$\mathcal{F}(t_0, t_1, t_2, \ldots) = \sum_{g \geq 0} \sum_{n \geq 0} \left\langle \frac{\gamma^n}{n!} \right\rangle_g$$

generating series of descendent integrals

$$\gamma = \sum_{i=0}^{\infty} t_i \tau_i$$

$$\mathcal{U} = \frac{\partial^2 \mathcal{F}}{\partial t_0^2}$$

Witten's conjecture: $\mathcal{U}$ satisfies KdV

KdV

$$\frac{\partial u}{\partial t_1} = u \frac{\partial u}{\partial t_0} + \frac{1}{12} \frac{\partial^3 u}{\partial t_0^3}$$

19th Century, to model shallow water waves

time

spatial coordinate

Proof via Kontsevich's matrix model uses analytic decomposition of moduli space into cells.

Is there any connection to Hurwitz?

Answer: Yes!

Okounkov-P, Gromov-Witten theory, Hurwitz numbers
Matrix models (written in 2001)

Connection to Hurwitz starts with ELSV formula:

$$H^0_{g,\mu} = \frac{(2g-2+|\mu|+\ell)!}{|\mathrm{Aut}(\mu)|} \prod_{i=1}^{\ell} \frac{\mu_i^{\mu_i}}{\mu_i!} \int_{\overline{M}_{g,\ell}} \frac{\sum_{k=0}^{g} (-1)^k \lambda_k}{\prod_{i=1}^{\ell} (1-\mu_i \psi_i)}$$

Hodge classes

μ is a partition of $|\mu|$

ℓ = length of μ

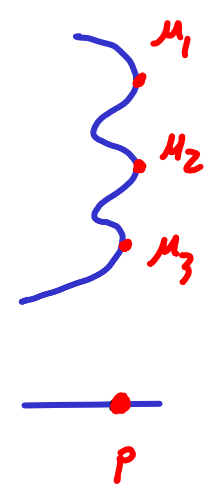
$H^0_{g,\mu}$ is the Hurwitz count of covers

$$\begin{array}{ccl} & C & \leftarrow \text{genus } g \\ \text{degree } |\mu| \rightsquigarrow \pi & \downarrow & \\ & \mathbb{P}^1 & \end{array}$$

with $2g-2+|\mu|+\ell$ simple branch points

plus a single point $p \in \mathbb{P}^1$ with

profile $\pi^{-1}(p)$ of shape μ.



Ekedahl
Lando
Shapiro
Vainstein
(2001)

Can also be proven very directly via Relative GW theory
Fantechi - P (2002), Graber-Vakil (2005)

The integral on the right side of ELSV:

$\lambda_k \in H^{2k}(\bar{M}_{g,\ell})$

leading term when we send $\mu_i \to \infty$

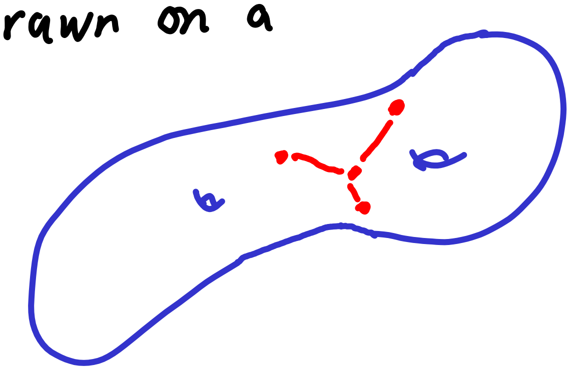
$$\int_{\bar{M}_{g,\ell}} \frac{\sum_{k=0}^{g} (-1)^k \lambda_k}{\prod_{i=1}^{\ell} (1-\mu_i \psi_i)} = \int_{\bar{M}_{g,\ell}} \frac{1}{\prod_{i=1}^{\ell}(1-\mu_i \psi_i)} + \text{lower terms}$$

We see that the asymptotics of $H^{\circ}_{g,\mu}$ carry the full information of descendents.

In the paper with Okounkov, we show how the Hurwitz asymptotics exactly yield Kontsevich's matrix model and therefore prove Witten's Conjecture.

Brief remarks for the curious:

(i) reinterpret $H^{\circ}_{g,\mu}$ in terms of branching graphs drawn on a topological surface.



(ii) As $\mu_i \to \infty$, we must study the leading terms in the counts of the branching graphs.

(iii) We exactly match Kontsevich's Comb model

Main New perspective for Study:

$$\boxed{\text{Hurwitz covers define a correspondence}}$$

genus g

degree d

$$\begin{array}{c} C \\ \pi \big\downarrow \\ \mathbb{P}^1 \end{array}$$

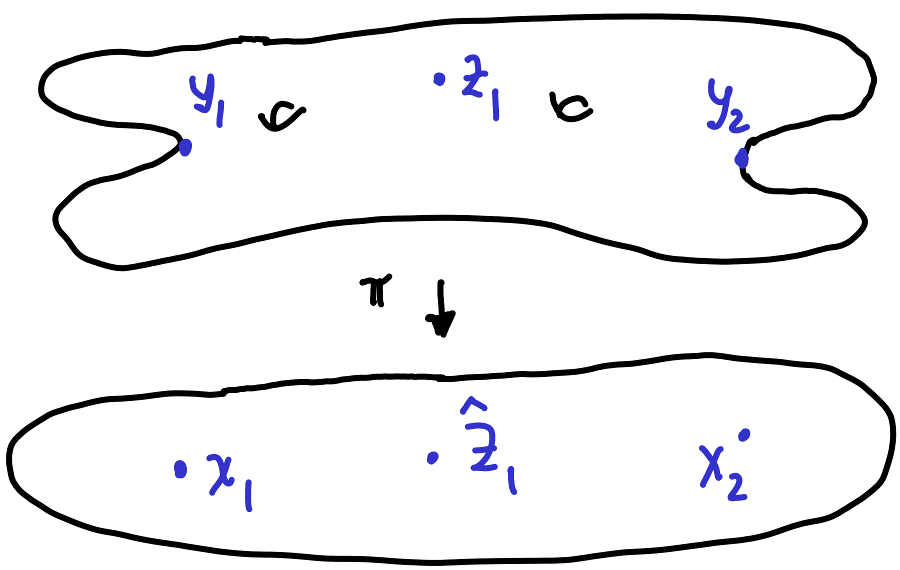
Hurwitz cover with C connected

ramification points $\quad \pi(y_i) = x_i$

$$\begin{array}{c} (C,\ y_1, \dots, y_b,\ z_1, \dots, z_n) \\ \pi \big\downarrow \\ (\mathbb{P}^1,\ x_1, \dots, x_b,\ \hat{z}_1, \dots, \hat{z}_n) \end{array}$$

extra points satisfying $\quad \pi(z_i) = \hat{z}_i$

branch points



$$\begin{tikzcd} & \mathcal{H}ur_{g,d,n} \arrow[dl, "\varepsilon_g"'] \arrow[dr, "\varepsilon_0"] & \\ \mathcal{M}_{g,b+n} & & \mathcal{M}_{0,b+n} \end{tikzcd}$$

moduli of such Hurwitz covers with n extra points

$b = 2g+2d-2$

All spaces have natural moduli compactifications

We have a correspondence

$$\begin{tikzcd} & \overline{\mathcal{H}ur}_{g,d,n} \arrow[dl, "\varepsilon_g"'] \arrow[dr, "\varepsilon_0"] & \\ \overline{\mathcal{M}}_{g,b+n} & & \overline{\mathcal{M}}_{0,b+n} \end{tikzcd}$$

Admissible covers
Harris-Mumford
1980s

Deligne-Mumford stable curves
1960s

- $$\mathcal{H}^{0}_{g,d} = \frac{\deg(\varepsilon_0)}{d^n}$$

Hurwitz Numbers are a small part of the information of the correspondence

- For a glimpse of the correspondence from another angle :

$$\varepsilon : \overline{\mathcal{H}ur}_{g,d,n} \to \overline{\mathcal{M}}_{g,n} \times \overline{\mathcal{M}}_{0,n}$$

$(\varepsilon_g, \varepsilon_0)$ followed by forgetting the b ramification and b branch points

$$\dim \overline{Hur}_{g,d,n} = 2g+2d-5+n$$

$$\dim \bar{\mathcal{M}}_{g,n} + \dim \bar{\mathcal{M}}_{0,n} = 3g-6+2n$$

dimensions usually different

Tevelev considered case

$$d = g+1 \ , \quad n = g+3$$

where both dimensions are $5g$

$$\boxed{\begin{array}{l} \text{Theorem (Tevelev, 2020) for } d=g+1,\ n=g+3 \\ \qquad \dfrac{\deg(\varepsilon)}{(4g)!} = 2^{g} \qquad [4g=b] \end{array}}$$

Motivated by study of Scattering amplitudes by

N. Arkani-Hamed. et al arXiv:1412.8475

Most general case where image and range of

$$\varepsilon : \overline{\mathcal{H}ur}_{g,d,n} \to \overline{\mathcal{M}}_{g,n} \times \overline{\mathcal{M}}_{0,n}$$

have the same dimension is

$$d = g+1+\ell , \quad n = g+3+2\ell \qquad \begin{cases} g \geq 0 \\ \ell \in \mathbb{Z} \end{cases}$$

We define $\mathrm{Tev}_{g,\ell} = \dfrac{\deg(\varepsilon)}{\overline{b}!}$

Tevelev's result is $\mathrm{Tev}_{g,0} = 2^g$

In the past month, with Alessio Cela and Johannes Schmitt, we have computed $T_{g,\ell}$ in all cases.

Theorem (Cela, P, Schmitt, 2021):

$$\mathrm{Tev}_{g,\ell} = 0 \quad \text{unless} \quad g \geq -2\ell$$

$$\mathrm{Tev}_{g,\ell} = 2^g - 2\sum_{i=0}^{-\ell-2} \binom{g}{i} + (-\ell-2)\binom{g}{-\ell-1} + \ell\binom{g}{-\ell}$$

Proof uses excess intersection calculations in the boundary of $\overline{\mathcal{H}ur}_{g,d,n}$

Historical Note: For $h \geq 0$,

1885

$\mathrm{Tev}_{2h,-h}$ = Castelnuovo's Count (1889) of linear series g^1_{h+1} on a genus $2h$ curve

= Catalan(h)

- The fundamental question is to compute the push-forward of the fundamental class

$$(\varepsilon_g \times \varepsilon_0)_* \left[\overline{Hur}_{g,d,n} \right] \in H^*\left(\overline{M}_{g,b+n} \times \overline{M}_{0,b+n} \right)$$

We know three aspects of the answer

(i) $(\varepsilon_g \times \varepsilon_0)_* \left[\overline{Hur}_{g,d,n} \right] \in R^*(\overline{M}_{g,b+n}) \otimes R^*(\overline{M}_{0,b+n})$

Faber-P (2005)

Tautological part

(ii) impractical algorithm to compute

via Relative GW theory (log), More natural is the double ramification cycle.

(iii) Pixton's formula for $DR_{g,(1^d,-1^d,0^n)}$

Janda, P, Pixton, Zvonkine (2017)

III Moduli of Meromorphic differentials

By now a large subject, I will just follow a single strand.

Many connections to Hurwitz covers

$\pi : C \to \mathbb{P}^1$

see Eskin-Okounkov's volume calculations

Let C be a Riemann Surface

We have parallel moduli problems

- Moduli of (C, f) $\Longrightarrow$ Hurwitz Covers

  f ← rational function

- Moduli of (C, ω) $\Longrightarrow$ flat Surfaces

  ω ← meromorphic differential

⋮

- Moduli of (C, ω^k) ← meromorphic k-differential

Theorem (Bae-Holmes-P-Schmitt-Schwarz, 2020):

Pixton's formula calculates the push-forwards of fundamental classes of the moduli spaces (C, ω^k) for all $k \geq 1$.

I will end the lecture by explaining (most) of the formula in the $k=1$ case.

Let $\mu = (m_1, \dots, m_n)$ be a vector of integers with $\sum_{i=1}^{n} m_i = 2g-2$ [insist $\exists\, m_i < 0$]

We want to calculate the class $\mathrm{Diff}_{g,\mu}$ of the locus* $(C, p_1, \dots, p_n) \in \overline{M}_{g,n}$ where $\omega_C \cong \mathcal{O}_C(\sum m_i p_i)$ holds.

* Closure or BCGGM space

The Pixton cycle $P_{ix_{g,\mu}}^{diff}$ is a sum over stable graphs Γ ← strata of $\overline{M}_{g,n}$

- An admissible weighting of Γ mod r is a function

$$w : H(\Gamma) \to \{0, 1, \ldots, r-1\}$$

Set of half edges of Γ

Satisfying

half edge corresponding to a marking

(i) $w(j) = m_j + 1 \mod r$

(ii) $w(h) + w(h') = 0 \mod r$

$\overset{h}{\bullet\!\!-\!\!-\!\!-}\overset{h'}{\bullet}$ edge

(iii) $\sum_{h \mapsto v} w(h) = 2g(v) - 2 + \mathrm{val}(v) \mod r$

v is a vertex

- $2^g \cdot \mathrm{Pix}^{\mathrm{diff}}_{g,\mu}$ is the r-constant codim g

part of

$$\sum_{\Gamma \in G_{g,n}} \sum_{w \in W_{\Gamma,r}} \frac{1}{|\mathrm{Aut}\,\Gamma|} \frac{1}{r^{h^1(\Gamma)}} \cdot i_{\Gamma *} \left[\prod_{v \in V(\Gamma)} \exp(-\kappa_1(v)) \prod_{i=1}^{n} \exp\left((m_i+1)^2 \psi_i\right) \prod_{e \in E(\Gamma)} \frac{1-\exp\left(-w(h)w(h')(\psi_h+\psi_{h'})\right)}{\psi_h+\psi_{h'}} \right]$$

all stable graphs for $\overline{\mathcal{M}}_{g,n}$

all admissible weightings mod r

Whole expression is a polynomial in r for $r \gg 0$.

Take the constant term!

$$\mathrm{Pix}^{\mathrm{diff}}_{g,\mu} \in H^{2g}(\overline{\mathcal{M}}_{g,n}) \leftarrow \text{formula}$$

$$\mathrm{Diff}_{g,\mu} \in H^{2g}(\overline{\mathcal{M}}_{g,n}) \leftarrow \text{geometric class}$$

Theorem (BHPSS, 2020)

$\mathrm{Pix}^{\mathrm{diff}}_{g,\mu}$ and $\mathrm{Diff}_{g,\mu}$ are related by a simple upper triangular transformation.

- Farkas-P, The moduli of twisted canonical divisors, J. Math. Jussieu (2018)
- BHPSS, Pixton's formula and Abel-Jacobi theory on the Picard stack arXiv:2004.08676
- Sage program Admcycles ← Link on Bonn webpage of Johannes Schmitt

The End

ETH archives 1913