K3 surfaces:

Curves, sheaves, Moduli

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I will wander over several different aspects of the study of K3 surfaces

The work of many people will be discussed.

Since K3 surfaces are almost impossibly beautiful. We will start with a picture:

3d print of a Kummer K3

Curve Counting

$X = S \times E \qquad c_1(X) = 0$

K3 surface

Elliptic curve

Gromov-Witten curve counting well defined:

- Reduced deformation theory of maps $C \xrightarrow{f} S$

$$\mathrm{Obs} = H^1(f^*T_S) \overset{\omega}{\cong} H^1(f^*\Omega_S) \xrightarrow{df} H^1(\omega_C) \cong \mathbb{C}$$

Reduced Obs $\subset$ Obs is the kernel of the composition

Bryan-Leung 1997

- E has translation symmetry, so curves occur in E-families.

Oberdieck-P 2014

We count families.

Well defined Gromov-Witten Count

$$N^{\bullet}_{g,\underbrace{h,d}}$$

genus

Curve class

$$(\beta, d) \in H_2(S \times E)$$

primitive on S, $\beta^2 = 2h-2$

degree on E

$$N(u,q,\tilde{q}) = \sum_{g \in \mathbb{Z}} \sum_{h \geq 0} \sum_{d \geq 0} N^{\bullet}_{g,h,d}\, u^{2g-2} q^{h-1} \tilde{q}^{d-1}$$

Theorem (Oberdieck-Pixton 2017):

$$\mathcal{N}(u, q, \tilde{q}) = -\frac{1}{\chi_{10}}$$

Igusa cusp form

predicted in 1999

Katz-Klemm-Vafa

χ_{10} is a weight 10 Siegel modular form

$$\chi_{10}(\Omega), \quad \Omega = \begin{pmatrix} \tau & z \\ z & \tilde{\tau} \end{pmatrix} \in \mathbb{H}_2$$

$$u = 2\pi z, \quad q = \exp(2\pi i \tau), \quad \tilde{q} = \exp(2\pi i \tilde{\tau})$$

$g=0, d=0 \Rightarrow$ Yau-Zaslow formula 1995

$d=0 \Rightarrow$ Katz-Klemm-Vafa formula

Spinning Black holes paper 1999

The Igusa cusp form $\chi_{10}(\Omega)$ is a weight 10 Siegel modular form on

genus 2

$$\Omega = \begin{pmatrix} \tau & z \\ z & \widetilde{\tau} \end{pmatrix} \in \mathbb{H}_2 \,,$$

$Sp(4,\mathbb{Z})$

where $\tau, \widetilde{\tau} \in \mathbb{H}_1$ lie in the Siegel upper half plane, $z \in \mathbb{C}$, and

$$\mathrm{Im}(z)^2 < \mathrm{Im}(\tau)\mathrm{Im}(\tilde{\tau}) \,.$$

Let $u = 2\pi z$. Define:

$$p = \exp(iu), \quad q = \exp(2\pi i \tau), \quad \tilde{q} = \exp(2\pi i \tilde{\tau}).$$

$\chi_{10}(\Omega)$ is a function of p, q, $\tilde{q}$.

Eisenstein $E_{2k}(q)$

Define the Jacobi theta function by

$$F(z,\tau) = u \exp\left(\sum_{k\geq 1} (-1)^k \frac{B_{2k}}{2k(2k!)} E_{2k} u^{2k} \right).$$

Define the Weierstrass $\wp$ function by

$$\wp(z,\tau) = -\frac{1}{u^2} + \sum_{k\geq 2} (-1)^k (2k-1) \frac{B_{2k}}{(2k)!} E_{2k} u^{2k-2}.$$

Define the coefficients $c(m)$ by

$$-24\wp(z,\tau)F(z,\tau)^2 = \sum_{n\geq 0} \sum_{k\in\mathbb{Z}} c(4n-k^2) p^k q^n.$$

Igusa cusp form $\chi_{10}(\Omega)$ following Gritsenko - Nikulin is

$$\chi_{10}(\Omega) = pq\tilde{q} \prod_{(k,h,d)} (1 - p^k q^h \tilde{q}^d)^{c(4hd-k^2)},$$

where the product is over all $k \in \mathbb{Z}$ and $h, d \geq 0$ satisfying one of:

- $h > 0$ or $d > 0$,
- $h = d = 0$ and $k < 0$.

The proof uses a lot of different ideas

- Study of elliptic fibrations

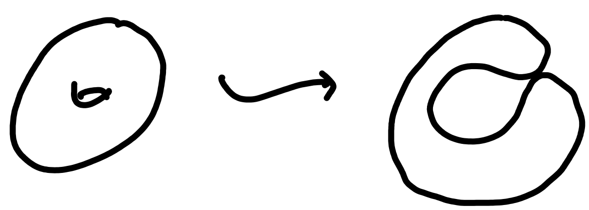
Stable pairs (sheaf counting) Wall crossing

$$\begin{array}{c} S \times E \\ \downarrow \\ S \end{array}$$

Jacobi forms

Oberdieck - Junliang Shen 2016

- GW / Stable pairs Correspondence

-  degeneration

Double ramification cycle formula of Pixton

Janda P Pixton Zvonkine 2016

- New holomorphic anomaly equations for S, E

Open direction: imprimitive classes

Complete conjecture Oberdieck-P 2015

Even in genus 0, the answer is subtle:

Klemm
Maulik
P
Scheidegger
2010

$$GW^{K3}_{0,d\beta} = \sum_{k|d} \left(\frac{d}{k}\right)^3 GW^{K3}_{0,\beta_k}$$

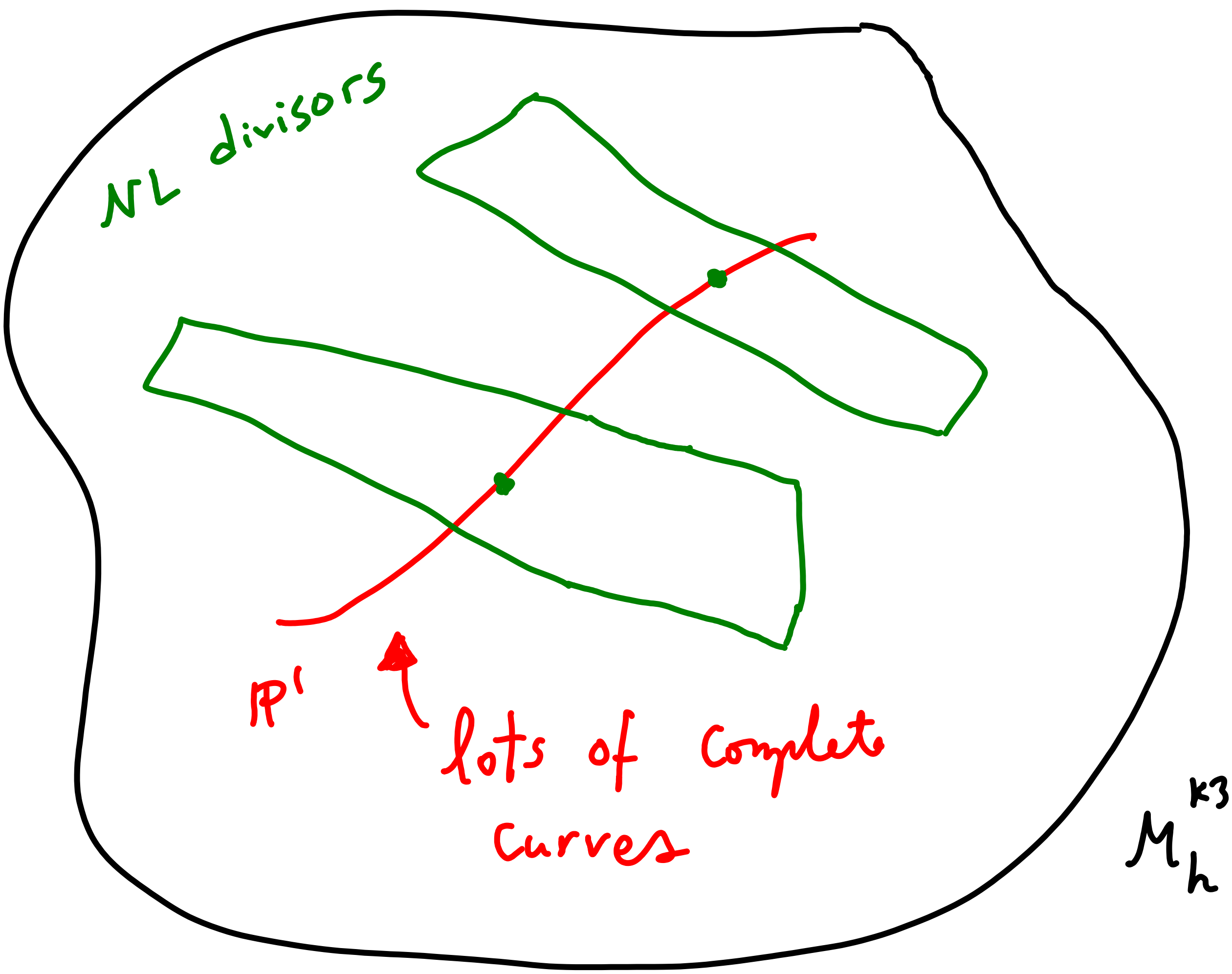
primitive

primitive

$\beta_k^2 = k^2 \beta^2$

The only known approach is by going to the moduli of K3 surfaces

Let $\mathcal{M}_h^{K3}$ be the moduli space of quasipolarized K3 surfaces of degree $2h-2$ $(h>1)$



The idea is that a complete curve

$$C \subset \mathcal{M}_h^{K3}$$

$$\Updownarrow$$

K3 fibered 3-fold $\begin{array}{c} X \\ \downarrow \pi \\ C \end{array}$

GW/NL Correspondence:

Maulik-P 2007

GW theory of X in fiber classes $=$

$\left(\text{GW theory of K3 surfaces} \right) \star \left(\text{intersections of } C \text{ with NL divisors} \right)$

determined by Mirror Symmetry in genus 0

STU Model

$\Downarrow$ Solve in genus 0

classical geometry determined in terms of modular forms

Borcherds, Kudla-Millson

STU model has simplest NL theory

Move to the geometry of $\mathcal{M}_h^{K3}$

Conjecture (Maulik-P 2007):

NL divisors generate $\mathrm{Pic}(\mathcal{M}_h^{K3}) \otimes \mathbb{Q}$.

Proven by Bergeron, Zhiyuan Li, Millson, Moeglin 2014

Shimura variety techniques

Opens the study of tautological classes on $\mathcal{M}_h^{K3}$.

But what are the tautological classes for the moduli of K3s?

There has been a lot of work on tautological classes on the moduli of curves $\overline{M}_g^{\text{Curves}}$:

$$R^*(\overline{M}_g^{\text{curves}}) \subset CH^*(\overline{M}_g^{\text{curves}})$$

What is the parallel construction?

$$R^*(M_h^{K3}) \overset{\text{def?}}{\subset} CH^*(M_h^{K3})$$

Idea (A) : $R_A^*(M_h^{K3}) \subset CH^*(M_h^{K3})$

the $\mathbb{Q}$-linear span of the classes of all NL subvarieties

Perhaps analogous to boundary strata in $\overline{M}_g^{\text{curves}}$.

But we know from $\overline{M}_g^{\text{curves}}$ that there are interior classes ψ_i and κ_j

The parallel construction is easy to imagine for M_h^{k3}:

quasi polarization

Let Λ be a lattice with $H \in \Lambda$

$$M_\Lambda \xrightarrow{i_\Lambda} M_h^{k3}$$

lattice quasi-polarised

generalization of NL divisors

Consider the universal family

$$\begin{array}{c} \mathcal{S} \\ \pi \downarrow \\ M_\Lambda \end{array}$$

Carries universal line bundles associated to elements of Λ

Subtle issue: Universal line bundles $\mathcal{L}$ are only defined up to twisting by pullbacks from M_Λ

How to find canonical universal lines:

For $L \in \Lambda$, $\quad \mathcal{D}_L \subset \mathcal{S}$ Define $\mathcal{L}$

divisor of rational curves $\quad \downarrow$ by normalizing by GW invariant

M_Λ

P-Q. Yin 2016

Marian
Oprea
P
2015

Idea (B): $R^*_A(\mathcal{M}_h^{k3}) \subset CH^*(\mathcal{M}_h^{k3})$

the $\mathbb{Q}$-linear span of the classes obtained from all push-forwards

$$i_{\Lambda *}\Big(\pi_*\big(c_1(\mathcal{L}_1)^{a_1}\cdots c_1(\mathcal{L}_k)^{a_k}\, c_1(T_\pi)^{b_1}\, c_2(T_\pi)^{b_2}\big)\Big)$$

$$\in CH^*(\mathcal{M}_h^{k3})$$

$\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_k$ corresponding to elements of Λ

$$\mathcal{S} \xrightarrow{\pi} \mathcal{M}_\Lambda$$

A lot more classes!

Theorem (P - Qizheng Yin 2016)

Bergeron-Li
in Cohomology
2017

$$NL^*(\mathcal{M}_h^{K3}) = R^*(\mathcal{M}_h^{K3})$$

idea (A) idea (B)

- $NL^*(\mathcal{M}_h^{K3})$ is finite $\mathbb{Q}$-dimensional

Brunier
Raum 2014

- Method of proof involves a new construction

$$\begin{tikzcd} & \overline{\mathcal{M}}_{0,4}(\pi, H) \arrow[dl, "\varepsilon"'] \arrow[r, "ev"] & \mathcal{S}^4 \arrow[d, "\pi"] \\ \overline{\mathcal{M}}_{0,4} & & \mathcal{M}_h^{K3} \end{tikzcd}$$

How does this help?

$$EV_* \, \varepsilon^*(\mathrm{WDVV}) \cap [\overline{\mathcal{M}}_{0,4}(\pi, H)]^{red}$$

The result is a relation in $CH^*(\mathcal{S}^4)$

which can be cut and pushed to $\mathcal{M}_h^{K3}$

Not enough for the Theorem!

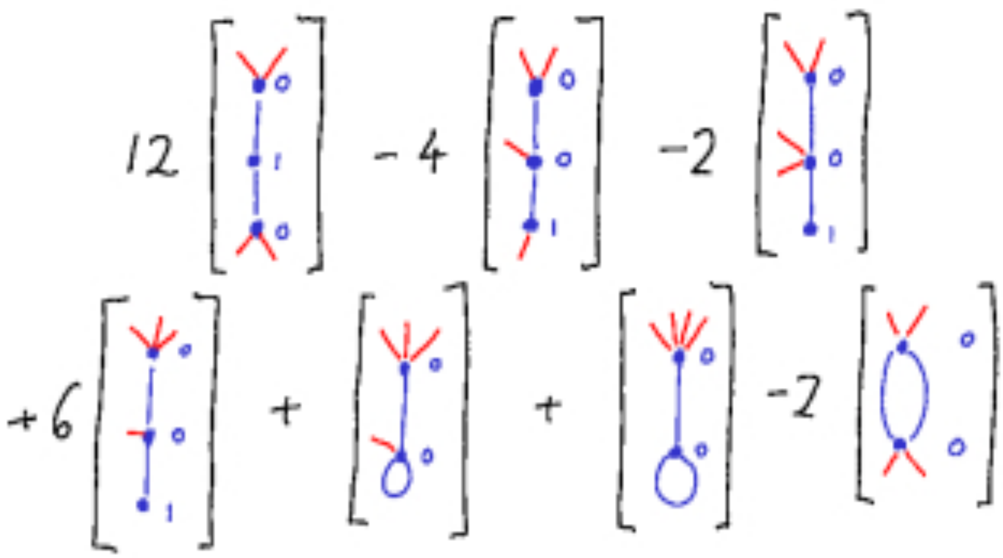
We need also genus 1

$$\begin{tikzcd}
\overline{\mathcal{M}}_{1,4}(\pi, H) \arrow[r, "EV"] \arrow[d, "\varepsilon"'] & \mathcal{S}^4 \arrow[d, "\pi"] \\
\overline{\mathcal{M}}_{1,4} & \mathcal{M}_h^{K3}
\end{tikzcd}$$

Corresponding relation in $CH^*(\mathcal{S}^4)$ is

$$EV_* \, \varepsilon^*(\mathrm{GETZLER}) \cap [\overline{\mathcal{M}}_{1,4}(\pi, H)]^{red}$$

Getzler 1996



$$= 0 \in H^4(\overline{\mathcal{M}}_{1,4})$$

also holds in $CH^2(\overline{\mathcal{M}}_{1,4})$

Remark:

relation in $CH^2(\overline{\mathcal{M}}_{1,4})$

Cut

$$\pi_{123\,*}\Big(H_4 \cdot \mathrm{Ev}_* \, \varepsilon^*(\mathrm{GETZLER}) \cap [\overline{\mathcal{M}}_{1,4}(\pi, H)]^{red} \Big)$$

on $\begin{array}{c} S^3 \\ \downarrow \\ \mathcal{M}_h^{k3} \end{array}$ yields a universal Beauville-Voisin diagonal decomposition:

$$\begin{aligned}(2h-2)\Delta_{123} = \; & H_1^2 \Delta_{23} + H_2^2 \Delta_{13} + H_3^2 \Delta_{12} \\ & - H_1^2 \Delta_{12} - H_1^2 \Delta_{13} - H_2^2 \Delta_{23} \\ & + \text{Corrections supported on NL divisors}\end{aligned}$$

tautological classes

Conjecture: Let S be a fixed K3 with polarization $H \in \mathrm{Pic}(S)$

$$\overline{M}_{g,n}(S,H) \xrightarrow{\mathrm{Ev}} S^n$$

Then $\quad \mathrm{Ev}_* \left[\overline{M}_{g,n}(S,H)\right]^{red} \in BV(S^n)$

Beauville-Voisin ring of Chow tautological classes generated by all diagonals and the pullbacks of Pic from the factors.

The above is a Gromov-Witten approach to relations in $R^*(\mathcal{M}_h^{K3})$

But in low dimensions, there is almost always a sheaf approach:

$$\begin{array}{c} S \\ \downarrow \pi \\ \mathcal{M}_h^{K3} \end{array} \qquad \mathrm{Quot}\left(\bigoplus_{1}^{N} \mathcal{O}_S, H, n\right)$$

quotients

$$\bigoplus_{1}^{N} \mathcal{O}_S \to Q \to 0$$

$[Q] = H$ Support on curves

$\chi(Q) = n$

Quot scheme

carries a perfect obstruction theory and a reduced virtual class

With Marian and Oprea, we started studying the virtual geometry, but immediately integrals arise over $\mathrm{Hilb}(S,n)$ which were not understood:

Tautological bundle

$$\int_{\mathrm{Hilb}(S,n)} s(\mathcal{H}^{[n]}) = ?$$

Segre class $\frac{1}{c(\mathcal{H}^{[n]})}$

Governed by Lehn's Conjecture 1999

Table of known Segre integrals

with Marian Oprea

Y surface, $B \to Y$ is a bundle of rank b

$$\int_{\mathrm{Hilb}(Y,n)} s(B^{[n]})$$

	K trivial surface	Arbitrary surface
rank $b=1$	✓ Lehn Conj MOP 2015	✓ Lehn Conj Voisin 2017 MOP 2017
Arbitrary rank	✓ MOP 2017	?

Conjecture: All functions are algebraic

Perfect knowledge in K3 case

a few ranks known

$b=2$ MOP 2017

Theorem (Marian - Oprea - P 2017)

Let S be a K3 surface

Let B be a K-theory class of rank b

Let $r = b+1$. Then,

$\chi(\mathcal{O}_S)$

$$\sum_{n=0}^{\infty} z^n \int_{\mathrm{Hilb}(S,n)} s\left(B^{[n]}\right) = A_0^{c_2(\beta)} \cdot A_1^{c_1^2(\beta)} \cdot A_2^{2}$$

$$A_0(z) = (1+rt)^{-r} \cdot (1+(1+r)t)^{r-1}$$

$$A_1(z) = (1+rt)^{\frac{r-1}{2}} \cdot (1+(1+r)t)^{-\frac{r}{2}+1}$$

$$A_2(z) = (1+rt)^{\frac{r^2-1}{2}} \cdot (1+(1+r)t)^{-\frac{r^2}{2}+r} \cdot (1+r(1+r)t)^{-\frac{1}{2}}$$

using $z = t(1+rt)^r$

Lets return to $R^*(\mathcal{M}_h^{k3})$.

One (almost) complete example:

$h = 2$ double covers of $\mathbb{P}^2$ branched along a sextic.

Computer algebra Not yet finished

Theorem*/Expectation (Si Fei, Oprea, P, Q. Yin hopefully 2020)

(1) $R^*(\mathcal{M}_2^{k3}) = CH^*(\mathcal{M}_2^{k3})$

(2) Betti Numbers of $R^*(\mathcal{M}_2^{k3})$ are :

Remember $\dim_{\mathbb{C}} \mathcal{M}_2^{k3} = 19 \ldots$

$$1 + 2q + 3q^2 + 5q^3 + 6q^4 + 8q^5$$
$$+ 10q^6 + 12q^7 + 13q^8 + 14q^9 + 12q^{10}$$
$$+ 10q^{11} + 8q^{12} + 6q^{13} + 5q^{14} + 3q^{15}$$
$$+ 2q^{16} + q^{17}$$

Related Cohomology Calculations by Kirwan Lee 1980s

Observations / Patterns

(i) $R^*(M_2^{k3})$ is NOT generated by divisors.

(ii) $R^{18}(M_2^{k3}) = R^{19}(M_2^{k3}) = 0$

Petersen 2018 in Cohomology

Conjecture: $R^{18}(M_h^{k3}) = R^{19}(M_h^{k3}) = 0$

We know

(iii) $R^{17}(\mathcal{M}_2^{k3}) \cong \mathbb{Q}$ $\quad$ $\dim R^{17}(\mathcal{M}_h^{k3}) \geq 1$

Conjecture: $R^{17}(\mathcal{M}_h^{k3}) \cong \mathbb{Q}$

Parallel to $R^{g-2}(\mathcal{M}_g^{curves}) \cong \mathbb{Q}$

(iv) Poincaré duality (with pairing into $R^{17}(\mathcal{M}_2^{k3})$) does NOT hold.

Wish: data for $R^*(\mathcal{M}_3^{k3})$!

The End