Cycles on the moduli of curves via Torelli

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Let A_g be the moduli space of PPAVs : A_g is a DM stack of dimension $\binom{g+1}{2}$.

The theory of tautological cycles on A_g was developed by van der Geer (1996).

$$\begin{array}{c} \mathbb{E} \\ \downarrow \\ A_g \end{array}$$

Hodge bundle with fiber $T_0^*(X)$ over $[X, 0]$

Abelian variety with zero.

- $R^*(A_g) \subset CH^*(A_g)$ is defined to be the $\mathbb{Q}$-subalgebra generated by $\lambda_1, \dots, \lambda_g$ where $\lambda_i = c_i(\mathbb{E})$.

- Presentation by van der Geer :

$$R^*(A_g) \cong \frac{\mathbb{Q}[\lambda_1, \dots, \lambda_g]}{(\lambda_g = 0,\ c(\mathbb{E}) \cdot c(\mathbb{E}^\vee) = 1)}$$

- $R^*(A_g)$ is a Gorenstein local ring with socle in codim $\binom{g}{2}$.

For the past few months, Canning, Oprea, and I have been investigating the NL loci in $\mathcal{A}_g$ and their relation to $R^*(\mathcal{A}_g)$. For more about these motivations, see my lecture in the North German AG Seminar (9 Feb 2023).

Today, I will go directly to the moduli of curves:

Consider the Torelli map

$$\mathrm{Tor}: \mathcal{M}_g^{ct} \longrightarrow \mathcal{A}_g \qquad \text{proper}$$

$$[C] \longmapsto [\mathrm{Jac}^0(C)] \qquad \text{PPAV}$$

Consider the cycle class $(g \geq 2)$

$$[A_1 \times A_{g-1}] \in CH^{g-1}(A_g).$$

My lecture is about

$$\mathrm{Tor}^*[A_1 \times A_{g-1}] \in CH^{g-1}(M_g^{ct}).$$

Question 1 : $[A_1 \times A_{g-1}] \in R^{g-1}(A_g)$?

Question 2 : $\mathrm{Tor}^*[A_1 \times A_{g-1}] \in R^{g-1}(M_g^{ct})$?

$$\mathrm{Tor}^* : R^{g-1}(A_g) \longrightarrow \Lambda^{g-1}(M_g^{ct})$$

Lambda ring

Question 3 : $\mathrm{Tor}^*[A_1 \times A_{g-1}] \in \Lambda^{g-1}(M_g^{ct})$?

I. What is $\mathrm{Tor}^{-1}(A_1 \times A_{g-1})$?

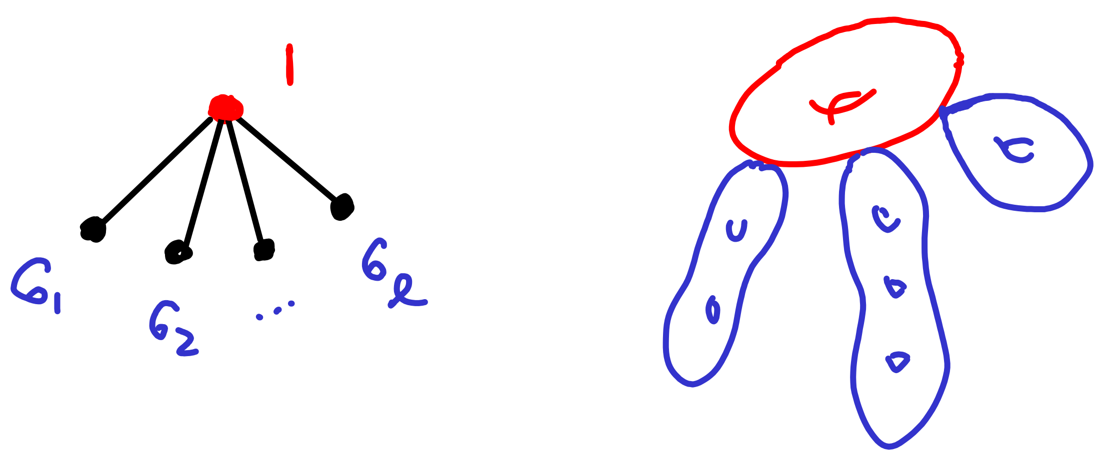
Defined by fiber product:

$$\begin{tikzcd} \mathrm{Tor}^{-1}(A_1 \times A_{g-1}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & A_1 \times A_{g-1} \arrow[d] \\ \mathcal{M}_g^{ct} \arrow[r] & A_g \end{tikzcd}$$

The irreducible components of $\mathrm{Tor}^{-1}(A_1 \times A_{g-1})$ are in bijective correspondence with partitions $\mathrm{Part}(g-1)$.

$$\mathrm{Tor}^{-1}(A_1 \times A_{g-1}) = \bigcup_{\sigma \in \mathrm{Part}(g-1)} M_\sigma$$

M_σ = Stratum corresponding to dual graph



A subtle point: $\mathrm{Tor}^{-1}(A_1 \times A_{g-1})$ is scheme theoretically reduced.

II. Excess intersection theory

Expected Codim of $\mathrm{Tor}^{-1}(A_1 \times A_{g-1})$ with respect to M_g^{ct} is $g-1$.

So all irreducible components are of excess dimension except $\sigma = (1,1,1,\cdots,1)$.

For the excess calculation, we have contributions from all strata of intersections of irreducible components.

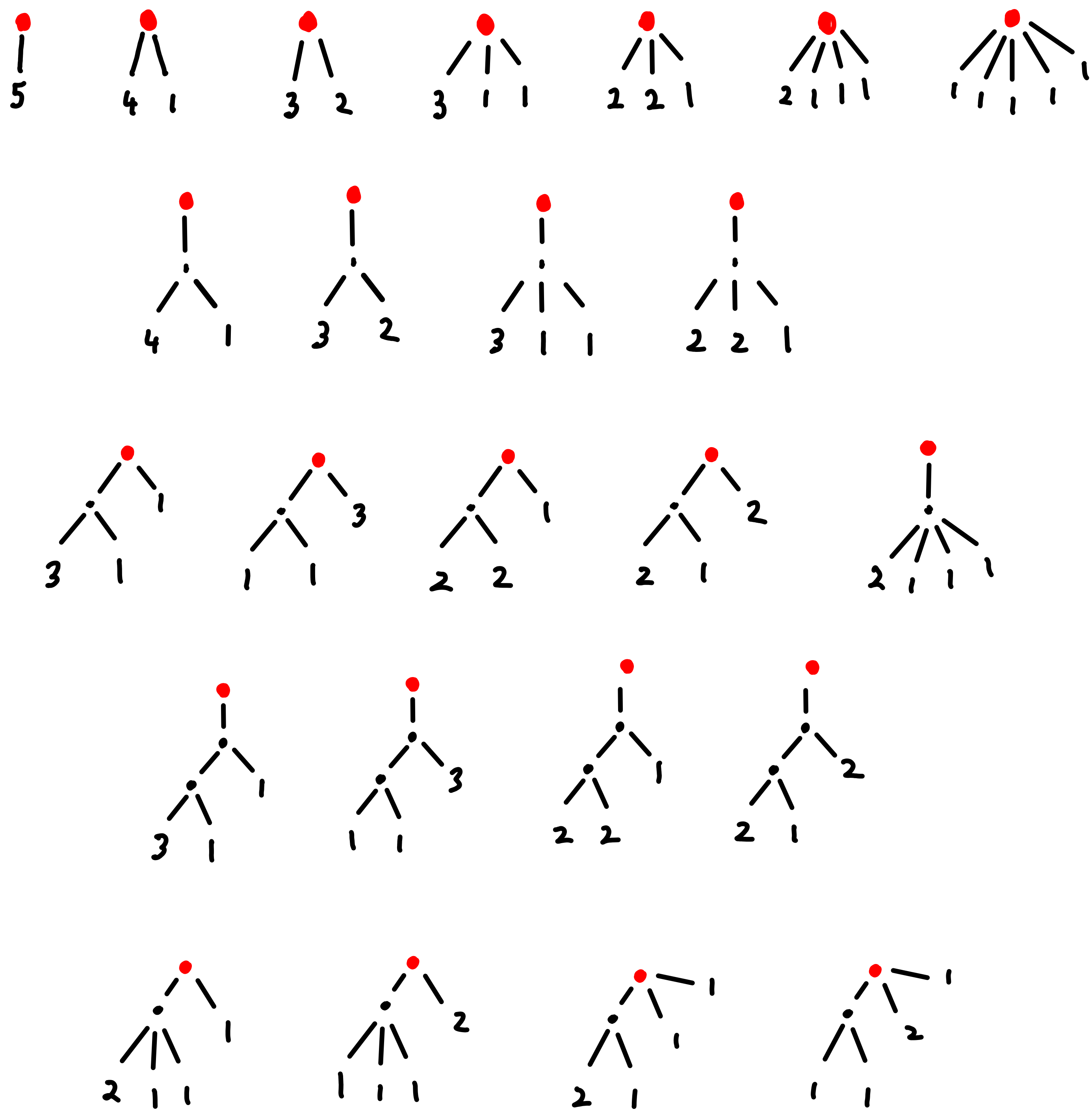
Intersection strata are indexed by extremal trees T of genus g:

Definition: T is a stable graph of genus g satisfying

(i) T is a rooted tree with root $\bullet$ of genus 1.

(ii) all vertices of positive genus other than the root are extremal.

Example: For $g=6$, there are 24 Strata to consider (with at most 5 edges)



Excess intersection theory $\Rightarrow$

Tautological classes on moduli

$$\mathcal{M}_T = \prod_{v \in \mathrm{Vert}(T)} \mathcal{M}^{ct}_{g(v),\, \mathrm{val}(v)}$$

$$\mathrm{Tor}^*[A_1 \times A_{g-1}] = \sum_{\substack{\text{all strata } T \text{ of} \\ \mathrm{Tor}^{-1}(A_1 \times A_{g-1})}} \mathrm{Cont}(T)$$

recursive formula for these excess contributions

Corollary: $\mathrm{Tor}^*[A_1 \times A_{g-1}] \in R^{g-1}(M_g^{ct})$.

Question 2: $\mathrm{Tor}^*[A_1 \times A_{g-1}] \in R^{g-1}(M_g^{ct})$?

ANSWER: Yes.

III Pixton recently solved our recursion for $\mathrm{Cont}(T)$.

Let T be an extremal tree of genus g. Consider the variable set

$$Z = \{ Z_e \}_{e \in \mathrm{Edge}(T)} .$$

Let $E \longrightarrow M_T$ be a vector bundle.

polynomial in Z and $c_i(E)$

$$\mathrm{Cont}(T) = \frac{P_T(Z, E)}{|\mathrm{Aut}(T)|}$$

automorphisms preserving root and genus labels

under specialization

Z_e = Normal bundle of smoothing node associated to e

$E = \mathrm{Tor}^*(\mathrm{Nor}_{A_1 \times A_{g-1}/A_g})$

Pixton's formula for $P_T(Z, E)$

$P_T(Z,E)$ is the degree $g-1-|\mathrm{Edge}(T)|$ part of the following polynomial:

$$\left[(-1)^{\ell} \frac{\prod_{v \in \mathrm{Vert}(T)} \left(1 + \sum_{e \in v \to \bullet} Z_e\right)^{\mathrm{val}(v)-2}}{\prod_{e \in \mathrm{Edge}(T)} Z_e} \right]_{Z^{\geq 0}} \cdot c(E)$$

ℓ is the number of extremal vertices of T (root is not extremal)

$e \in v \to \bullet$ denotes edge on the unique minimal path to the root

discard all polar terms of the Laurent series

Theorem: We have an explicit formula:

$$i_T : M_T \to M_g^{ct}$$

$$\mathrm{Tor}^*[A_1 \times A_{g-1}] = \sum_T \frac{i_{T*} P_T(Z, E)}{|\mathrm{Aut}(T)|}.$$

Sum over all extremal trees of genus g with at most $g-1$ edges

$\underline{\text{IV}}$ We now address <u>Question 3</u>:

$$\mathrm{Tor}^*[A_1 \times A_{g-1}] \in \Lambda^{g-1}(M_g^{ct})?$$

<u>Proposition</u>: If $[A_1 \times A_{g-1}] \in R^{g-1}(A_g)$,

then we must have:

$$[A_1 \times A_{g-1}] = \frac{(-1)^{g+1}\, g}{6\, B_{2g}} \lambda_{g-1}$$

in $CH^{g-1}(A_g)$.

Proposition also known to Faber, unpublished.

Proof: using the presentation of $R^*(A_g)$ and

$$[A_1 \times A_{g-1}] \cdot \lambda_{g-1} = 0,$$

We see $[A_1 \times A_{g-1}]$ must be proportional to λ_{g-1}.

The proportionality is determined by comparing

use formula and Hodge integrals of Faber-P

$$\int_{\overline{M}_g} \lambda_g \lambda_{g-2} \mathrm{Tor}^*[A_1 \times A_{g-1}]$$

and

$$\int_{\overline{M}_g} \lambda_g \lambda_{g-1} \lambda_{g-2}.$$

A refinement of Question 3:

$$\mathrm{Tor}^*[A_1 \times A_{g-1}] - \frac{(-1)^{g+1} g}{6 B_{2g}} \lambda_{g-1} = 0 \; ?$$

We can test whether the vanishing holds with admcycles.

$$\Delta_g = \mathrm{Tor}^*[A_1 \times A_{g-1}] - \frac{(-1)^{g+1} g}{6 B_{2g}} \lambda_{g-1}$$

$$\left.\begin{aligned} \Delta_2 &= 0 \in R^1(M_2^{ct}) \\ \Delta_3 &= 0 \in R^2(M_3^{ct}) \end{aligned}\right\}$$

must vanish since CHOW is tautological for A_2, A_3

admcycles calculates:

$$\left.\begin{array}{l} \Delta_4 = 0 \in R^3(M_4^{ct}) \\[2ex] \Delta_5 = 0 \in R^4(M_5^{ct}) \end{array}\right\} \text{first nontrivial vanishings}$$

$g=6$ is the first really interesting case:

$$\Delta_6 = \mathrm{Tor}^*[A_1 \times A_5] - \frac{2730}{691}\lambda_5 .$$

Pixton's Conjecture for $R^*(M_6^{ct})$ predicts:

$$\underset{\text{dim } 71}{R^4(M_6^{ct})} \times \underset{\text{dim } 72}{R^5(M_6^{ct})} \longrightarrow \mathbb{Q}$$

pairing of rank 71

Assuming Pixton's Conjecture, admcycles finds

- $\Delta_6 \neq 0 \in R^5(M_6^{ct})$.
- Δ_6 is the generator of the kernel of the pairing.

Canning, Larson, and Schmitt prove Pixton's Conjecture for $R^*(M_6^{ct})$, so $\Delta_6 \neq 0 \in R^5(M_6^{ct})$.

Furthermore, admcycles shows

$$\mathrm{Tor}^*[A_1 \times A_5] \notin \Lambda^5(M_6^{ct}).$$

We conclude

$$[A_1 \times A_5] \notin R^5(A_6),$$

the first interesting non-tautological algebraic cycle on A_6.

<u>Question 1</u>: $[A_1 \times A_{g-1}] \in R^{g-1}(A_g)$?

ANSWER: No for $g=6$.

<u>Question 3</u>: $\mathrm{Tor}^*[A_1 \times A_{g-1}] \in \Lambda^{g-1}(M_g^{ct})$?

ANSWER: No for $g=6$.

V Tautological classes on M_6^{ct}

To prove Pixton's Conjecture for $R^*(M_6^{ct})$, Canning, Larson, and Schmitt need only prove

$$\dim R^5(M_6^{ct}) = 72.$$

Their method:

- Canning-Larson have proven earlier that

$$R^4(\overline{M}_{5,2}) = CH^4(\overline{M}_{5,2}).$$

- Therefore they obtain

$$R^4(\overline{M}_{5,2}) \to R^5(\overline{M}_6) \to R^5(M_6^{ct}) \to 0.$$

Pixton's Conjecture for $R^5(\overline{M}_6)$ predicts

$$\dim R^5(\overline{M}_6) = 988.$$

- Assuming Pixton's Conjecture for $R^5(\overline{M}_6)$, admcycle calculates

$$\operatorname{Im} R^4(\overline{M}_{5,2}) \subset R^5(\overline{M}_6).$$

dim 916

Pixton's Conjecture for $R^5(\overline{M}_6)$

$$\Downarrow$$

Pixton's Conjecture for $R^5(M_6^{ct})$

- admcycles verifies

$$\dim R^5(\overline{M}_6) = 988$$

using Pixton's relations

and the pairing

$$R^5(\overline{M}_6) \times R^{10}(\overline{M}_6) \longrightarrow \mathbb{Q},$$

a long computation.

VI Stable maps

Pixton suggests a connection to GW theory.

- Define $P_{g,1}$ by:

$$\begin{tikzcd} P_{g,1} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & A_1 \times A_{g-1} \arrow[d] \\ M^{ct}_{g,1} \arrow[r] & A_g \end{tikzcd},$$

We have added a marking.

• Let

$$\begin{tikzcd} \mathcal{E} \arrow[d, "\pi"'] \\ \mathcal{M}^{ct}_{1,1} \arrow[u, bend right, "\Delta_0"'] \end{tikzcd}$$

be the universal elliptic curve with zero section Δ_0.

Let $\mathcal{M}^{ct}_{g,1}(\pi, 1) \to \mathcal{M}^{ct}_{1,1}$ be the moduli space of stable maps to the fibers of π of fiber degree 1.

Let $ev_1 : \mathcal{M}^{ct}_{g,1}(\pi, 1) \to \mathcal{E}$

be the evaluation map.

Define $Q_{g,1}$ by:

$$Q_{g,1} = ev_1^{-1}(s_0) \subset \mathcal{M}^{ct}_{g,1}(\pi, 1).$$

Theorem: $P_{g,1} \cong Q_{g,1}$

Surely the isomorphism respects the virtual classes on both sides, but we have not proven this yet.

VII Further results / directions

- $\Delta_g \in R^{g-1}(M_g^{ct})$ is always in the kernel of the λ_g-Pairing:

$$R^{g-2}(M_g^{ct}) \times R^{g-1}(M_g^{ct}) \longrightarrow \mathbb{Q}.$$

Proved by an argument of Pixton together with relations we know in $CH^*(A_g)$.

- Natural to Conjecture

$$\Delta_g \neq 0 \in R^{g-1}(M_g^{ct})$$

for all $g \geq 6$.

- Vanishing

Consider the cycle class

$$C^k[A_1 \times A_{g-1}] = \sum_T \frac{i_{T*}\, P_T^k(Z, E)}{|\mathrm{Aut}(T)|}$$

Sum over all extremal trees of genus g with at most $g-1+k$ edges

in $R^{g-1+k}(M_g^{ct})$,

where $P_T^K(z,E)$ is the degree

$$g-1-|\mathrm{Edge}(T)|+K$$

part of the polynomial:

$$\left[(-1)^{\ell}\,\frac{\displaystyle\prod_{v\in \mathrm{Vert}(T)}\Big(1+\sum_{e\in v\to\bullet} z_e\Big)^{\mathrm{val}(v)-2}}{\displaystyle\prod_{e\in \mathrm{Edge}(T)} z_e}\right]_{z\geq 0}\cdot\, c(E)$$

Pixton conjectures, for all $k>0$,

$$C^k[A_1\times A_{g-1}]=0\in R^{g-1+k}(M_g^{ct}).$$

VIII further NL loci

If a curve of genus g admits a degree d map to an elliptic curve,

$$C \to E,$$

then the NS group of $\mathrm{Jac}^0(C)$ contains an extra class obtained by pull-back from E.

Let $NL_d \to \mathcal{A}_g$ be the corresponding Noether-Lefschetz locus.

- $NL_1 \cong A_1 \times A_{g-1}$.

- Define $P^d_{g,1}$ by:

$$
\begin{tikzcd}
P^d_{g,1} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & NL_d \arrow[d] \\
\mathcal{M}^{ct}_{g,1} \arrow[r] & A_g
\end{tikzcd}
$$

Define $Q^d_{g,1}$ by:

$$Q^d_{g,1} = ev_1^{-1}(s_0) \subset \mathcal{M}^{ct}_{g,1}(\pi, d).$$

I expect we will have

$$P^d_{g,1} \cong Q^d_{g,1} \,.$$

- If, furthermore, the virtual classes are respected, we can study aspects of the cycle class

$$[NL_d] \in CH^{g-1}(A_g)$$

via Gromov-Witten theory.