Tevelev degrees and Hurwitz moduli spaces

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September 2021

Abstract

We interpret the degrees which arise in Tevelev's study of scattering amplitudes in terms of moduli spaces of Hurwitz covers. Via excess intersection theory, the boundary geometry of the Hurwitz moduli space yields a simple recursion for the Tevelev degrees (together with their natural two parameter generalization). We find exact solutions which specialize to Tevelev's formula in his cases and connect to the projective geometry of lines and Castelnuovo's classical count of g_d^1's in other cases. For almost all values, the calculation of the two parameter generalization of the Tevelev degree is new. A related count of refined Dyck paths is solved along the way.

Contents

1 Introduction

1.1 Hurwitz numbers

The counting of Hurwitz covers of $\mathbb{P}^1$ with fixed degree and branching data has been studied for more than a century [16]. The question is connected to complex geometry, topology, and the representation theory of the symmetric group. In the last few decades, there has been an resurgence of interest in Hurwitz covers motivated by connections to Gromov-Witten theory and the geometry of the moduli space of curves, see [4, 9, 11, 12, 15, 20, 21].

The moduli space $\mathcal{H}_{g,d,n}$ parameterizes Hurwitz covers

$$\pi : C \to \mathbb{P}^1$$

where C is a complete, nonsingular, irreducible curve of genus g with n distinct markings $p_1, \ldots, p_n \in C$, the map π is of degree d with $2g + 2d - 2$ distinct simple ramification points

$$q_1, \ldots, q_{2g+2d-2} \in C \,,$$

and all the points

$$\pi(p_1), \ldots, \pi(p_n), \pi(q_1), \ldots, \pi(q_{2g+2d-2}) \in \mathbb{P}^1$$

are distinct. There is a natural compactification

$$\mathcal{H}_{g,d,n} \subset \overline{\mathcal{H}}_{g,d,n}$$

by admissible covers [15].

The moduli space of admissible covers has two canonical maps determined by the domain and range of the cover:

$$
\begin{array}{ccc}
 & \overline{\mathcal{H}}_{g,d,n} & \\
 {}^{\epsilon_g}\swarrow & & \searrow^{\epsilon_0} \\
\overline{\mathcal{M}}_{g,2g+2d-2+n} & & \overline{\mathcal{M}}_{0,2g+2d-2+n}
\end{array}
$$

The Hurwitz number of [16], in connected form, is defined using the degree of the second map,

$$\mathsf{Hur}_{g,d} = \frac{\deg(\epsilon_0)}{d^n} \,.$$

2

The denominator d^n removes the dependence on n.[1]

1.2 Tevelev degrees

A different degree was introduced by Tevelev in his study of scattering amplitudes of stable curves [28] motivated in part by [1]. Tevelev's degree, from the point of view of the moduli spaces of Hurwitz covers, is defined as follows. Consider the map

$$\tau_{g,d,n} : \overline{\mathcal{H}}_{g,d,n} \to \overline{\mathcal{M}}_{g,n} \times \overline{\mathcal{M}}_{0,n}$$

defined by ϵ_g and ϵ_0 forgetting the ramification points q_j of the domain of the cover and the branch points $\pi(q_j)$ of the range. When

$$d = g + 1 \quad \text{and} \quad n = g + 3 \,,$$

both the domain and range of $\tau_{g,g+1,g+3}$ have dimension $5g$. The number of forgotten ramification points on the domain is $4g$. Tevelev's degree is

$$\mathsf{Tev}_g = \frac{\deg(\tau_{g,g+1,g+3})}{(4g)!} \,.$$

The denominator $(4g)!$ reflects the possible orderings of the forgotten ramification points. One of the central results of [28] is the remarkably simple formula

$$\mathsf{Tev}_g = 2^g \tag{1}$$

for all $g \geq 0$. Tevelev provides several paths to the proof of (1) involving beautiful aspects of the classical geometry of curves.[2]

Our goal here is to study Tevelev's degrees using the boundary geometry of the moduli space of Hurwitz covers. To start, we observe the dimension constraint required for the existence of a degree holds more generally. For $\ell \in \mathbb{Z}$, let

$$d[g, \ell] = g + 1 + \ell \quad \text{and} \quad n[g, \ell] = g + 3 + 2\ell \,.$$

Consider the associated τ-map (again forgetting ramification and branch points),

$$\tau_{g,\ell} : \overline{\mathcal{H}}_{g,d[g,\ell],n[g,\ell]} \to \overline{\mathcal{M}}_{g,n[g,\ell]} \times \overline{\mathcal{M}}_{0,n[g,\ell]} \,.$$

The domain and range of $\tau_{g,\ell}$ are both of dimension $5g + 4\ell$. We can therefore define

$$\mathsf{Tev}_{g,\ell} = \frac{\deg(\tau_{g,\ell})}{(2g + 2d[g,\ell] - 2)!} \,,$$

where we once again divide by the possible orderings of the forgotten ramification points.

Our first result is that the simplicity of the degree that Tevelev found for $\ell = 0$ continues to hold for all non-negative ℓ.

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[1] The standard definition of the Hurwitz number is for the $n = 0$ geometry.

[2] The translation of Tevelev's definition to ours requires a change of language. See Section 2 for a discussion.

Theorem 1. *For all $g \geq 0$ and $\ell \geq 0$, we have*

$$\mathsf{Tev}_{g,\ell} = 2^g \,.$$

The proof of Theorem 1 involves a recursion which arises from the analysis of the excess intersection theory of a particular fiber of $\tau_{g,\ell}$. The study of another naturally related degree is necessary for the argument.

For $1 \leq r \leq d$, let $\overline{\mathcal{H}}_{g,d,n,r}$ be the moduli space of admissible covers with n markings *of which r lie in the same fiber of the cover.* See Figure 1 for an illustration.
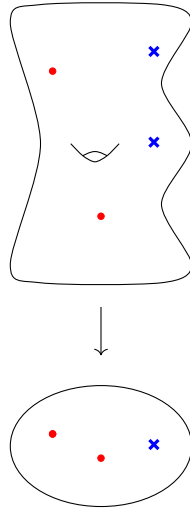

Figure 1: A curve $C \to \mathbb{P}^1$ in $\overline{\mathcal{H}}_{1,d,4,2}$ with $n = 4$ markings in C of which $r = 2$ lie in the fibre over the same point of $\mathbb{P}^1$.

Since r markings lie in the same fiber, the range of the map ϵ_0 is altered,

$$\epsilon_0 : \overline{\mathcal{H}}_{g,d,n,r} \to \overline{\mathcal{M}}_{0,2g+2d-2+n-r+1} \,.$$

For $\ell \in \mathbb{Z}$, let the τ-map

$$\tau_{g,\ell,r} : \overline{\mathcal{H}}_{g,d[g,\ell],n[g,\ell],r} \to \overline{\mathcal{M}}_{g,n[\mathsf{g},\ell]} \times \overline{\mathcal{M}}_{0,n[g,\ell]-r+1} \,.$$

be obtained as before by forgetting all the ramification points of the domain and branch points of the range of the cover. The domain and range of $\tau_{g,\ell,r}$ are both of dimension $5g + 4\ell - r + 1$. The degrees

$$\mathsf{Tev}_{g,\ell,r} = \frac{\deg(\tau_{g,\ell,r})}{(2g + 2d[g,\ell] - 2)!} \,.$$

appear in the proof of Theorem 1.

In case $r = 1$, we recover the previously defined Tevelev degree

$$\mathsf{Tev}_{g,\ell,1} = \mathsf{Tev}_{g,\ell} \,.$$

In case $\ell = 0$ and $r = 1$, we recover Tevelev's original count

$$\mathsf{Tev}_{g,0,1} = \mathsf{Tev}_g \,.$$

We therefore have a two parameter variation of the Tevelev degree.

The study of $\mathsf{Tev}_{g,\ell,r}$ separates into two main cases depending upon whether ℓ is non-negative or non-positive.

1.3 Results for $\ell \geq 0$

The following two results completely determine the degrees $\mathsf{Tev}_{g,\ell,r}$ in all cases where $\ell \geq 0$.

Theorem 2. *For all $g \geq 0$, $\ell = 0$, and $1 \leq r \leq g+1$, we have*

$$\mathsf{Tev}_{g,0,r} = 2^g - \sum_{i=0}^{r-2} \binom{g}{i} \,.$$

Theorem 3. *For all $g \geq 0$, $\ell > 0$, and $1 \leq r \leq g+1+\ell$, we have:*

$$
\begin{aligned}
\mathsf{Tev}_{g,\ell,r} &= 2^g & &\text{if } \ell \geq r \,, \\
\mathsf{Tev}_{g,\ell,r} &= \mathsf{Tev}_{g,0,r-\ell} & &\text{if } \ell < r \,.
\end{aligned}
$$

We have already seen that the $\ell = 0$ and $r = 1$ case recovers Tevelev's result. If we take $r = g + 1 + \ell$, then Theorems 2 and 3 yield

$$\mathsf{Tev}_{g,\ell,g+1+\ell} = 1 \tag{2}$$

for $\ell \geq 0$. For a Hurwitz covering

$$[\pi : C \to \mathbb{P}^1] \in \mathcal{H}_{g,d[g,\ell],n[g,\ell],r=d[g,\ell]} \,,$$

a full fiber of π is specified by the r markings. The r markings determine a line bundle L on C. The evaluation (2) can also be obtained by studying a classical transformation related to the associated complete linear series $\mathbb{P}(H^0(C,L))$. See Section 4.3 for a discussion.

1.4 Results for $\ell \leq 0$

The calculation of $\mathsf{Tev}_{g,\ell,r}$ for $\ell \leq 0$ has a more intricate structure. As before, we require

$$1 \leq r \leq d[g,\ell] = g + 1 + \ell \,. \tag{3}$$

Since the range of $\tau_{g,\ell,r}$ is

$$\overline{\mathcal{M}}_{g,n[\mathrm{g},\ell]} \times \overline{\mathcal{M}}_{0,n[g,\ell]-r+1} \,,$$

we also require

$$n[g, \ell] - r + 1 = g + 3 + 2\ell - r + 1 \geq 3 \,. \tag{4}$$

If either (3) or (4) is not satisfied, then $\mathsf{Tev}_{g,\ell,r} = 0$ by definition.

In fact, $r \geq 1$ and condition (4) can be written together as

$$1 \leq r \leq g + 1 + 2\ell \tag{5}$$

which implies the upper bound of (3) when $\ell \leq 0$.

Suppose $\ell \leq 0$ and $r \geq 1$ are both fixed. Let

$$g[\ell, r] = r - 2\ell - 1$$

be the minimum genus permitted by (5). We express the solutions to the associated Tevelev counts for all genera as an infinite vector

$$\mathsf{T}_{\ell,r} = \left(\mathsf{Tev}_{g[\ell,r],\ell,r} \,,\; \mathsf{Tev}_{g[\ell,r]+1,\ell,r} \,,\; \mathsf{Tev}_{g[\ell,r]+2,\ell,r} \,,\; \mathsf{Tev}_{g[\ell,r]+3,\ell,r} \,, \ldots \right) \,.$$

The jth component[3] of $\mathsf{T}_{\ell,r}$ is

$$\mathsf{T}_{\ell,r}[j] = \mathsf{Tev}_{g[\ell,r]+j,\ell,r} \,.$$

We will write a formula for $\mathsf{T}_{\ell,r}$.

Define the infinite vector E_s for $s \geq 1$ as

$$\mathsf{E}_s = \left(2^{s-1} - \sum_{i=0}^{s-2} \binom{s-1}{i} \,,\; 2^s - \sum_{i=0}^{s-2} \binom{s}{i} \,,\; 2^{s+1} - \sum_{i=0}^{s-2} \binom{s+1}{i} \,,\; \ldots \right) \,.$$

The jth component[4] of E_s is

$$\mathsf{E}_s[j] = 2^{s+j-1} - \sum_{i=0}^{s-2} \binom{s+j-1}{i} \,.$$

The first few vectors are

$$
\begin{array}{lllllll}
\mathsf{E}_1 & = & (& 1, & 2, & 4, & 8, & 16, & \ldots &) \\
\mathsf{E}_2 & = & (& 1, & 3, & 7, & 15, & 31, & \ldots &) \\
\mathsf{E}_3 & = & (& 1, & 4, & 11, & 26, & 57, & \ldots &)
\end{array}
$$

The leading components are always 1,

$$\mathsf{E}_s[0] = 1 \,.$$

Moreover, the components of the full set of vectors E_s is easily seen to satisfy a Pascal-type addition law

$$\mathsf{E}_{s+1}[j+1] = \mathsf{E}_{s+1}[j] + \mathsf{E}_s[j+1] \,.$$

---

[3] We start the vector index at 0, so $\mathsf{T}_{\ell,r} = \left(\mathsf{T}_{\ell,r}[0], \mathsf{T}_{\ell,r}[1], \mathsf{T}_{\ell,r}[2], \mathsf{T}_{\ell,r}[3], \ldots \right)$

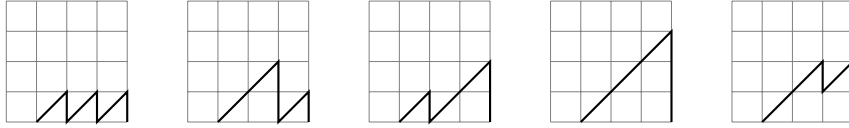[4] Again, the vector index is started at 0.

Figure 2: Possible paths from $(0,1)$ to $(-3,1)$. There are exactly $C_3 = 5$ possible paths, where C_3 is the third Catalan number. Our paths are equivalent to Dyck paths [27].

By Theorem 2, we have $\mathsf{E}_s[j] = \mathsf{Tev}_{s+j-1,0,s}$, so

$$\mathsf{T}_{0,r} = \mathsf{E}_r \,. \tag{6}$$

Our first result for $\ell \leq 0$ expresses $\mathsf{T}_{\ell,r}$ as a finite linear combination of the vectors E_s. The coefficients are non-negative integers and are given by a simple path counting formula.

The path counting occurs on the integer lattice $\mathbb{Z}^2$. We are only interested in the lattice points

$$\mathcal{A} = \{\, (\ell, r) \,|\, \ell \leq 0, r \geq 1 \,\} \subset \mathbb{Z}^2 \,. \tag{7}$$

Our paths start at the point $(0,1)$, must stay on the lattice points of $\mathcal{A}$ and take steps only by the vectors

$$\mathsf{U} = (0,1) \quad \text{and} \quad \mathsf{D} = (-1,-1) \,.$$

Let $\mathsf{P}(\ell, r)$ be the set of such paths from $(0,1)$ to (ℓ, r), see Figure 2.

Let $\gamma \in \mathsf{P}(\ell, r)$. The *index* $\mathsf{Ind}(\gamma)$ is the number of points of γ which meet the boundary $\partial \mathcal{A}$,

$$\partial \mathcal{A} = \{\, (\ell,1) \,|\, \ell \leq 0 \,\} \ \cup \ \{\, (0,r) \,|\, r \geq 1 \,\} \subset \mathcal{A} \,.$$

Theorem 4. *Let $\ell \leq 0$ and $r \geq 1$. Then,*

$$\mathsf{T}_{\ell,r} = \sum_{\gamma \in \mathsf{P}(\ell,r)} \mathsf{E}_{\mathsf{Ind}(\gamma)} \,.$$

Theorem 4 takes the simplest form in case $(\ell, r) = (0, r)$. The set $\mathsf{P}(0, r)$ contains a unique path[5] of index r. Theorem 4 then just recovers (6).

For all $\ell \leq 0$ and $r \geq 1$, the set $\mathsf{P}(\ell, r)$ is finite. Moreover, the index of a path $\gamma \in \mathsf{P}(\ell, r)$ is bounded by

$$\mathsf{Ind}(\gamma) \leq r - \ell + 1 \,.$$

The following result is then a consequence of Theorem 4.

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[5]The unique path in the $(\ell, 1) = (0, 1)$ case is degenerate and consists only of the single point $(0, 1)$.

Corollary 5. *Let $\ell \leq 0$ and $r \geq 1$. Then, $\mathsf{T}_{\ell,r}$ lies in the $\mathbb{Z}_{\geq 0}$-linear span of the vectors $\mathsf{E}_1, \ldots, \mathsf{E}_{r-\ell+1}$.*

The first examples[6] with $\ell < 0$ and $r = 1$ are:

$$\mathsf{T}_{-1,1} = \mathsf{E}_3, \quad \mathsf{T}_{-2,1} = 2\mathsf{E}_4, \quad \mathsf{T}_{-3,1} = 2\mathsf{E}_4 + 3\mathsf{E}_5, \quad \mathsf{T}_{-4,1} = 4\mathsf{E}_4 + 6\mathsf{E}_5 + 4\mathsf{E}_6.$$

In fact, if $\ell < 0$, then $\mathsf{T}_{\ell,r}$ lies in the span of $\mathsf{E}_3, \ldots, \mathsf{E}_{r-\ell+1}$ since every path in $\mathsf{P}(\ell, r)$ has index at least 3.

The leading coefficient of $\mathsf{T}_{\ell,1}$ for $\ell \leq 0$ has an alternative geometric interpretation: $\mathsf{T}_{\ell,1}[0]$ is the number of $g^1_{|\ell|+1}$'s carried by a general curve of genus $2|\ell|$. The above examples[7] show

$$\mathsf{T}_{-1,1}[0] = 1, \quad \mathsf{T}_{-2,1}[0] = 2, \quad \mathsf{T}_{-3,1}[0] = 5, \quad \mathsf{T}_{-4,1}[0] = 14,$$

which are recognizable as Catalan numbers. By the well-known counting of Dyck paths [10, 27],

$$\mathsf{T}_{\ell,1}[0] = |\mathsf{P}(\ell, 1)| = \frac{1}{|\ell| + 1} \binom{|2\ell|}{|\ell|},$$

which, of course, agrees with Castelnuovo's famous count of linear series [6].

The coefficients of the expansion of $\mathsf{T}_{\ell,r}$ in terms of E_s determined by Theorem 4 may be viewed as a refined Dyck path counting problem.[8] An exact solution in terms of binomials is presented in Section 5.3. For example, the coefficients of the expansion

$$\mathsf{T}_{\ell,1} = \sum_{s=3}^{-\ell+2} c^s_{\ell,1} \mathsf{E}_s$$

for $\ell < -1$ are[9]

$$c^s_{\ell,1} = \frac{(s-2)(s-3)}{|\ell| - 1} \binom{|2\ell| - s}{|\ell| + 2 - s}.$$

In fact, the summation over paths in Theorem 4 can be exactly evaluated to yield closed formulas for the Tevelev degrees.

Theorem 6. *The Tevelev degrees for $\ell \leq 0$ and $g \geq r - 2\ell - 1$ are*

• *for $g \geq 0$ and $r = 1$,*

$$\mathsf{T}_{g,\ell,1} = 2^g - 2\sum_{i=0}^{-\ell-2} \binom{g}{i} + (-\ell-2)\binom{g}{-\ell-1} + \ell\binom{g}{-\ell},$$

• *for $g \geq 0$ and $r > 1$,*

$$\mathsf{T}_{g,\ell,r} = 2^g - 2\sum_{i=0}^{-\ell-2} \binom{g}{i} + (-\ell+r-3)\binom{g}{-\ell-1} + (\ell-1)\binom{g}{-\ell} - \sum_{i=-\ell+1}^{r-\ell-2} \binom{g}{i}.$$

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[6] A table of values of $\mathsf{T}_{\ell,r}$ can be found in Section 5.

[7] Remember $\mathsf{E}_s[0] = 1$.

[8] See [10] for the study of various refined Dyck path counting problems.

[9] The coefficient formula for $\ell = -1$ and $s = 3$ is also valid if the 0's are cancelled.

The $r = 1$ formula of Theorem 6 can be viewed as the $r = 1$ specialization[10] of the $r > 1$ formula. In fact, the formulas for $\ell \geq 0$ of Theorems 1-3 can also be seen as specializations of the $r > 1$ formula of Theorem 6. The $r > 1$ formula of Theorem 6 therefore represents a complete calculation of the Tevelev degrees $\mathsf{T}_{g,\ell,r}$. A unified point of view for all ℓ is presented in Section 6.

1.5 Hurwitz cycles

Both Hurwitz numbers and Tevelev degrees are aspects of a more general question. Via the diagram

$$
\begin{array}{ccc}
& \overline{\mathcal{H}}_{g,d,n} & \\
{}^{\epsilon_g}\swarrow & & \searrow {}^{\epsilon_0} \\
\overline{\mathcal{M}}_{g,2g+2d-2+n} & & \overline{\mathcal{M}}_{0,2g+2d-2+n} \ ,
\end{array}
$$

we always have a map

$$\tau : \overline{\mathcal{H}}_{g,d,n} \to \overline{\mathcal{M}}_{g,2g+2d-2+n} \times \overline{\mathcal{M}}_{0,2g+2d-2+n} \ .$$

The dimension of the range of τ usually exceeds the dimension of the domain. A basic question here is to compute the push-forward of the fundamental class:

$$\tau_*[\overline{\mathcal{H}}_{g,d,n}] \in \mathsf{CH}^*\big(\overline{\mathcal{M}}_{g,2g+2d-2+n} \times \overline{\mathcal{M}}_{0,2g+2d-2+n}\big) \ . \tag{8}$$

It follows from the main result of [12] that the push-forward is given by tautological classes[11]

$$\tau_*[\overline{\mathcal{H}}_{g,d,n}] \in \mathsf{R}^*\big(\overline{\mathcal{M}}_{g,2g+2d-2+n}\big) \otimes \mathsf{R}^*\big(\overline{\mathcal{M}}_{0,2g+2d-2+n}\big) \ ,$$

but very few complete formulas are known.[12] Hurwitz numbers and Tevelev degrees give slivers of cohomological information about the push-forward (8).

1.6 Subsequent developments

Cavalieri, Markwig, and Ranganathan [7] have proposed a tropical calculation of the Tevelev degrees $\mathsf{T}_{g,\ell}$. Another calculation of the Tevelev degrees is given by Farkas and Lian in [13] via degenerations and the Schubert calculus. Farkas

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[10]Care has to be taken for the meaning of $r = 1$ specialization of the last sum.

[11]Formally, the results in [12] only show that the push-forward to $\overline{\mathcal{M}}_{g,2g+2d-2+n}$ is tautological. However, using the cellular structure of $\overline{\mathcal{M}}_{2g+2d-2+n}$, the Chow groups on the right hand side of (8) split as the tensor product of Chow groups of the factors. Combining the results of [12] with techniques from [19], each individual term in the tensor product decomposition of (8) can then be seen to be tautological.

[12]There are several other natural variations. We could consider more complicated ramification profiles of the Hurwitz covers. Another idea is to consider relative stable map spaces. The result along these lines which is closest to the spirit of question (8) is Pixton's formula for the double ramification cycle [17]. For further aspects of the study of Hurwitz moduli spaces and their associated cycles on the moduli spaces of curves, see [3, 18, 19, 22, 23, 25, 29].

and Lian analyse a different fiber of $\tau_{g,\ell,r}$ than we do, but in the end recover the same formulas.

More generally, Farkas and Lian consider an enumerative Tevelev degree for maps to $\mathbb{P}^n$ and calculate it in high degree cases. Motivated by their construction, a parallel Tevelev degree can be defined for all target varieties in Gromov-Witten theory using the virtual fundamental class of the moduli space of stable maps [5]. Calculations of the Gromov-Witten Tevelev degrees for flag varieties and complete intersections are obtained [5] by quantum cohomology techniques.

Together, the papers [13] and [5] place the study of Tevelev degrees in a much wider setting. The common theme is that the Tevelev degrees are surprisingly computable in closed form.

Acknowledgments

2 Tevelev degrees

Tevelev [28] arrives at the numbers Tev_g from a slightly different point of view. Below we describe his perspective and how it relates to definitions presented in Section 1.2.

To start, let $g \geq 0$ and $n = g + 3$. Fix a general curve

$$(C, p_1, \ldots, p_n) \in \mathcal{M}_{g,n}\,.$$

Then a line bundle $\mathcal{L}$ on C of degree $d = g + 1$ satisfies

$$\chi(C, \mathcal{L}) = h^0(C, \mathcal{L}) - h^1(C, \mathcal{L}) = g + 1 + 1 - g = 2$$

by Riemann-Roch. For a general such line bundle $\mathcal{L} \in \mathrm{Pic}^{g+1}(C)$, the first cohomology vanishes, so $\mathcal{L}$ has precisely two sections. For general $\mathcal{L}$, the two sections do not have a common zero and therefore define a degree d map

$$\varphi_{\mathcal{L}} : C \to \mathbb{P}^1 = \mathbb{P}(H^0(C, \mathcal{L})^\vee)\,.$$

Tevelev [28] constructs a rational *scattering amplitude map* Λ,

$$\Lambda : \mathrm{Pic}^{g+1}(C) \dashrightarrow \mathcal{M}_{0,n}, \quad \mathcal{L} \mapsto \left(\mathbb{P}^1, \varphi_{\mathcal{L}}(p_1), \ldots, \varphi_{\mathcal{L}}(p_n)\right). \qquad (9)$$

In [28, Definition 1.7], Λ is shown to be dominant and generically finite, and Tev_g is defined to be the degree of Λ. The result

$$\mathsf{Tev}_g = 2^g$$

is [28, Theorem 1.14].

To relate Tevelev's definition to the definition of Section 1.2, let $\mathcal{P}ic^d_{g,n}$ be the *universal Picard stack* over $\mathcal{M}_{g,n}$ parameterizing the data

$$(C, p_1, \ldots, p_n, \mathcal{L})$$

of a nonsingular, n-pointed genus g curve C together with a line bundle $\mathcal{L}$ of degree $d = g + 1$. The morphism

$$\mathcal{H}_{g,d,n} \to \mathcal{P}ic^d_{g,n},$$
$$\left(\pi : (C, p_1, \ldots, p_n) \to \mathbb{P}^1\right) \mapsto (C, p_1, \ldots, p_n, \mathcal{L} = \pi^*\mathcal{O}_{\mathbb{P}^1}(1))$$

is dominant and generically finite of degree $b!$, where

$$b = 2g - 2 + 2d = 4g$$

is the number of (simple) ramification points of a Hurwitz cover π. Indeed, for $\mathcal{L} \in \mathcal{P}ic^d_{g,n}$ general, the fiber will be the map $\pi = \varphi_{\mathcal{L}}$ together with the $b!$ many choices of ordering the b simple ramification points.

To conclude, consider the map

$$\tau_{g,d,n} : \mathcal{H}_{g,d,n} \to \mathcal{M}_{g,n} \times \mathcal{M}_{0,n}$$

from Section 1.2. For $(C, p_1, \ldots, p_n) \in \mathcal{M}_{g,n}$ general and every $\mathcal{L}$ in the fibre of the map Λ of (9) over a general point $(\mathbb{P}^1, q_1, \ldots, q_n) \in \mathcal{M}_{0,n}$, we have precisely $b!$ points in the fibre

$$\tau_{g,d,n}^{-1}((C, p_1, \ldots, p_n), (\mathbb{P}^1, q_1, \ldots, q_n)).$$

Hence, we obtain

$$\mathsf{Tev}_g = \frac{\deg(\tau_{g,d,n})}{(4g)!}$$

as defined in Section 1.2.

3   Recursion via fiber geometry

In this section, we prove a recursion satisfied by the Tevelev degrees (see Proposition 7 below). In the proof, we need to perform some intersection computations involving the Hurwitz stacks above. In the description of this computation, we need some information on the boundary stratification of the spaces of Hurwitz covers from [25, 18] which we recall below.

3.1 The boundary stratification of Hurwitz stacks

Recall that the space $\overline{\mathcal{H}}_{g,d,n,r}$ parameterizes tuples

$$\left(C \xrightarrow{\pi} D, p_1, \ldots, p_n \in C \right),$$

where π is a degree d admissible cover from the genus g curve C to the genus 0 curve D and $p_1, \ldots, p_n$ are smooth points of C such that $p_{n-r+1}, \ldots, p_n$ map to the same point under π. Let Γ, Γ' be the dual graphs of C, D, respectively. Then the combinatorics of the cover π can be described by a graph cover $\Gamma \to \Gamma'$. The data of such a cover is given by

- maps $V(\Gamma) \to V(\Gamma')$, $H(\Gamma) \to H(\Gamma')$ from the vertices/half-edges of Γ to those of Γ', corresponding to the action of π on the components and nodes of C,

- local degrees at vertices and half-edges of Γ, describing the degree of π on the components, and the ramification order at the nodes of C, respectively.

Associated to the graph cover $\Gamma \to \Gamma'$, there exists a gluing morphism

$$\overline{\mathcal{H}}_{(\Gamma,\Gamma')} \to \overline{\mathcal{H}}_{g,d,n,r} \tag{10}$$

parameterizing the closure of the locus in $\overline{\mathcal{H}}_{g,d,n,r}$ with dual graph cover $\Gamma \to \Gamma'$. Here, the space $\overline{\mathcal{H}}_{(\Gamma,\Gamma')}$ is a fibre product of spaces of Hurwitz covers (one for each vertex of Γ) *over* a product of moduli spaces of stable curves (one for each vertex of Γ'). The reason for this fibre product structure is that for any two vertices v_1, v_2 in Γ mapping to the same vertex v in Γ', the associated subcurves C_{v_1}, C_{v_2} of C must cover the *same* subcurve D_v of D. For more details see [25, Section 3.4] in the setting of Galois covers and [18, Section 2.3.4] in the setting of Hurwitz covers.

In the computation below, we will see that the pullback of a boundary stratum in $\overline{\mathcal{M}}_{g,n}$ under the map

$$\epsilon_g : \overline{\mathcal{H}}_{g,d,n,r} \to \overline{\mathcal{M}}_{g,n}$$

has an effective description via a disjoint union of gluing morphisms (10), see [18, Proposition 3.2]. For the combinatorial description of this disjoint union, we need to make precise the concept that a given stable graph Γ is a specialization of a graph Γ_0. This is encoded in the notion of a Γ_0-structure on Γ, which is specified by maps

$$\varphi_V : V(\Gamma) \twoheadrightarrow V(\Gamma_0) \text{ and } \varphi_H : H(\Gamma_0) \hookrightarrow H(\Gamma),$$

such that the graph obtained from Γ by contracting all edges formed by half-edges not in the image of φ_H is isomorphic to Γ_0 via the map φ_V. See [14, Appendix A] for a formal definition and further discussions of this notion.

Then, when pulling back the boundary stratum of $\overline{\mathcal{M}}_{g,n}$ associated to the stable graph Γ_0 under the map ϵ_g, we obtain the disjoint union of boundary strata $\overline{\mathcal{H}}_{(\Gamma,\Gamma')}$ such that Γ carries a Γ_0-structure satisfying that the composition

$$H(\Gamma_0) \xrightarrow{\varphi_H} H(\Gamma) \to H(\Gamma')$$

is surjective. However, instead of introducing these result for pullbacks of boundary strata under ϵ_g in full generality here, we recall the relevant formulas as they come up in the proof.

3.2 Recursive formulas for Tevelev degrees

The basic property satisfied by the Tevelev degree which is used in the proofs of Theorems 1-4 is the following recursion.

Proposition 7. *Let $g, r \geq 1$. Let $\ell \in \mathbb{Z}$ satisfy*

$$n[g, \ell] - r + 1 = g + 3 + 2\ell - r + 1 \ \geq \ 3\,.$$

Then, we have the recursion

$$\mathsf{Tev}_{g,\ell,r} = \mathsf{Tev}_{g-1,\ell,\max(1,r-1)} + \mathsf{Tev}_{g-1,\ell+1,r+1}\,. \tag{11}$$

Proof. Up to a combinatorial factor, $\mathsf{Tev}_{g,\ell,r}$ is defined as the degree of the map

$$\tau_{g,\ell,r} : \overline{\mathcal{H}}_{g,d[g,\ell],n[g,\ell],r} \to \overline{\mathcal{M}}_{g,n[\mathrm{g},\ell]} \times \overline{\mathcal{M}}_{0,n[g,\ell]-r+1}\,.$$

To prove the recursion (11), we compute the degree by analysing the fiber over a particular point

$$(C, D) \in \overline{\mathcal{M}}_{g,n[\mathrm{g},\ell]} \times \overline{\mathcal{M}}_{0,n[g,\ell]-r+1}\,.$$

The degree of $\tau_{g,\ell,r}$ is simply the degree of the zero cycle

$$\tau^*_{g,\ell,r}[(C,D)] \in \mathsf{CH}_0\left(\overline{\mathcal{H}}_{g,d[g,\ell],n[g,\ell],r}\right)\,.$$

The actual fiber $\tau^{-1}_{g,\ell,r}[(C,D)]$ will have excess dimension, so some care must be taken in the analysis.

- The pointed curve C in $\overline{\mathcal{M}}_{g,n[\mathrm{g},\ell]}$ is chosen to have the following form:
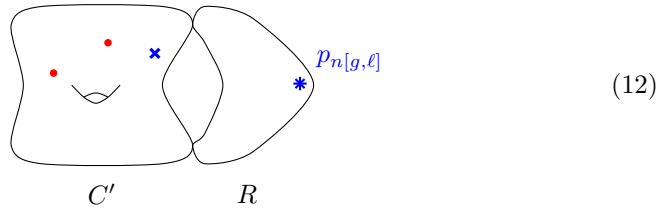
$$\tag{12}$$

The curve C consists of a general curve $C' \in \overline{\mathcal{M}}_{g-1,n[\mathrm{g},\ell]-1+2}$ attached at two points to a rational curve R. All the markings lie on C' except for the last marking $p_{n[g,\ell]}$ which lies on R. The marking $p_{n[g,\ell]}$ is the last of the distinguished subset of r markings.

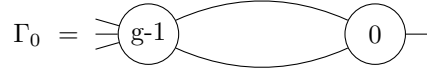• The pointed curve D in $\overline{\mathcal{M}}_{0,n[g,\ell]-r+1}$ is any general point (in particular, the curve D is nonsingular).

To compute the cycle $\tau^*_{g,\ell,r}[(C,D)]$, let

$$\Gamma_0 \quad = \quad \text{}$$

be the stable graph of the curve (12), and let

$$\xi_{\Gamma_0} : \overline{\mathcal{M}}_{\Gamma_0} = \overline{\mathcal{M}}_{g-1,n[\mathrm{g},\ell]-1+2} \times \overline{\mathcal{M}}_{0,3} \to \overline{\mathcal{M}}_{g,n[\mathrm{g},\ell]}$$

be the associated gluing map, sending the point $(C', R) \in \overline{\mathcal{M}}_{\Gamma_0}$ to the curve C. We consider the following commutative diagram:

$$
\begin{array}{ccc}
\coprod \overline{\mathcal{H}}_{(\Gamma,\Gamma')} & \longrightarrow & \overline{\mathcal{H}}_{g,d[g,\ell],n[g,\ell],r} \\
\downarrow & & \downarrow \\
\coprod \overline{\mathcal{M}}_{\widehat{\Gamma}_0} & \xrightarrow{\xi_{\widehat{\Gamma}_0}} & \overline{\mathcal{M}}_{g,n[\mathrm{g},\ell]+b[g,\ell]} \\
\downarrow & & \downarrow \\
\overline{\mathcal{M}}_{\Gamma_0} & \xrightarrow{\xi_{\Gamma_0}} & \overline{\mathcal{M}}_{g,n[\mathrm{g},\ell]}
\end{array}
\qquad (13)
$$

The upper right vertical arrow is the map remembering the domain of the admissible cover together with the $n[g,\ell]$ markings and the

$$b[g,\ell] = 2g - 2 + 2d[g,\ell] = 4g + 2\ell$$

simple ramification points. The bottom right vertical arrow is the map forgetting these extra markings, so the composition on the right is the familiar map ϵ_g. The disjoint union on the left side of the diagram is over all stable graphs $\widehat{\Gamma}_0$ that can be obtained by distributing $b[g,\ell]$ legs to the two vertices of Γ_0. The lower square in the diagram is then not quite Cartesian, but the disjoint union of the $\overline{\mathcal{M}}_{\widehat{\Gamma}_0}$ maps properly and birationally to the corresponding fibre product. This property will be sufficient for the intersection theoretic computations below (see [2, Lemma C.7] for a formal argument).

The upper square is a fibre diagram on the level of sets, as proven in [18, Proposition 3.2]. Here, the disjoint union is over boundary strata $\overline{\mathcal{H}}_{(\Gamma,\Gamma')}$ of $\overline{\mathcal{H}}_{g,d[g,\ell],n[g,\ell],r}$ which, as we saw in Section 3.1, parameterize maps from a curve with stable graph Γ mapping to a curve of stable graph Γ'. The index set of the disjoint union consists of isomorphism classes of $\widehat{\Gamma}_0$-structures on Γ such that the induced map

$$E(\widehat{\Gamma}_0) \to E(\Gamma) \to E(\Gamma')$$

is surjective. The lift of the upper fibre diagram to the level of schemes is described in [18, Proposition 3.3].

Our strategy is now as follows. To compute

$$\tau_{g,\ell,r}^*[(C,D)] = \epsilon_g^*[C] \cdot \epsilon_0^*[D]\,,$$

we consider $[C] = (\xi_{\Gamma_0})_*[(C',R)]$ and use the diagram (13) to decompose $\epsilon_g^*[C]$ into terms supported on the spaces $\overline{\mathcal{H}}_{(\Gamma,\Gamma')}$. Only three types of contributions survive after taking the product with $\epsilon_0^*[D]$. We begin by describing these three surviving cases and computing their contributions. Multiplicities and excess intersections appear.

Contribution 1. The Hurwitz cover degenerates as indicated in diagram (14) with the degrees of the map written in green.[13] The last r markings are in blue, and the last marking is starred.

$$(14)$$
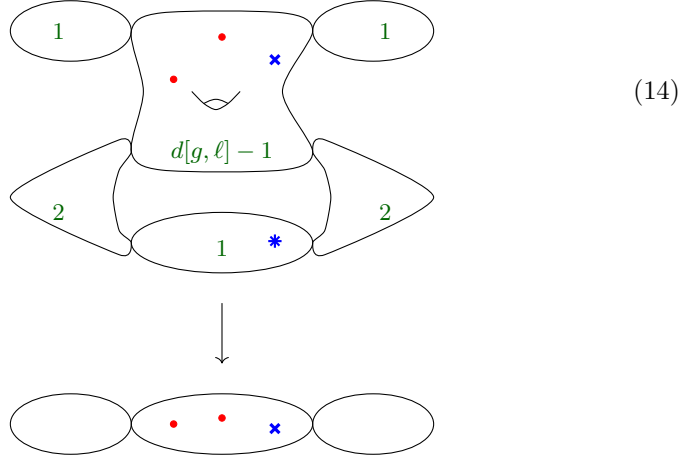
Each of the two rational components mapping with degree 2 must carry two of the ramification points. The total number of loci of the form (14) is therefore given by

$$\frac{1}{2}\binom{b[d,\ell]}{2,2,b[d,\ell]-4} = \frac{1}{8}\frac{b[d,\ell]!}{(b[d,\ell]-4)!}\,, \tag{15}$$

with the factor $1/2$ corresponding to the fact that the two pairs of branch points going to the outer components can be exchanged without changing the locus.

For $r > 1$, the locus is parameterized by the space

$$\overline{\mathcal{H}}_{(\Gamma,\Gamma')} = \overline{\mathcal{H}}_{g-1,d[g,\ell]-1,n[g,\ell]-1+2,r-1} \times \underbrace{\overline{\mathcal{H}}_{0,2,2,2} \times \overline{\mathcal{H}}_{0,2,2,2}}_{=\{\mathrm{pt}\}}\,. \tag{16}$$

---

[13]In the picture (14), the preimage of the rightmost component of the bottom curve consists of a single rational curve mapping with degree 2 and $d-2$ rational curves mapping with degree 1. Here, and in the following pictures, we only draw one representative of such rational components mapping with degree 1.

The unique point of $\overline{\mathcal{H}}_{0,2,2,2}$ is given by the map

$$(\mathbb{P}^1, 1, -1) \xrightarrow{z \mapsto z^2} \mathbb{P}^1 \,,$$

and the cover has no remaining automorphisms.

On the other hand, for $r = 1$, the position of marking $p_{n[g,\ell]}$ is no longer determined by the map and markings on the genus $g - 1$ component. If $r = 1$, let

$$\widehat{\epsilon}_0 : \overline{\mathcal{H}}_{g-1,d[g,\ell]-1,n[g,\ell]-1+2,r-1} \to \overline{\mathcal{M}}_{0,4g+2\ell-4+n[g,\ell]-1}$$

be the map remembering both the $4g + 2\ell - 4$ branch points and the images of the $n[g, \ell] - 1$ markings on the target of the cover. Then the locus fits into a diagram

$$
\begin{array}{ccc}
\overline{\mathcal{H}}_{(\Gamma,\Gamma')} & \xrightarrow{\hspace{3cm}} & \overline{\mathcal{M}}_{0,4g+2\ell-4+n[g,\ell]} \\
\downarrow & & \downarrow \\
\overline{\mathcal{H}}_{g-1,d[g,\ell]-1,n[g,\ell]-1+2,1} & \xrightarrow{\hspace{1.5cm}} & \overline{\mathcal{M}}_{0,4g+2\ell-4+n[g,\ell]-1}
\end{array}
\tag{17}
$$

where the right vertical arrow is the map forgetting the last marking.

Since, independent of r, the map (14) is unramified at all the nodes, we see from [18, Proposition 3.3] that the locus $\overline{\mathcal{H}}_{(\Gamma,\Gamma')}$ appears with multiplicity 1 in the upper fibre diagram (13). On the other hand, there are 8 different $\widehat{\Gamma}_0$-structures on the graph Γ of the domain curve in (14) that satisfy the assumptions of [18, Proposition 3.2]. Indeed, the image of the first edge of $\widehat{\Gamma}_0$ can be chosen freely among the 4 nonseparating edges of Γ, and then the second edge must go to one of the 2 edges on the other side. So overall, in the disjoint union in the top left of the diagram (13), there are

$$\frac{b[d, \ell]!}{(b[d, \ell] - 4)!}$$

many components that can contribute.

By the strategy explained above, we see that the contribution to the degree of $\tau_{g,\ell,r}^*[(C, D)]$ coming from each of the components is given by the degree of the map

$$\overline{\mathcal{H}}_{(\Gamma,\Gamma')} \to \underbrace{\overline{\mathcal{M}}_{g-1,n[g,\ell]+1}}_{=\overline{\mathcal{M}}_{\Gamma_0}} \times \overline{\mathcal{M}}_{0,n[g,\ell]-r+1}
\tag{18}$$

• For $r > 1$, the locus $\overline{\mathcal{H}}_{(\Gamma,\Gamma')}$ is given by (16) and the above map fits as the top horizontal arrow into a commutative diagram

$$
\begin{array}{ccc}
\overline{\mathcal{H}}_{g-1,d[g,\ell]-1,n[g,\ell]+1,r-1} & \xrightarrow{\hspace{1.5cm}} & \overline{\mathcal{M}}_{g-1,n[g,\ell]+1} \times \overline{\mathcal{M}}_{0,n[g,\ell]-r+1} \\
\downarrow & & \downarrow \\
\overline{\mathcal{H}}_{g-1,d[g,\ell]-1,n[g,\ell]-1,r-1} & \xrightarrow{\hspace{1.5cm}} & \overline{\mathcal{M}}_{g-1,n[g,\ell]-1} \times \overline{\mathcal{M}}_{0,n[g,\ell]-r+1}
\end{array}
$$

mapping birationally onto the corresponding fibre product. So the degree of the top map is the same as the degree of the bottom map. Using

$$d[g, \ell] - 1 = d[g - 1, \ell], \quad n[g, \ell] - 1 = n[g - 1, \ell],$$

the bottom map is precisely $\tau_{g-1,\ell,r-1}$.

• For $r = 1$, we must distinguish two more cases:

⋄ For $n[g, \ell] \geq 4$, the map (18) appears as the top horizontal arrow in the commutative diagram

$$
\begin{array}{ccc}
\overline{\mathcal{H}}_{(\Gamma,\Gamma')} & \longrightarrow & \overline{\mathcal{M}}_{g-1,n[g,\ell]+1} \times \overline{\mathcal{M}}_{0,n[g,\ell]} \\
\downarrow & & \downarrow \\
\overline{\mathcal{H}}_{g-1,d[g,\ell]-1,n[g,\ell]-1,1} & \longrightarrow & \overline{\mathcal{M}}_{g-1,n[g,\ell]-1} \times \overline{\mathcal{M}}_{0,n[g,\ell]-1}
\end{array}
$$

mapping birationally to the associated fibre product, and so the degree of (18) is given by the degree of the bottom map $\tau_{g-1,\ell,1}$.

⋄ For $n[g, \ell] = 3$, the map (18) factors through the left vertical map in (17). Since the vertical map drops dimension by 1, we know that the morphism (18) can no longer be dominant and thus yields a zero contribution. The vanishing here is compatible with our original definition, since $\mathsf{Tev}_{g-1,\ell,1}$ is defined to be zero since $n[g - 1, \ell] = 2$.

Overall, we see that the total part of the degree arising from Contribution 1 is given by

$$\frac{b[d, \ell]!}{(b[d, \ell] - 4)!} \deg \tau_{g-1,\ell,\max(1,r-1)} = b[d, \ell]! \cdot \mathsf{Tev}_{g-1,\ell,\max(1,r-1)}.$$

Contribution 2. The second type of locus that can contribute parameterizes covers of the form illustrated in diagram (19).
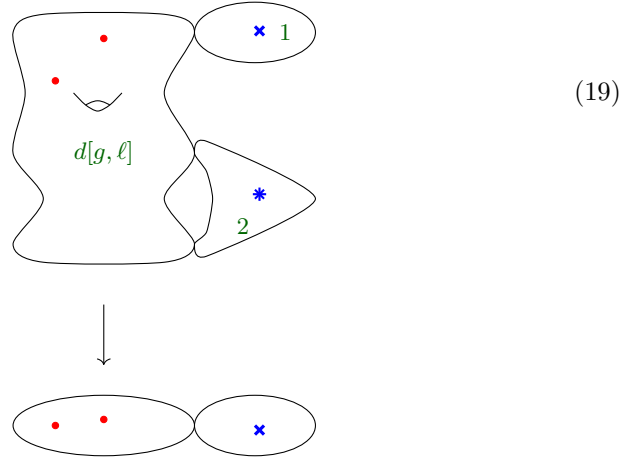
$$(19)$$

The rational component mapping with degree 2 must carry two of the ramification points. The total number of loci of the form (19) is given by

$$\binom{b[d, \ell]}{2} = \frac{1}{2} \frac{b[d, \ell]!}{(b[d, \ell] - 2)!} \,. \tag{20}$$

The locus $\overline{\mathcal{H}}_{(\Gamma, \Gamma')}$ is parameterized by the space

$$\overline{\mathcal{H}}_{g-1, d[g, \ell], n[g, \ell]+1, r+1} \times \overline{\mathcal{H}}_{0,2,3,2} \,. \tag{21}$$

The $r+1$ index in (21) is explained by the following basic important observation: all $r+1$ nodes of the genus $g-1$ domain component connecting to rational curves containing blue markings map to the same point (which is the node of the range curve).

The space $\overline{\mathcal{H}}_{0,2,3,2}$, parameterizes double covers

$$f : (\mathbb{P}^1, p, z, z', q_1, q_2) \to \mathbb{P}^1 \tag{22}$$

such that $f(z) = f(z')$ and such that q_1, q_2 are ramification points of the cover. The natural map

$$\delta : \overline{\mathcal{H}}_{0,2,3,2} \to \overline{\mathcal{M}}_{0,4}$$

sending the point (22) to

$$(\mathbb{P}^1, f(p), f(z), f(q_1), f(q_2)) \in \overline{\mathcal{M}}_{0,4}$$

has degree 2, corresponding to the two choices z, z' of preimage of $f(z)$.

By the same argument as before, the loci (19) appear with multiplicity 1 in the upper fibre diagram (13). On the other hand, there are 2 $\widehat{\Gamma}_0$-structures on the graph Γ of the domain curve in (19), corresponding to the two choices of sending the edges of $\widehat{\Gamma}_0$ to the two nonseparating edges e, e' of Γ. However, these two $\widehat{\Gamma}_0$-structures are isomorphic (since the cover $\Gamma \to \Gamma'$ has an automorphism exchanging the two edges e, e'). So overall, in the disjoint union in the top left of the diagram (13) there are

$$\frac{1}{2} \cdot \frac{b[d, \ell]!}{(b[d, \ell] - 2)!}$$

many components that can contribute.

However, a new phenomenon (not seen in Contribution 1) appears here: a dimension count shows that the loci (21) only have codimension 1 in their ambient space, whereas the gluing map ξ_{Γ_0} had codimension 2. The excess class is given by

$$-\psi_w \otimes 1 - 1 \otimes \psi_z$$

where w, z are the preimages of one of the nonseparating nodes in (19), see [25, Section 4.2] and [18, Proposition 3.3]. The class ψ_z on $\overline{\mathcal{H}}_{0,2,3,2}$ is the pullback of ψ_2 on $\overline{\mathcal{M}}_{0,4}$ under the map δ above. We calculate

$$\int_{\overline{\mathcal{H}}_{0,2,3,2}} \psi_z = \deg(\delta) \cdot \int_{\overline{\mathcal{M}}_{0,4}} \psi_2 = 2 \cdot 1 = 2 \,.$$

The contribution to the degree of $\tau^*_{g,\ell,r}[(C,D)]$ coming from each locus above is given by the degree of the excess class capped with the preimage of a point under the map

$$\overline{\mathcal{H}}_{g-1,d[g,\ell],n[g,\ell]+1,r+1} \times \overline{\mathcal{H}}_{0,2,3,2} \to \overline{\mathcal{M}}_{g-1,n[g,\ell]+1} \times \overline{\mathcal{M}}_{0,n[g,\ell]-r+1}.$$

Since the above map does not depend on the factor $\overline{\mathcal{H}}_{0,2,3,2}$ in the domain, the only nonzero contribution can come from the term $-1 \otimes \psi_z$. After eliminating the factor $\overline{\mathcal{H}}_{0,2,3,2}$ by integrating $-1 \otimes \psi_z$ (and obtaining a factor -2 from the degree of $-\psi_z$), we are left with the degree of $\tau_{g-1,\ell+1,r+1}$.

The total part of the degree arising from Contribution 2 is given by

$$\frac{1}{2} \cdot \frac{b[d,\ell]!}{(b[d,\ell]-2)!} \cdot (-2) \cdot \deg \tau_{g-1,\ell+1,r+1} = -b[d,\ell]! \cdot \mathsf{Tev}_{g-1,\ell+1,r+1}.$$

Contribution 3. The third type of locus that can contribute parameterizes covers of the form illustrated in diagram (23).
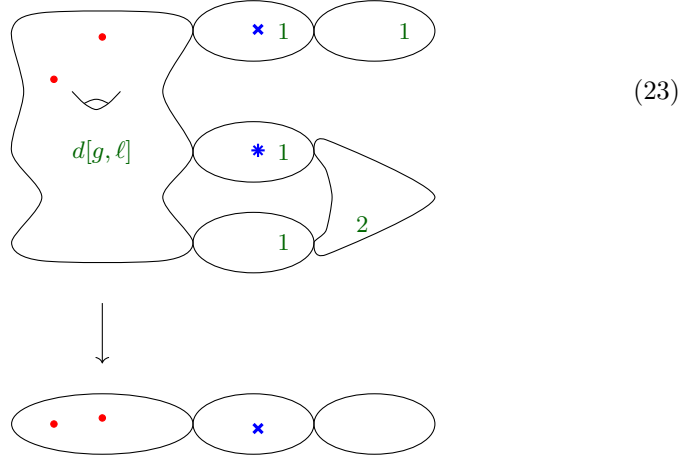
(23)

The rational component mapping with degree 2 must carry two of the ramification points. The total number of loci of the form (19) is given by

$$\binom{b[d,\ell]}{2} = \frac{1}{2} \frac{b[d,\ell]!}{(b[d,\ell]-2)!}. \tag{24}$$

The locus $\overline{\mathcal{H}}_{(\Gamma,\Gamma')}$ is parameterized by the space

$$\overline{\mathcal{H}}_{g-1,d[g,\ell],n[g,\ell]+1,r+1} \times \underbrace{\overline{\mathcal{H}}_{0,2,2,2}}_{=\{\mathrm{pt}\}}. \tag{25}$$

Again the multiplicity is 1, but there are 4 non-isomorphic $\widehat{\Gamma}_0$-structures on the graph Γ: one of the two edges of $\widehat{\Gamma}_0$ must go to the edge of Γ connecting the

genus $g-1$ vertex to the vertex containing the last marking. The other can go to either one of the two edges incident to the vertex with the degree 2 map. So overall, in the disjoint union in the top left of the diagram (13) there are

$$2\frac{b[d,\ell]!}{(b[d,\ell]-2)!}$$

components that can contribute.

By the strategy explained above, the contribution to the degree of $\tau_{g,\ell,r}^*[(C,D)]$ coming from each locus above is given by the degree of the map

$$\overline{\mathcal{H}}_{g-1,d[g,\ell],n[g,\ell]+1,r+1} \to \overline{\mathcal{M}}_{g-1,n[g,\ell]+1} \times \overline{\mathcal{M}}_{0,n[g,\ell]-r+1}$$

The total part of the degree arising from Contribution 3 is given by

$$\frac{1}{2} \cdot 2 \cdot \frac{b[d,\ell]!}{(b[d,\ell]-2)!} \cdot 2 \cdot \deg \tau_{g-1,\ell+1,r+1} = 2b[d,\ell]! \cdot \mathsf{Tev}_{g-1,\ell+1,r+1} \,.$$

Exclusion of other components.

To conclude, we must prove that Contributions 1, 2 and 3 are the only loci in the diagram (13) that can contribute.

Let $\overline{\mathcal{H}}_{(\Gamma,\Gamma')}$ be a locus in (13) dominating the moduli space

$$\overline{\mathcal{M}}_{g-1,n[g,\ell]+1} \times \overline{\mathcal{M}}_{0,n[g,\ell]-r+1} \,.$$

There must then be a vertex $v_{g-1} \in V(\Gamma)$ such that the corresponding factor of $\overline{\mathcal{H}}_{(\Gamma,\Gamma')}$ parameterizes covers from a genus $g-1$ curve. Since the $n[g,\ell]-1$ first markings will go to the vertex v_{g-1} after forgetting the ramification points and stabilizing, their images in Γ' will similarly stabilize to a unique vertex v_0. Therefore, in the cover $\Gamma \to \Gamma'$, the vertex v_{g-1} maps to v_0. Next, we observe that the map

$$\overline{\mathcal{H}}_{(\Gamma,\Gamma')} \to \overline{\mathcal{M}}_{g-1,n[g,\ell]+1} \times \overline{\mathcal{M}}_{0,n[g,\ell]-r+1} \tag{26}$$

only depends upon the factors of $\overline{\mathcal{H}}_{(\Gamma,\Gamma')}$ lying over the vertex v_0. Indeed, the map to $\overline{\mathcal{M}}_{g-1,n[g,\ell]+1}$ only depends on the factor for v_{g-1}. On the other hand, since the stabilization of Γ', after forgetting the branch points, consists of the single vertex v_0, the factors of $\overline{\mathcal{H}}_{(\Gamma,\Gamma')}$ over v_0 determine the component of the map (26) to $\overline{\mathcal{M}}_{0,n[g,\ell]-r+1}$.

Since the factors over v_0 form a finite cover of the moduli space $\overline{\mathcal{M}}_{0,n(v_0)}$ associated to v_0, the valence $n(v_0)$ must satisfy

$$n(v_0) - 3 \geq \dim \overline{\mathcal{M}}_{g-1,n[g,\ell]+1} \times \overline{\mathcal{M}}_{0,n[g,\ell]-r+1} = b[g,\ell] + n[g,\ell] - r - 4$$

in order for the map (26) to be dominant.

A short computation shows that the valence condition is only possible in one of the following cases:

(i) Γ' has two other vertices w_1, w_2, both of valence 3,

(ii) Γ' has one other vertex w, of valence 4,

(iii) Γ' has one other vertex w, of valence 3.

We further observe that for a simple branch point at the vertex v_0 whose ramification point is *not* on the component associated to v_{g-1}, the position of the branch point does not affect the image under the map (26). Therefore, in cases (i) and (ii) all the simple ramification points over v_0 are located at the vertex v_{g-1}, whereas in case (iii) at most one of them can be on a separate component.

Finally, we observe that for every leaf of the graph Γ' which is not equal to v_0 (so one of the vertices w, w_1, w_2 above), the leaf must contain at least two of the simple branch points. Indeed, by the condition that there is a $\widehat{\Gamma}_0$ structure on $\Gamma \to \Gamma'$, there must be a circular path from v_{g-1} to itself in Γ whose edges surject to the edges of Γ'. Thus over a leaf there must be a vertex with at least two edges adjacent, which needs at least two ramification points on that vertex.

Case (i) There are two possibilities Γ'_1 and Γ'_2 for the graph Γ'.

$$\Gamma'_1 \;=\; \boxed{v_0} \text{———} \boxed{w_1} \text{———} \boxed{w_2}$$
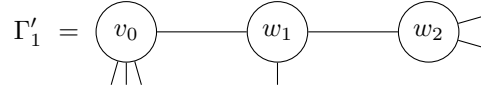
By the general comments above, the two markings at w_2 must be branch points, and we see that there are at least $b[g, \ell] - 3$ simple ramification points at v_{g-1}. The configuration is only possible if the degree at the vertex v_{g-1} remains at $d[g, \ell]$, otherwise the number of ramification points would drop by at least 4. The marking at w_1 is then forced to be the image of the marking $p_{n[g,\ell]}$, since otherwise this marking would be contained in the genus $g-1$ component of the curve, contradicting the shape (12) we have fixed before. We conclude that only the pair (Γ, Γ'_1) of Contribution 3 has the specified shape.

$$\Gamma'_2 \;=\; \boxed{w_1} \text{———} \boxed{v_0} \text{———} \boxed{w_2}$$

The markings at w_1, w_2 belonging to branch points, so we have precisely $b[g, \ell] - 4$ branch points at v_0. Then the component v_{g-1} must have degree $d[g, \ell] - 1$, so that there is an additional vertex of genus 0 over v_0 mapping with degree 1. We are then forced to be in the case of Contribution 1.

Case (ii) The shape of the graph Γ' is as follows:

$$\Gamma' \;=\; \boxed{v_0} \text{———} \boxed{w}$$

21
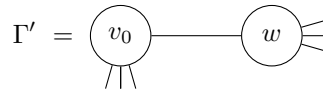
Since there are at least $b[g, \ell] - 3$ simple branch points at v_0, the component v_{g-1} must map with full degree $d[g, \ell]$. Therefore, one of the three markings at w is forced to be the image of marking $p_{n[g,\ell]}$. Moreover, the point $p_{n[g,\ell]}$ must lie on the unique loop in Γ which goes through the vertex over w containing the two ramification points. We are in the case of Contribution 2.

Case (iii) The graph Γ' has the following shape:
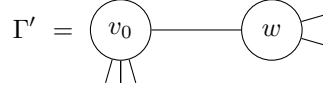
The markings at w are branch points, so the remaining $b[g, \ell] - 2$ branch points are at v_0. As we have seen before, in case (iii), at most one of the associated ramification points can be on a component not equal to v_{g-1}, which still forces $b[g, \ell] - 3$ ramification points on v_{g-1}. The component v_{g-1} must map with full degree $d[g, \ell]$. Hence we arrive at a contradiction, since the marking $p_{n[g,\ell]}$ lies over v_0 and thus in v_{g-1}. The curve can not have the shape (12).

Contributions 1,2, and 3 are therefore the only nonvanishing terms in the intersection. After summing the three contributions to the degree of $\tau_{g,\ell,r}$ and dividing by $b[d, \ell]!$, we obtain the recursion stated in the Proposition. $\qquad\square$

4 Proof of Theorems 1-3 for $\ell \geq 0$

4.1 The genus 0 case

The recursion of Proposition 7 reduces the calculation of the degree of $\tau_{g,\ell,r}$ to the genus $g = 0$ case. Indeed, for $n[g, \ell] - r + 1 \geq 3$, we can apply the recursion and lower the genus and, for $n[g, \ell] - r + 1 < 3$, $\mathsf{Tev}_{g,\ell,r}$ is defined to be 0. In genus 0, since

$$1 \leq r \leq d[0, \ell] = \ell + 1$$

by (3), we must have $\ell \geq 0$. The Tevelev degrees in genus 0 for $\ell \geq 0$ are determined by the following result.

Proposition 8. *For $\ell \geq 0$ and $1 \leq r \leq \ell + 1$ we have*

$$\mathsf{Tev}_{0,\ell,r} = 1.$$

Proof. Let $(\mathbb{P}^1, p_1, \ldots, p_n)$ be a genus 0 curve with n distinct markings, for

$$n = n[0, \ell] = 3 + 2\ell \geq 3\,.$$

Let $\mathcal{M}_{d,r}$ be the moduli space of maps

$$\pi : (\mathbb{P}^1, p_1, \ldots, p_n) \to \mathbb{P}^1$$

of degree $d[0, \ell] = 1 + \ell$ satisfying

$$\pi(p_{n-r+1}) = \ldots = \pi(p_n) = [1, 0] \in \mathbb{P}^1 \, .$$

The moduli space $\mathcal{M}_{d,r}$ is the nonsingular $n - r$ dimensional quasi-projective subvariety

$$\mathcal{M}_{d,r} \subset \mathbb{P}(H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d)) \oplus H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d)))$$

defined by pairs of sections (s_0, s_1) of $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d))$ satisfying:

(i) s_0 and s_1 have no common zeros,

(ii) s_1 vanishes at the r points $p_{n-r+1}, \ldots, p_n$.

Conditions (ii) are linear, so $\mathcal{M}_{d,r} \subset \mathbb{P}^{2d+1-r} = \mathbb{P}^{n-r}$.

The tangent space of $\mathcal{M}_{d,r}$ at $[\pi] \in \mathcal{M}_{d,r}$ is

$$\mathsf{Tan}_\pi = H^0(\mathbb{P}^1, \pi^*(\mathsf{Tan}_{\mathbb{P}^1})(-p_{n-r+1} - \ldots - p_n)) \, .$$

The degree of the line bundle $\pi^*(\mathsf{Tan}_{\mathbb{P}^1})(-p_{n-r+1} - \ldots - p_n)$ is

$$2(1 + \ell) - r = n - r - 1 \, .$$

Since the points $p_1, \ldots, p_{n-r}$ impose independent conditions on sections of $\pi^*(\mathsf{Tan}_{\mathbb{P}^1})(-p_{n-r+1} - \ldots - p_n)$, the differential of the evaluation map

$$\mathsf{ev} : \mathcal{M}_{d,r} \to (\mathbb{P}^1)^{n-r} \, , \qquad [\pi] \mapsto (\pi(p_1), \ldots, \pi(p_{n-r}))$$

is surjective.

Let $x_1 \ldots, x_{n-r} \in \mathbb{P}^1$ be general points. For $1 \leq i \leq n - r$, let

$$\mathsf{H}_i \subset \mathcal{M}_{d,r}$$

be the locus of maps π satisfying

$$\pi(p_i) = x_i \, .$$

The closure $\overline{\mathsf{H}}_i \subset \mathbb{P}^{2d+n-r}$ is a linear space of codimension 1. Since

$$\mathsf{ev}^{-1}(x_1, \ldots, x_{n-r}) \subset \mathcal{M}_{d,r} \tag{27}$$

is both transverse and linear, $\mathsf{ev}^{-1}(x_1, \ldots, x_{n-r})$ is either empty or a reduced point. Since the image of ev is dense, $\mathsf{ev}^{-1}(x_1, \ldots, x_{n-r})$ must be a reduced point.[14]

Finally, a dimension count shows the single reduced point (27) corresponds to a map with only simple ramification points. Therefore, $\mathsf{Tev}_{0,\ell,r} = 1$. $\qquad \square$

---

[14]Alternatively, the empty case is easily seen to be impossible. The hyperplane closures have to intersect in the linear space

$$\overline{\mathcal{M}}_{d,r} = \mathbb{P}^{n-r} \subset \mathbb{P}(H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d)) \oplus H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d))) \, .$$

Using the standard description of

$$\partial \overline{\mathcal{M}}_{d,r} = \overline{\mathcal{M}}_{d,r} \setminus \mathcal{M}_{d,r}$$

via maps of lower degrees, the intersection of $\overline{\mathsf{H}}_1 \cap \ldots \cap \overline{\mathsf{H}}_{n-r}$ is easily seen to be disjoint from $\partial \overline{\mathcal{M}}_{d,r}$ using the genericity of $q_1, \ldots, q_{n-r}$.

4.2   Induction argument

We are ready to completely calculate the Tevelev degrees in case $\ell \geq 0$. Because of the form of the recursion (11), we consider all the cases with $\ell \geq 0$ at once instead of proving Theorems 1, 2, and 3 separately.

Theorem. *For $g \geq 0$, $\ell \geq 0$ and $1 \leq r \leq \ell + 1 + g$, we have*

$$\mathsf{Tev}_{g,l,r} = \begin{cases} 2^g - \sum_{j=0}^{r-2} \binom{g}{j} & \text{for } \ell = 0, \\ \mathsf{Tev}_{g,0,\max(r-l,1)} & \text{for } \ell \leq r, \\ 2^g & \text{for } \ell \geq r. \end{cases}$$

Proof. We proceed by induction on g. The $g = 0$ case is covered by Proposition 8. Suppose $g > 0$.

Step 1 We consider the case $\ell \geq r \geq 1$. If we start with $\ell \geq r$, the same inequality persists for the recursion, and we have

$$\mathsf{Tev}_{g,\ell,r} = \mathsf{Tev}_{g-1,\ell,\max(1,r-1)} + \mathsf{Tev}_{g-1,\ell+1,r+1} = 2^{g-1} + 2^{g-1} = 2^g \, .$$

Step 2 We consider case $0 \leq \ell \leq r$. We have already checked the case $\ell = r$. Suppose $\ell < r$. If we start with $\ell < r$, then the non-strict inequality $\ell \leq r$ persists in the first step of the recursion. We have

$$\mathsf{Tev}_{g,\ell,r} = \mathsf{Tev}_{g-1,\ell,\max(r-1,1)} + \mathsf{Tev}_{g-1,\ell+1,r+1}$$

$$= 2^{g-1} - \sum_{j=0}^{\max(r-3-l,-1-\ell)} \binom{g-1}{j} + 2^{g-1} - \sum_{j=0}^{r-2-l} \binom{g-1}{j}$$

$$= 2^g - \sum_{j=0}^{r-2-l} \left[\binom{g-1}{j-1} + \binom{g-1}{j} \right]$$

$$= 2^g - \sum_{j=0}^{r-2-l} \binom{g}{j} \, ,$$

where in the first equality we are using the inductive step applied to g.

The proof is complete. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

4.3   Lines in projective space

By the formulas of Theorems 2 and 3, we have

$$\mathsf{Tev}_{g,\ell,g+1+\ell} = 1 \, . \tag{28}$$

The Tevelev degree is defined via the map

$$\tau_{g,\ell,g+1+\ell} : \overline{\mathcal{H}}_{g,g+1+\ell,g+3+2\ell,g+1+\ell} \to \overline{\mathcal{M}}_{g,g+3+2\ell} \times \overline{\mathcal{M}}_{0,3+\ell} \, .$$

There is a simple geometric reason for the Tevelev degree to be 1 here.

Let $(C, p_1, \ldots, p_{g+3+2\ell})$ be a general nonsingular pointed curve. The last $g + 1 + \ell$ markings (associated to r) determine a line bundle L on C. Consider the complete linear series $\mathbb{P}(H^0(C, L))$ of dimension $1 + \ell$ by Riemann-Roch.

- There is a distinguished element

$$\zeta = [p_{3+\ell} + \ldots + p_{g+3+2\ell}] \in \mathbb{P}(H^0(C, L)).$$

- There are $2 + \ell$ distinguished hyperplanes

$$\mathsf{H}_1, \ldots, \mathsf{H}_{2+\ell} \subset \mathbb{P}(H^0(C, L))$$

determined by the points $p_1, \ldots, p_{2+\ell} \in C$.

- There is a projective space $\mathbb{P}(\mathsf{Tan}_\zeta)$ of dimension ℓ parameterizing lines

$$\mathsf{P} \subset \mathbb{P}(H^0(C, L))$$

passing through ζ.

For a general such line P, we obtain a $3 + \ell$ pointed curve of genus 0,

$$(\mathsf{P}, \zeta, \mathsf{P} \cap \mathsf{H}_1, \ldots, \mathsf{P} \cap \mathsf{H}_{2+\ell}).$$

The associated rational map

$$\mathbb{P}(\mathsf{Tan}_\zeta) \dashrightarrow \mathcal{M}_{0,3+\ell} \tag{29}$$

is easily seen to be birational (of degree 1). The degree is $\mathsf{Tev}_{g,\ell,g+1+\ell}$.

The map (29) has the flavour of a Gale transformation. In the $\ell = 1$ case, a direct connection to the Gale transformation can be made. Whether the birationality of (29) can be explained for all ℓ in terms of the Gale transformation is a question left to the reader.

5 Path counting

5.1 Notation

We consider here the case $\ell \leq 0$ and prove Theorem 4. Before presenting the proof, we briefly recall the notation of Section 1.4.

For $\ell \leq 0$ and $r \geq 1$, we have

$$n[g, \ell] = 3 + g + 2\ell.$$

After imposing $n[g, \ell] - (r - 1) \geq 3$, we obtain the condition

$$g \geq r - 2\ell - 1.$$

Following the notation of Section 1.4,

$$g[\ell, r] = r - 2\ell - 1.$$

We also have

$$d[g[\ell, r], \ell] = g[\ell, r] + 1 + \ell = r - \ell > 0.$$

Define the infinite vectors $\mathsf{T}_{\ell, r}$ and E_r by

$$\mathsf{T}_{\ell, r}[j] = \mathsf{Tev}_{g[\ell, r] + j, \ell, r} \quad \text{and} \quad \mathsf{E}_r[j] = 2^{r+j-1} - \sum_{i=0}^{r-2} \binom{r+j-1}{i}$$

for $j \geq 0$. We set

$$\mathsf{T}_{\ell, r}[j] = 0 \quad \text{and} \quad \mathsf{E}_r[j] = 0$$

for $j < 0$. For all $j \in \mathbb{Z}$ and $r \geq 1$, we have

$$\mathsf{E}_{r+1}[j+1] = \mathsf{E}_{r+1}[j] + \mathsf{E}_r[j+1]. \tag{30}$$

5.2 Proof of Theorem 4 for $\ell \leq 0$

By Theorems 2 and 3 for $\ell \geq 0$, we already know

$$\mathsf{T}_{0, r} = \mathsf{E}_r = \sum_{\gamma \in \mathsf{P}(0, r)} \mathsf{E}_{\mathsf{Ind}(\gamma)}$$

holds.

We prove the formula of Theorem 4 for the coefficient

$$\mathsf{T}_{\ell, r}[j] \quad \text{for} \quad j \geq 0$$

by induction on $g = g[\ell, r] + j$. When $g[\ell, r] + j = 0$, we must have $(\ell, r) = (0, 1)$ and $j = 0$. So the base of the induction is established.

Suppose now $g[\ell, r] + j > 0$. The proof of the induction step is split into two cases:

Case 1. Suppose $r = 1$ and $\ell \leq 0$.

• If $j = 0$, we have

$$\begin{aligned}
\mathsf{T}_{\ell, 1}[0] &= \mathsf{Tev}_{g[\ell, 1], \ell, 1} \\
&= \mathsf{Tev}_{g[\ell, 1] - 1, \ell + 1, 2} \\
&= \mathsf{T}_{\ell + 1, 2}[0] \\
&= \sum_{\gamma \in \mathsf{P}(\ell + 1, 2)} \mathsf{E}_{\mathsf{Ind}(\gamma)}[0] \\
&= \sum_{\gamma \in \mathsf{P}(\ell, 1)} \mathsf{E}_{\mathsf{Ind}(\gamma)}[0].
\end{aligned}$$

The second equality follows from (11) since

$$\mathsf{Tev}_{g[\ell,1]-1,\ell,1} = \mathsf{T}_{\ell,1}[-1] = 0 \,.$$

The last equality follows from $\mathsf{E}_{\mathsf{Ind}(\gamma)}[0] = 1$ for all $\gamma \in \mathsf{P}(\ell,1) \cup \mathsf{P}(\ell+1,2)$ and the fact that $\mathsf{P}(\ell+1,2)$ and $\mathsf{P}(\ell,1)$ are in bijection.

• If $j > 0$, we have

$$
\begin{aligned}
\mathsf{T}_{\ell,1}[j] &= \mathsf{Tev}_{g[\ell,1]+j,\ell,1} \\
&= \mathsf{Tev}_{g[\ell,1]+j-1,\ell,1} + \mathsf{Tev}_{g[\ell,1]+j-1,\ell+1,2} \\
&= \mathsf{T}_{\ell,1}[j-1] + \mathsf{T}_{\ell+1,2}[j] \\
&= \sum_{\gamma\in\mathsf{P}(\ell,1)} \mathsf{E}_{\mathsf{Ind}(\gamma)}[j-1] + \sum_{\gamma\in\mathsf{P}(\ell+1,2)} \mathsf{E}_{\mathsf{Ind}(\gamma)}[j] \\
&= \sum_{\gamma\in\mathsf{P}(\ell,1)} \mathsf{E}_{\mathsf{Ind}(\gamma)}[j-1] + \sum_{\gamma\in\mathsf{P}(\ell,1)} \mathsf{E}_{\mathsf{Ind}(\gamma)-1}[j] \\
&= \sum_{\gamma\in\mathsf{P}(\ell,1)} \mathsf{E}_{\mathsf{Ind}(\gamma)}[j] \,.
\end{aligned}
$$

In the fifth equality, we use the natural bijection between $\mathsf{P}(\ell+1,2)$ and $\mathsf{P}(\ell,1)$ given by completing a path that ends in $(\ell+1,2)$ to a path that ends $(\ell,1)$ by stepping with D. In the last equality, we use (30).

Case 2. Suppose $r > 1$ and $\ell \leq 0$.

For all $j \geq 0$, we have a similar chain of equalities:

$$
\begin{aligned}
\mathsf{T}_{\ell,r}[j] &= \mathsf{Tev}_{g[\ell,r]+j,\ell,r} \\
&= \mathsf{Tev}_{g[\ell,r]+j-1,\ell,r-1} + \mathsf{Tev}_{g[\ell,r]+j-1,\ell+1,r+1} \\
&= \mathsf{T}_{\ell,r-1}[j] + \mathsf{T}_{\ell+1,r+1}[j] \\
&= \sum_{\gamma\in\mathsf{P}(\ell,r-1)} \mathsf{E}_{\mathsf{Ind}(\gamma)}[j] + \sum_{\gamma\in\mathsf{P}(\ell+1,r+1)} \mathsf{E}_{\mathsf{Ind}(\gamma)}[j] \\
&= \sum_{\gamma\in\mathsf{P}(\ell,r)} \mathsf{E}_{\mathsf{Ind}(\gamma)}[j] \,,
\end{aligned}
$$

which completes the induction step. $\qquad\square$

5.3   Refined Dyck path counting

For $\ell \leq 0$ and $r \geq 1$, we can write

$$\mathsf{T}_{\ell,r} = \sum_{s=1}^{r-\ell+1} c_{\ell,r}^s \mathsf{E}_s \tag{31}$$

by Theorem 4 and Corollary 5. By definition,

$$\mathsf{E}_1[j] = 2^j \,.$$

ℓ \ r	1	2	3	4	5
-1	E_3	$2\mathsf{E}_3$	$2\mathsf{E}_3 + \mathsf{E}_4$	$2\mathsf{E}_3 + \mathsf{E}_4 + \mathsf{E}_5$	$2\mathsf{E}_3 + \mathsf{E}_4 + \mathsf{E}_5 + \mathsf{E}_6$
-2	$2\mathsf{E}_4$	$2\mathsf{E}_3 + 3\mathsf{E}_4$	$4\mathsf{E}_3 + 4\mathsf{E}_4 + \mathsf{E}_5$	$6\mathsf{E}_3 + 5\mathsf{E}_4 + 2\mathsf{E}_5 + \mathsf{E}_6$	$8\mathsf{E}_3 + 6\mathsf{E}_4 + 3\mathsf{E}_5 + 2\mathsf{E}_6 + \mathsf{E}_7$
-3	$2\mathsf{E}_4 + 3\mathsf{E}_5$	$4\mathsf{E}_3 + 6\mathsf{E}_4 + 4\mathsf{E}_5$	$10\mathsf{E}_3 + 11\mathsf{E}_4 + 6\mathsf{E}_5 + \mathsf{E}_6$	$18\mathsf{E}_3 + 17\mathsf{E}_4 + 9\mathsf{E}_5 + 3\mathsf{E}_6 + \mathsf{E}_7$	$28\mathsf{E}_3 + 24\mathsf{E}_4 + 13\mathsf{E}_5 + 6\mathsf{E}_6 + 3\mathsf{E}_7$
-4	$4\mathsf{E}_4 + 6\mathsf{E}_5 + 4\mathsf{E}_6$	$10\mathsf{E}_3 + 15\mathsf{E}_4 + 12\mathsf{E}_5 + 5\mathsf{E}_6$	$28\mathsf{E}_3 + 32\mathsf{E}_4 + 21\mathsf{E}_5 + 8\mathsf{E}_6 + \mathsf{E}_7$	$56\mathsf{E}_3 + 56\mathsf{E}_4 + 34\mathsf{E}_5 + 14\mathsf{E}_6 + 4\mathsf{E}_7 + \mathsf{E}_8$	$96\mathsf{E}_3 + 88\mathsf{E}_4 + 52\mathsf{E}_5 + 24\mathsf{E}_6 + 10\mathsf{E}_7 + 4\mathsf{E}_8 + \mathsf{E}_9$
-5	$10\mathsf{E}_4 + 15\mathsf{E}_5 + 12\mathsf{E}_6 + 5\mathsf{E}_7$	$28\mathsf{E}_3 + 42\mathsf{E}_4 + 36\mathsf{E}_5 + 20\mathsf{E}_6 + 6\mathsf{E}_7$	$84\mathsf{E}_3 + 98\mathsf{E}_4 + 70\mathsf{E}_5 + 34\mathsf{E}_6 + 10\mathsf{E}_7 + \mathsf{E}_8$	$180\mathsf{E}_3 + 186\mathsf{E}_4 + 122\mathsf{E}_5 + 58\mathsf{E}_6 + 20\mathsf{E}_7 + 5\mathsf{E}_8 + \mathsf{E}_9$	$330\mathsf{E}_3 + 315\mathsf{E}_4 + 198\mathsf{E}_5 + 97\mathsf{E}_6 + 40\mathsf{E}_7 + 15\mathsf{E}_8 + 5\mathsf{E}_9 + \mathsf{E}_{10}$

Table 1: Values of $\mathsf{T}_{\ell,r}$ for $\ell < 0$

For $s \geq 2$, define the vectors E'_s by

$$\mathsf{E}'_s[j] = \mathsf{E}_s[j] - 2^{s-1}\mathsf{E}_1[j].$$

Then, $\mathsf{E}'_s[j]$ a polynomial in j of degree $s - 2$. The vectors E_s are therefore independent, and the coefficients $c^s_{\ell,r}$ in (31) are unique.

By Theorem 4, the coefficient $c^s_{\ell,r}$ is the number of paths $\gamma \in \mathsf{P}(\ell, r)$ of index s and therefore defines a refined Dyck path counting problem. The solution is given by the following result.

Proposition 9. *Let $\ell \leq 0$ and $r \geq 1$.*

■ *If $\ell = 0$ and $s \in \mathbb{Z}$, then[15] $c^s_{0,r} = \delta^s_r$.*

■ *If $\ell = -1$ and $s \in \mathbb{Z}$, then $c^s_{-1,r} = \delta^{s-\delta_{r,1}}_{r+1}$.*

■ *If $\ell < -1$ and $3 \leq s \leq r - \ell + 1$, then*

$$c^s_{\ell,r} = \frac{(s-2)(s+r-4)}{|\ell| - 2 + r}\binom{|2\ell| + r - s - 1}{|\ell| + 2 - s} + \binom{|2\ell| + r - s - 1}{|\ell| - 1} - \binom{|2\ell| + r - s - 1}{|\ell| + r - 2}$$

and $c^s_{\ell,r} = 0$ otherwise.

Before starting the proof, we require additional notation and an auxiliary result. For $(u, v), (u', v') \in \mathcal{A}$ and $k \in \mathbb{Z}$, let

$$\mathcal{D}(k; u, v, u', v')$$

be the set of paths γ in $\mathcal{A}$ from (u, v) to (u', v') which take steps U and D and meet the $r = 1$ axis

$$\{(\ell, 1) \mid \ell \leq 0\} \subset \mathcal{A}$$

exactly k times. Let $d(k; u, v, u', v') = |\mathcal{D}(k; u, v, u', v')|$.

---

[15] Here, δ^s_r is the Kronecker delta.

Lemma 10. *Let $(u, v), (u', v') \in \mathcal{A}$. Let $t, j, k \in \mathbb{Z}$ satisfy $1 \leq t < k$, $u' \leq u$ and $j \leq -u$. The following equalities then hold:*

(i) $d(k; u + j, v, u' + j, v') = d(k; u, v, u', v')$,

(ii) $d(k; 0, 1, u', v') = d(k; 0, v', u' - v' + 1, 1)$,

(iii) $d(k; u, v, u', v') = d(k - t; u, v + t, u', v')$,

(iv) $d(k; 0, 1, u', 1) = \binom{-2u'-k}{-u'-k+1} \frac{k-1}{-u'}$.

Proof. Identity (i) is trivial. Identities (ii)-(iv) have been proven, for example, in [26] by B. Scott. We sketch here his proof adapted to our notation.

• Identity (ii). Every Dyck path in $\mathcal{A}$ from $(0, 1)$ to (u', v') can be written as a sequence of $-u'$ letters D and $-u' + v' - 1$ letters U. For any such sequence, changing every D with an U and every U with a D yields a Dyck path in $\mathcal{A}$ from $(0, v')$ and $(u' - v' + 1, 1)$. Moreover, the number of intersections with the $r = 1$ axis of the old and the new paths are the same. Therefore, we have bijection between $\mathcal{D}(k; 0, 1, u', v')$ and $\mathcal{D}(k; 0, v', u' - v' + 1, 1)$.

• Identity (iii). Let $\gamma \in \mathcal{D}(k; u, v, u', v')$. For $1 \leq i \leq t$, let $(\ell_i, 1)$ be the first t points at which γ meets the $r = 1$ axis. Here,

$$\ell_1 < \ldots < \ell_t .$$

Each of the points $(\ell_i, 1)$ is immediately followed by a step U in γ. Removing these t steps U and translating γ upwards t units yields a path in $\mathcal{D}(k - t; u, v + t, u', v')$. The operation is a bijection.

• Identity (iv). Using Identity (iii), we have

$$d(k; 0, 1, u', 1) = d(1; 0, k, u', 1)$$
$$= \binom{-2u' - k}{-u' - k + 1} - \binom{-2u' - k}{-u'}$$
$$= \binom{-2u' - k}{-u' - 1} \frac{k - 1}{-u'} .$$

The second equality is by directly counting the paths using a standard reflection trick (see [8, p.22]). □

Proof of Proposition 9 . For $\ell = 0, -1$ the result is trivial, so we assume $\ell < -2$. We write

$$c_{\ell,r}^s = |\{\gamma \in \mathsf{P}(\ell, r) : \mathsf{Ind}(\gamma) = s\}|$$
$$= \sum_{h=1}^{s-1} |\{\gamma \in \mathsf{P}(\ell, r) : |\gamma \cap \{\ell = 0\}| = h + 1 \text{ and } |\gamma \cap \{r = 1\}| = s - h\}|$$
$$= \sum_{h=1}^{s-1} d(s - h - 1; -1, h, \ell, r).$$

By Lemma 10, for $h \in \{1, ..., s-2\}$, the following chain of equalities holds:

$$
\begin{aligned}
d(s-h-1; -1, h, \ell, r) &= d(s-2; -1, 1, \ell, r) \\
&= d(s-2; 0, 1, \ell+1, r) \\
&= d(s-2; 0, r, \ell+2-r, 1) \\
&= d(s+r-3; 0, 1, \ell+2-r, 1) \\
&= \binom{|2\ell|+r-s-1}{|\ell|-s+2} \frac{s+r-4}{|\ell|-2+r}.
\end{aligned}
$$

For $h = s-1$, a simple direct count yields

$$
\begin{aligned}
d(0; -1, s-1, \ell, r) &= d(0; 0, s-1, \ell+1, r) \\
&= \binom{|2\ell|+r-s-1}{|\ell|-1} - \binom{|2\ell|+r-s-1}{|\ell|-2+r}
\end{aligned}
$$

which is the number of allowed paths from $(0, s-2)$ to $(\ell+1, r-1)$.
The result follows by summing all the terms. $\qquad\square$

5.4   Fixed j

An interesting direction is to compute $\mathsf{T}_{\ell,r}[j]$ for fixed j in terms of ℓ and r.
The simplest example is Castelnuovo's formula for $\ell < 0$:

$$
\mathsf{T}_{\ell,1}[0] = |\mathsf{P}(\ell,1)| = \frac{1}{|\ell|+1} \binom{|2\ell|}{|\ell|}.
$$

Proposition 11. *For $\ell < 0$, we have*

$$
\mathsf{T}_{\ell,1}[1] = |\mathsf{P}(\ell,1)| + \frac{3}{2|\ell|+1} \binom{2|\ell|+1}{|\ell|-1} + \frac{4(2|\ell|-1)(2|\ell|+1)(\binom{2|\ell|-2}{|\ell|-1} - \binom{2|\ell|-2}{|\ell|}))}{(|\ell|+1)(|\ell|+2)}.
$$

Proof. The case $l = -1$ can be checked by hand, so we assume $l < -1$. Using Theorem 4, we have

$$
\mathsf{T}_{\ell,1}[1] = \sum_{\gamma \in \mathsf{P}(\ell,1)} \mathsf{E}_{\mathsf{Ind}(\gamma)}[1] = |\mathsf{P}(\ell,1)| + \sum_{\gamma \in \mathsf{P}(\ell,1)} \mathsf{Ind}(\gamma).
$$

For $\gamma \in \mathsf{P}(\ell,1)$, define a *return step* of γ to be a step $\mathsf{D} = (-1,-1)$ of γ ending in the subset of $\mathcal{A}$ with $r = 1$. Let $r(\gamma)$ to be the number of return steps in γ. Then,

$$
\mathsf{Ind}(\gamma) = r(\gamma) + |\gamma \cap \{\ell = 0\}|. \tag{32}
$$

By Section 6.6 of [10], the first term of (32) is

$$
\sum_{\gamma \in \mathsf{P}(\ell,1)} r(\gamma) = \frac{3}{2|\ell|+1} \binom{2|\ell|+1}{|\ell|-1}.
$$

30

The second term of (32) is

$$\sum_{\gamma \in \mathsf{P}(\ell,1)} |\gamma \cap \{\ell = 0\}|$$

$$= \sum_{k=2}^{|\ell|+1} k \left[\binom{2|\ell| - (k-1)}{|\ell|} - \binom{2|\ell| - (k-1)}{|\ell|+1} - \left(\binom{2|\ell| - k}{|\ell|} - \binom{2|\ell| - k}{|\ell|+1} \right) \right]$$

$$= \sum_{k=2}^{|\ell|+1} k \left[\binom{2|\ell| - k}{|\ell| - 1} - \binom{2|\ell| - k}{|\ell|} \right]$$

$$= \frac{4(2|\ell| - 1)(2|\ell| + 1)(\binom{2|\ell|-2}{|\ell|-1} - \binom{2|\ell|-2}{|\ell|})}{(|\ell| + 1)(|\ell| + 2)} .$$

After summing the terms, we obtain the result. $\qquad\square$

6 Proof of Theorem 6

As discussed at the end of Section 1.4, the formulas of Theorem 6 cover the Tevelev degrees $\mathsf{T}_{g,\ell,r}$ in *all* cases. The master result which specializes to Theorem 6 and also covers the $\ell \geq 0$ cases takes the following form.

Theorem 12. *Let $g \geq 0$, $\ell \in \mathbb{Z}$, and $r \geq 1$.*

■ *If $g < r - 2\ell - 1$, then*
$$\mathsf{Tev}_{g,\ell,r} = 0 . \tag{33}$$

■ *If $g \geq r - 2\ell - 1$, then the Tevelev degrees are*

- *for $r = 1$,*

$$\mathsf{Tev}_{g,\ell,r} = 2^g - 2 \sum_{i=0}^{-\ell-2} \binom{g}{i} + (-\ell - 2)\binom{g}{-\ell - 1} + \ell \binom{g}{-\ell}, \tag{34}$$

- *for $r > 1$,*

$$\mathsf{Tev}_{g,\ell,r} = 2^g - 2 \sum_{i=0}^{-\ell-2} \binom{g}{i} + (-\ell + r - 3)\binom{g}{-\ell - 1} + (\ell - 1)\binom{g}{-\ell} - \sum_{i=-\ell+1}^{r-\ell-2} \binom{g}{i} . \tag{35}$$

Proof. We prove the complete statment by induction on g. The main tool is the recursion of Proposition 7.

We start with the genus 0 case. For $g = 0$, $\ell \geq 0$, and $1 \leq r \leq \ell + 1$, we have

$$\mathsf{Tev}_{0,l,r} = 1$$

by Proposition 8. In all other $g = 0$ cases,

$$\mathsf{Tev}_{0,l,r} = 0 .$$

31

For $\ell < 0$, the Theorem predicts $\mathsf{Tev}_{0,l,r} = 0$ by (33). For $\ell \geq 0$ and $r = 1$, formula (34) specializes to

$$1 + \ell \binom{0}{\ell} = 1.$$

Similarly, for $\ell \geq 0$ and $r > 1$, formula (35) specializes to

$$1 + (\ell - 1)\binom{g}{-\ell} - \sum_{i=-\ell+1}^{r-\ell-2} \binom{g}{i} = \begin{cases} 1 - 1 = 0 & \text{for } \ell = 0 \\ 1 - \sum_{i=-\ell+1}^{r-\ell-2} \delta_{i,0} & \text{for } \ell > 0. \end{cases}$$

In the $\ell > 0$ case, the expression vanishes if and only if $0 \leq r - \ell - 2$ and returns 1 otherwise, completing the verification of the case $g = 0$.

Let $g > 0$. Then, $\mathsf{Tev}_{g,\ell,r} = 0$ if

$$n[g, \ell] - r + 1 = g + 3 + 2\ell - r + 1 \ < \ 3.$$

Equivalently

$$g < r - 2\ell - 1,$$

recovering condition (33). If $g \geq r - 2\ell - 1$, we have the recursion

$$\mathsf{Tev}_{g,\ell,r} = \mathsf{Tev}_{g-1,\ell,\max(1,r-1)} + \mathsf{Tev}_{g-1,\ell+1,r+1} \tag{36}$$

by Proposition 7.

After substituting the $g - 1$ formulas (by induction), $\mathsf{Tev}_{g,\ell,1}$ is given by

$$2^{g-1} - 2 \sum_{i=0}^{-\ell-2} \binom{g-1}{i} + (-\ell + 1 - 3)\binom{g-1}{-\ell-1} + \ell\binom{g-1}{-\ell}$$

$$+2^{g-1} - 2 \sum_{i=0}^{-\ell-3} \binom{g-1}{i} + (-\ell + 1 - 3)\binom{g-1}{-\ell-2} + \ell\binom{g-1}{-\ell-1} - \underbrace{\sum_{i=-\ell}^{1-\ell-2} \binom{g-1}{i}}_{=0}.$$

A straightforward check shows that the horizontally aligned terms combine to the terms in (34) by the universal formula

$$\binom{a-1}{b-1} + \binom{a-1}{b} = \binom{a}{b} \quad \text{for } a, b \in \mathbb{Z}.$$

The check uses the formal manipulation

$$\sum_{i=0}^{-\ell-3} \binom{g-1}{i} = \sum_{i=0}^{-\ell-3} \binom{g-1}{i} + \underbrace{\binom{g-1}{-1}}_{=0} = \sum_{i=-1}^{-\ell-3} \binom{g-1}{i}$$

$$= \sum_{i=0}^{-\ell-2} \binom{g-1}{i-1}.$$

Similarly, in case $r > 1$, recursion (36) shows that $\mathsf{Tev}_{g,\ell,r}$ is given by

$$2^{g-1} - 2\sum_{i=0}^{-\ell-2}\binom{g-1}{i} + (-\ell+r-4)\binom{g-1}{-\ell-1} + (\ell-1)\binom{g-1}{-\ell} - \sum_{i=-\ell+1}^{r-\ell-3}\binom{g-1}{i}$$

$$+2^{g-1} - 2\sum_{i=0}^{-\ell-3}\binom{g-1}{i} + (-\ell+r-3)\binom{g-1}{-\ell-2} + \ell\binom{g-1}{-\ell-1} - \sum_{i=-\ell}^{r-\ell-2}\binom{g-1}{i}$$

The first two summands of each row combine to the corresponding terms in (35) as before.

To combine the middle two summand pairs, we transfer a term $\binom{g-1}{-l-1}$ from the second to the first row. For the last summation, we use

$$\sum_{i=-\ell+1}^{r-\ell-3}\binom{g-1}{i} + \sum_{i=-\ell}^{r-\ell-2}\binom{g-1}{i}$$

$$= \sum_{i=-\ell+1}^{r-\ell-3}\binom{g-1}{i} + \sum_{i=-\ell}^{r-\ell-3}\binom{g-1}{i} + \binom{g-1}{r-\ell-2}$$

$$= \sum_{i=-\ell+1}^{r-\ell-2}\binom{g-1}{i} + \sum_{i=-\ell}^{r-\ell-3}\binom{g-1}{i}$$

$$= \sum_{i=-\ell+1}^{r-\ell-2}\binom{g-1}{i} + \binom{g-1}{i-1}$$

$$= \sum_{i=-\ell+1}^{r-\ell-2}\binom{g}{i}.$$

We obtain precisely the last term of (35), concluding the proof. $\qquad\square$

If $g = -2\ell$, $\ell < 0$, and $r = 1$, Theorem 12 yields

$$\mathsf{Tev}_{-2\ell,\ell,1} = 2^{|2\ell|} - 2\sum_{i=0}^{|\ell|-2}\binom{|2\ell|}{i} + (|\ell|-2)\binom{|2\ell|}{|\ell|-1} - |\ell|\binom{|2\ell|}{|\ell|},$$

which equals the Catalan number $C_{|\ell|}$ as required by Castelnuovo's count.

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