

A FOCK SPACE APPROACH TO SEVERI DEGREES

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ABSTRACT. The classical Severi degree counts the number of algebraic curves of fixed genus and class passing through points in a surface. We express the Severi degrees of $\mathbb{P}^1 \times \mathbb{P}^1$ as matrix elements of the exponential of a single operator M_S on Fock space. The formalism puts Severi degrees on a similar footing as the more developed study of Hurwitz numbers of coverings of curves. The pure genus 1 invariants of the product $E \times \mathbb{P}^1$ (with E an elliptic curve) are solved via an exact formula for the eigenvalues of M_S to initial order. The Severi degrees of \mathbb{P}^2 are also determined by M_S via the $\frac{(-1)^{d-1}}{d^2}$ disk multiple cover formula for Calabi-Yau 3-fold geometries.

1. INTRODUCTION

Let \mathbb{P}^2 be the complex projective plane. The study of the following classical problem in enumerative geometry was initiated in the late 19th century by Chasles, Zeuthen, and Schubert:

How many algebraic curves in \mathbb{P}^2 of geometric genus g and degree d pass through $3d + g - 1$ general points?

More precisely, plane curves $C \subset \mathbb{P}^2$ of degree d are parameterized by the projective space $\mathbb{P}(H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d)))$ of dimension $\binom{d+2}{2} - 1$. Let $r \leq \binom{d-1}{2}$ be a non-negative integer. The *Severi variety*,

$$(1) \quad S_d^r \subset \mathbb{P}(H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d))),$$

parameterizing reduced irreducible curves with exactly r nodes (and no other singularities), is a nonsingular quasi-projective subvariety of codimension r , see [1, 11]. The degree of S_d^r in the embedding (1) is the classical *Severi degree* N_d^r .

The Severi degree N_d^r can be interpreted as the number of degree d reduced irreducible plane curves with exactly r nodes passing through

$$n = \binom{d+2}{2} - 1 - r$$

general points of \mathbb{P}^2 . Such curves have geometric genus $g = \binom{d-1}{2} - r$. Severi degrees can also be indexed by geometric genus

$$N_{g,d} = N_d^r$$

and be interpreted as the numbers of degree d reduced irreducible plane curves of geometric genus g passing through $n = 3d + g - 1$ general points of \mathbb{P}^2 .

The modern study of the Severi variety began with the proof of the irreducibility of S_d^r by J. Harris [10] using the geometry of stable curves in an essential way — foreshadowing the stable map techniques which would be applied years later. The Severi degrees $N_{g,d}$ of \mathbb{P}^2 were calculated by degeneration techniques by Z. Ran [21]. An elegant treatment was given by L. Caporaso and J. Harris [5]. There are several more recent solutions in Gromov-Witten theory via localization [8],

relative invariants [12, 14, 15], and Virasoro constraints [7]. Tropical counts also calculate Severi degrees [17].

With all of these methods available, what is there left to ask? For comparison, consider the enumeration of Hurwitz covers of curves. A very useful approach to Hurwitz counts is via the matrix elements in Fock space of the exponential of the operator M_H associated to the insertion of a 2-cycle. Since M_H is diagonalized by Schur functions, a fundamental connection to the representation theory of the symmetric group is made. The formalism also incorporates, in a basic way, the cut-and-join structure of Hurwitz numbers. Fock space methods play a crucial role in the study of the associated Toda hierarchy [18, 19, 20]. Is there a natural Fock space formalism for Severi degrees?

A direct approach to Severi degrees via the matrix elements in Fock space of a new operator M_S is presented here. While the surface $\mathbb{P}^1 \times \mathbb{P}^1$ is the most natural to study, the formalism simultaneously captures the Severi degrees of \mathbb{P}^2 blown-up in at most 3 points. Severi degrees for elliptic fibrations such as $E \times \mathbb{P}^1$ occur via the trace. We hope the results presented here will lead to further study of M_S (and the full algebra of operators discussed in Section 9.2).

One avenue for further study is the relationship between this approach for curve counting and tropical geometry. In [2], Block and Göttsche explore connections between this approach to Severi degrees and tropical curve counting techniques, such as those of Brugallé and Mikhalkin [3]. In doing so, they extend curve counting via Fock space to a larger class of toric surfaces, those described by an h-transverse lattice polygon. Furthermore, they extend consideration from Severi degrees to the refined Severi degrees defined and studied by Göttsche and Shende and by Block and Göttsche.

2. HURWITZ THEORY

2.1. Fock space. We start by reviewing the Fock space formalism for the Hurwitz theory of covers of curves. The results provide direct motivation for our treatment of Severi degrees.

We begin by recalling the definition of the Heisenberg algebra \mathcal{H} . \mathcal{H} is an infinite dimensional Lie algebra. A basis of \mathcal{H} is given by the operators 1 and $\{\alpha_k\}$, where $k \in \mathbb{Z} \setminus \{0\}$. The Lie bracket is given by

$$(2) \quad [\alpha_k, \alpha_l] = k \delta_{k+l,0}.$$

\mathcal{H} has an infinite dimensional irreducible representation \mathcal{F} , called the Fock space representation. \mathcal{F} contains a distinguished vector called the vacuum vector v_\emptyset . As a vector space \mathcal{F} is freely generated over \mathbb{Q} by the elements

$$\alpha_{-k_1} \dots \alpha_{-k_n} v_\emptyset, \quad k_i \in \mathbb{Z}_{>0},$$

where by the commutation relations, the order of $\alpha_{-k_1} \dots \alpha_{-k_n}$ does not matter.

The Lie algebra \mathcal{H} acts on \mathcal{F} in the following way. The creation operators

$$\alpha_{-k}, \quad k \in \mathbb{Z}_{>0},$$

act via

$$\alpha_{-k}(\alpha_{-k_1} \dots \alpha_{-k_n} v_\emptyset) = \alpha_{-k} \alpha_{-k_1} \dots \alpha_{-k_n} v_\emptyset.$$

The annihilation operators

$$\alpha_k, \quad k \in \mathbb{Z}_{>0},$$

kill the vacuum,

$$\alpha_k \cdot v_\emptyset = 0, \quad k > 0,$$

and their action on all other vectors is determined by the commutation relations (2). For example,

$$\alpha_1(\alpha_{-2}v_\emptyset) = \alpha_1\alpha_{-2}v_\emptyset = \alpha_{-2}\alpha_1v_\emptyset = 0$$

while

$$\alpha_1(\alpha_{-1}v_\emptyset) = \alpha_1\alpha_{-1}v_\emptyset = ([\alpha_1, \alpha_{-1}] + \alpha_{-1}\alpha_1)v_\emptyset = v_\emptyset.$$

Given a partition $\mu = \mu_1 + \dots + \mu_r$, let $Aut(\mu)$ denote the subgroup of S_r which leaves the partition unchanged. For example, if $\mu = 2 + 3 + 3 + 5 + 5 + 5$, $Aut(\mu) = S_2 \times S_3$.

Let $\mathfrak{z}(\mu)$ denote the combinatorial factor

$$\mathfrak{z}(\mu) = |\text{Aut}(\mu)| \cdot \prod_{i=1}^{\ell(\mu)} \mu_i .$$

We will work with the natural basis of \mathcal{F} given by the vectors

$$|\mu\rangle = \frac{1}{\mathfrak{z}(\mu)} \prod_{i=1}^{\ell(\mu)} \alpha_{-\mu_i} v_\emptyset$$

indexed by partitions μ . An inner product is defined on \mathcal{F} by

$$(3) \quad \langle \mu | \nu \rangle = \frac{u^{-\ell(\mu)}}{\mathfrak{z}(\mu)} \delta_{\mu\nu} .$$

The variable u will play the role of the genus parameter.

The operator in Hurwitz theory corresponding to the insertion of a 2-cycle is well-known to be given by

$$(4) \quad M_H(u) = \frac{1}{2} \sum_{k,l>0} \left[u \alpha_{k+l} \alpha_{-k} \alpha_{-l} + \alpha_{-k-l} \alpha_k \alpha_l \right] .$$

The operator M_H is self-adjoint with respect to the inner product (3).

2.2. Hurwitz numbers. The *Hurwitz number* $H_{g,d}^\bullet$ is the automorphism weighted count of genus g , degree d covers of \mathbb{P}^1 with simple ramification over $2d+2g-2$ fixed points of \mathbb{P}^1 . The superscript \bullet indicates the domain of the cover is *not* required to be connected. If $2d + 2g - 2 < 0$, then $H_{g,d}^\bullet$ vanishes by definition.

The partition function for the Hurwitz numbers of \mathbb{P}^1 is

$$Z^{\mathbb{P}^1} = 1 + \sum_{g \in \mathbb{Z}} u^{g-1} \sum_{d>0} H_{g,d}^\bullet \frac{t^{2d+2g-2}}{(2d+2g-2)!} Q^d .$$

Define the vector $\mathbf{v} \in \mathcal{F}$ by

$$\mathbf{v} = \sum_{d \geq 0} | (1^d) \rangle .$$

The partition function is expressed in terms of matrix elements by the formula

$$Z^{\mathbb{P}^1} = \langle \mathbf{v} | Q^{|\cdot|} \exp(tM_H(u)) | \mathbf{v} \rangle .$$

Here, $|\cdot|$ denotes the energy operator with eigenvalue $|\mu|$ on the basis vector $|\mu\rangle$. The simplicity of Hurwitz theory is largely due to the diagonal form of M_H in the basis of Schur functions.

The partition function Z^E for Hurwitz covers of an elliptic curve E is well-known to be given by the trace,

$$Z^E = \text{tr}(Q^{| \cdot |} \exp(tM_H(u))).$$

For example, the genus 1 part is given by the $u = 0$ specialization,

$$(5) \quad Z_1^E = \text{tr}(Q^{| \cdot |} \exp(tM_H(0))) = \sum_{\mu} Q^{|\mu|}.$$

Since $M_H(0)$ is nilpotent, all the eigenvalues of $M_H(0)$ vanish and the second equality is obtained. The sum on the right is over all partitions μ .

3. SEVERI THEORY

Let S be a nonsingular projective surface. The moduli space of stable maps

$$\overline{M}_{g,n}^{\bullet}(S, \beta)$$

from genus g , n -pointed curves to S representing the class $\beta \in H_2(S, \mathbb{Z})$ has virtual dimension

$$\dim_{\mathbb{C}} [\overline{M}_{g,n}^{\bullet}(S, \beta)]^{vir} = \int_{\beta} c_1(S) + g - 1 + n.$$

The superscript \bullet indicates the domain is possibly disconnected, but with no connected components collapsed to points of S . Let

$$\text{ev}_i : \overline{M}_{g,n}^{\bullet}(S, \beta) \rightarrow S$$

be the evaluation at the i^{th} marked point. We refer the reader to [6, 13] for an introduction to stable maps and Gromov-Witten theory.

A Gromov-Witten analogue of the Severi degree is defined by the following construction. Let

$$n = \int_{\beta} c_1(S) + g - 1$$

be the virtual dimension of the unpointed space $\overline{M}_g^{\bullet}(S, \beta)$. Let

$$N_{g,\beta}^{\bullet} = \int_{[\overline{M}_{g,n}^{\bullet}(S,\beta)]^{vir}} \prod_{i=1}^n \text{ev}_i^*(\mathfrak{p}),$$

where $\mathfrak{p} \in H^4(S, \mathbb{Z})$ is the point class. If $n < 0$, then $N_{g,\beta}^{\bullet}$ vanishes by definition.

For an arbitrary surface S , the Gromov-Witten invariant $N_{g,\beta}^{\bullet}$ may be completely unrelated to the classical Severi degree. Indeed, for Enriques surfaces, the Gromov-Witten invariants are often fractional, and, for K3 surfaces, the Gromov-Witten invariants vanish altogether. However, for the surfaces \mathbb{P}^2 and $\mathbb{P}^1 \times \mathbb{P}^1$ of main interest here, $N_{g,\beta}^{\bullet}$ is well-known to coincide with the (disconnected) classical Severi degree. In genus 0, the Gromov-Witten invariants of \mathbb{P}^2 and $\mathbb{P}^1 \times \mathbb{P}^1$ are enumerative by the convexity of the surfaces [6]. In fact, the stationary Gromov-Witten invariants in all positive genera are also enumerative for both surfaces. While excess components in the moduli spaces of stable maps to these surfaces appear in positive genus, the point conditions can not be satisfied on the excess components [9].

When we discuss Severi degrees for a surface, we will *always* mean the corresponding Gromov-Witten invariants $N_{g,\beta}^{\bullet}$. The Severi degrees of $\mathbb{P}^1 \times \mathbb{P}^1$ will play a special role.

4. FOCK SPACE

To start, we write the cohomology of \mathbb{P}^1 as the standard direct sum

$$H^*(\mathbb{P}^1, \mathbb{Q}) = \mathbb{Q} \cdot \mathbf{1} \oplus \mathbb{Q} \cdot \mathbf{p}$$

where $\mathbf{1}$ and \mathbf{p} are the unit and point classes respectively.

The Lie algebra $\mathcal{H}[\mathbb{P}^1]$ is generated by the operators 1 , $\{\alpha_k[\mathbf{1}]\}$, and $\{\alpha_k[\mathbf{p}]\}$, where $k \in \mathbb{Z} \setminus \{0\}$. The Lie bracket is given by

$$(6) \quad [\alpha_k[\mathbf{1}], \alpha_l[\mathbf{p}]] = k \delta_{k+l,0}$$

with all other commutators vanishing.

The Fock space $\mathcal{F}[\mathbb{P}^1]$ is freely generated over \mathbb{Q} by the elements

$$\alpha_{-k_1}[\mathbf{1}] \dots \alpha_{-k_n}[\mathbf{1}] \alpha_{-\ell_1}[\mathbf{p}] \dots \alpha_{-\ell_m}[\mathbf{p}] v_\emptyset, \quad k_i, \ell_j \in \mathbb{Z}_{>0},$$

where by the commutation relations, the order of $\alpha_{-k_1}[\mathbf{1}] \dots \alpha_{-k_n}[\mathbf{1}] \alpha_{-\ell_1}[\mathbf{p}] \dots \alpha_{-\ell_m}[\mathbf{p}]$ does not matter.

$\mathcal{H}[\mathbb{P}^1]$ acts on $\mathcal{F}[\mathbb{P}^1]$ in a way analogous to the action of \mathcal{H} on \mathcal{F} . The creation operators

$$\alpha_{-k}[\mathbf{1}] \text{ and } \alpha_{-k}[\mathbf{p}], \quad k \in \mathbb{Z}_{>0},$$

act via

$$\alpha_{-k}[\mathbf{1}] \left(\prod_{i,j} \alpha_{-k_i}[\mathbf{1}] \alpha_{-\ell_j}[\mathbf{p}] v_\emptyset \right) = \alpha_{-k}[\mathbf{1}] \prod_{i,j} \alpha_{-k_i}[\mathbf{1}] \alpha_{-\ell_j}[\mathbf{p}] v_\emptyset,$$

and similarly for $\alpha_{-k}[\mathbf{p}]$. The annihilation operators

$$\alpha_k[\mathbf{1}] \text{ and } \alpha_k[\mathbf{p}], \quad k \in \mathbb{Z}_{>0},$$

kill the vacuum

$$\alpha_k[\mathbf{1}](v_\emptyset) = 0 \text{ and } \alpha_k[\mathbf{p}](v_\emptyset) = 0, \quad k > 0,$$

and their action on any element $\prod_{i,j} \alpha_{-k_i}[\mathbf{1}] \alpha_{-\ell_j}[\mathbf{p}] v_\emptyset$ is determined by the commutation relations (6).

A natural basis of $\mathcal{F}[\mathbb{P}^1]$ is given by the vectors

$$|\mu, \nu\rangle = \frac{1}{\mathfrak{z}(\mu)\mathfrak{z}(\nu)} \prod_{i=1}^{\ell(\mu)} \alpha_{-\mu_i}[\mathbf{1}] \prod_{j=1}^{\ell(\nu)} \alpha_{-\nu_j}[\mathbf{p}] v_\emptyset$$

indexed by all pairs of partitions μ and ν (of possibly different sizes). As before, $\mathfrak{z}(\mu)$ denotes the combinatorial factor

$$\mathfrak{z}(\mu) = |\text{Aut}(\mu)| \cdot \prod_{i=1}^{\ell(\mu)} \mu_i.$$

An inner product is defined by

$$\langle \mu, \nu | \mu', \nu' \rangle = \frac{u^{-\ell(\mu)} u^{-\ell(\nu)}}{\mathfrak{z}(\mu) \mathfrak{z}(\nu)} \delta_{\mu\nu'} \delta_{\nu\mu'}.$$

We define a new operator M_S on the Fock space $\mathcal{F}[\mathbb{P}^1]$ by the following formula,

$$M_S(u, Q) = \sum_{k>0} \alpha_{-k}[\mathbf{p}] \alpha_k[\mathbf{p}] + Q \sum_{|\mu|=|\nu|>0} u^{\ell(\mu)-1} \alpha_{-\mu}[\mathbf{1}] \alpha_\nu[\mathbf{1}].$$

The second sum is over all pairs of nontrivial partitions μ and ν of equal size, and

$$\alpha_{-\mu}[\mathbf{1}] = \frac{1}{|\text{Aut}(\mu)|} \prod_{i=1}^{\ell(\mu)} \alpha_{-\mu_i}[\mathbf{1}],$$

$$\alpha_{\nu}[\mathbf{1}] = \frac{1}{|\text{Aut}(\nu)|} \prod_{i=1}^{\ell(\nu)} \alpha_{\nu_i}[\mathbf{1}].$$

As before, u is the genus variable. The new variable Q will be related to the curve class.

The operator M_S is self-adjoint with respect to the inner product (4) and hence diagonalizable. The Hurwitz operator M_H occurs as a summand in the second term of M_S . Unlike M_H , the operator M_S does not appear to be diagonalizable over \mathbb{Q} .

5. $\mathbb{P}^1 \times \mathbb{P}^1$

Let the variables Q_1 and Q_2 correspond to the curve classes of the fibers of the first and second projections to \mathbb{P}^1 respectively. The partition function for Severi degrees of $\mathbb{P}^1 \times \mathbb{P}^1$ is

$$Z^{\mathbb{P}^1 \times \mathbb{P}^1} = 1 + \sum_{g \in \mathbb{Z}} u^{g-1} \sum_{(d_1, d_2)} N_{g, (d_1, d_2)}^{\bullet} \frac{t^{2d_1 + 2d_2 + g - 1}}{(2d_1 + 2d_2 + g - 1)!} Q_1^{d_1} Q_2^{d_2}$$

where the second sum is over all non-negative d_i satisfying $(d_1, d_2) \neq (0, 0)$. The vector

$$\mathbf{v} = \sum_{d_1 \geq 0} |(1^{d_1}), \emptyset \rangle$$

in $\mathcal{F}[\mathbb{P}^1]$ plays an important role.

Theorem 1. $Z^{\mathbb{P}^1 \times \mathbb{P}^1} = e^{tQ_2/u} \langle \mathbf{v} | Q_1^{|\cdot|} \exp(tM_S(u, Q_2)) | \mathbf{v} \rangle$.

Here, $|\cdot|$ is the energy operator on $\mathcal{F}[\mathbb{P}^1]$ with eigenvalue $|\mu| + |\nu|$ on the basis vector $|\mu, \nu\rangle$. The prefactor $e^{tQ_2/u}$ could have been included in the definition of M_S , but we have chosen not to.

6. PROOF OF THEOREM 1

6.1. Overview. We prove Theorem 1 via the degeneration formula for relative Gromov-Witten invariants [12, 14, 15]. Consider the Gromov-Witten invariant $N_{g, (d_1, d_2)}^{\bullet}$ counting genus g curves through

$$n = 2d_1 + 2d_2 + g - 1$$

points on $\mathbb{P}^1 \times \mathbb{P}^1$. Let C be a chain of $n + 2$ rational curves. Let S be the surface $C \times \mathbb{P}^1$ viewed as a union of $n + 2$ copies of $\mathbb{P}^1 \times \mathbb{P}^1$. We degenerate

$$(7) \quad \mathbb{P}^1 \times \mathbb{P}^1 \rightsquigarrow C \times \mathbb{P}^1$$

and distribute the original n point conditions by placing one on each of the middle n components. We will refer to the $n + 2$ components of the degeneration as S_0, \dots, S_{n+1} and the $n + 1$ relative divisors as D_0, \dots, D_n . The matrix M_S arises from explicit calculations on components of the degeneration (7).

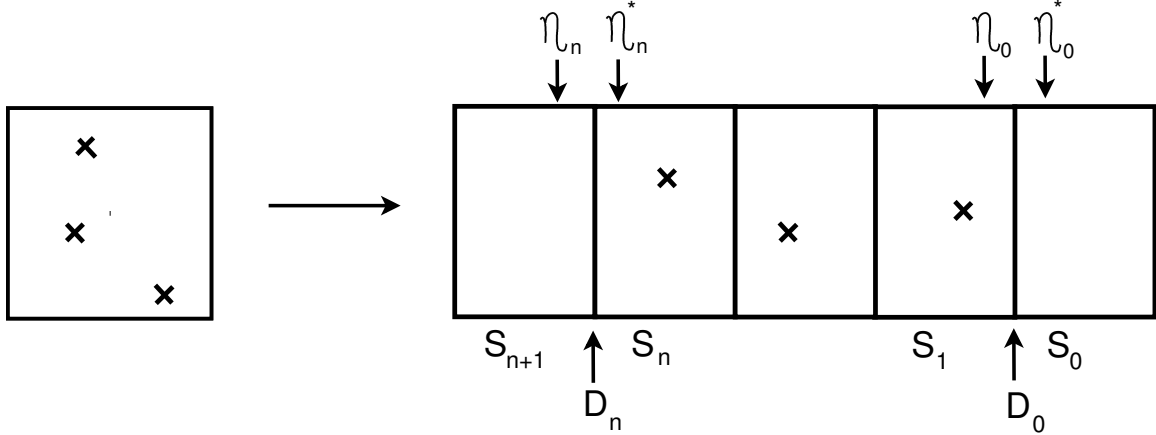


FIGURE 1. The first and last components carry no point conditions, all the rest have one point condition each. We take $(1, 0)$ to be the class of a horizontal line and $(0, 1)$ the class of a vertical line.

6.2. Stable relative maps. A moduli space of stable relative maps is defined for each component S_i of the above degeneration. Relative conditions along the divisors D_i are specified by partitions weighted by the cohomology of \mathbb{P}^1 . For S_i where $1 \leq i \leq n$, let

$$M^i = \overline{M}_{g_i, 1}^\bullet(D_i \setminus \mathbb{P}^1 \times \mathbb{P}^1 / D_{i-1}, (d_1, d_2^i), \eta_i^*, \eta_{i-1}, \Gamma^i)$$

denote the moduli space of stable relative degree (d_1, d_2^i) maps of a genus g_i curve to $\mathbb{P}^1 \times \mathbb{P}^1$ of graph type Γ^i satisfying relative conditions η_{i-1} along the divisor D_{i-1} and η_i^* along D_i . Note the superscript of Γ^i and d_2^i is an index, not an exponent.

The graph type Γ^i fixes the topology of the map. Each vertex of Γ^i corresponds to a component of the domain curve and is labeled with the genus of that component. For each relative condition on that domain curve the vertex is given a half-edge labeled with the corresponding relative condition. The unique marked point is assigned to a single component of Γ^i (which satisfies the incidence condition).

The outside components S_0 and S_{n+1} play a special role. Following the above conventions, let

$$M^0 = \overline{M}_{g_0, 0}^\bullet(D_0 \setminus \mathbb{P}^1 \times \mathbb{P}^1, (d_1, d_2^0), \eta_0^*, \Gamma^0)$$

and let

$$M^{n+1} = \overline{M}_{g_{n+1}, 0}^\bullet(\mathbb{P}^1 \times \mathbb{P}^1 / D_n, (d_1, d_2^{n+1}), \eta_n, \Gamma^{n+1}).$$

For all the above moduli spaces M^i , we will view the relative markings on the domain of the map as *ordered*.

6.3. Partition notation. We take all our partitions to be ordered partitions.

Definition 1. Let ρ be a partition of d and let $\rho(k)$ be the number of parts of size k in ρ , so $d = \sum_{k=1}^{\infty} \rho(k)k$.

Let $\rho = \rho_1 + \cdots + \rho_m$ and $\lambda = \lambda_1 + \cdots + \lambda_n$ be two partitions, and $d = |\rho| + |\lambda|$. We say

$$\rho[\mathbf{1}] + \lambda[\mathbf{p}] = \rho_1[\mathbf{1}] + \cdots + \rho_m[\mathbf{1}] + \lambda_1[\mathbf{p}] + \cdots + \lambda_n[\mathbf{p}]$$

is a cohomology weighted partition of d , weighted by the cohomology of \mathbb{P}^1 .

Let \cup denote concatenation of partitions,

$$\rho \cup \lambda = \rho_1 + \cdots + \rho_m + \lambda_1 + \cdots + \lambda_n.$$

Definition 2. Let $\eta = \rho[\mathbf{1}] + \lambda[\mathbf{p}]$ be a partition weighted by the cohomology of \mathbb{P}^1 . Let

$$m(\eta) = \prod_i \rho_i \prod_j \lambda_j, \quad \text{Aut}(\eta) = \text{Aut}(\rho) \times \text{Aut}(\lambda), \quad \eta^* = \lambda[\mathbf{1}] + \rho[\mathbf{p}].$$

6.4. Degeneration. By the degeneration formula of [12, 14, 15],

$$N_{g,(d_1,d_2)}^\bullet = \sum_{d_2^i, \eta_i, \Gamma^i} \left(\int_{[M^{n+1}]} 1 \right) \frac{m(\eta_n)}{|\text{Aut}(\eta_n)|} \prod_{i=1}^n \left[\left(\int_{[M^i]} \text{ev}_1^*(\mathbf{p}) \right) \frac{m(\eta_{i-1})}{|\text{Aut}(\eta_{i-1})|} \right] \left(\int_{[M^0]} 1 \right).$$

The sum is over all degree splittings

$$d_2^0 + \cdots + d_2^{n+1} = d_2,$$

relative conditions η_0, \dots, η_n , and compatible graph types $\Gamma^0, \dots, \Gamma^{n+1}$ which connect to form a genus g curve. The relative conditions η_i^* are set by Definition 2. On the right side, $[M^i]$ denotes the virtual fundamental class of the moduli space M^i .

Equivalently, we can write the partition function of the Severi degrees of the surface $\mathbb{P}^1 \times \mathbb{P}^1$ as

$$(8) \quad Z^{\mathbb{P}^1 \times \mathbb{P}^1} = 1 + \sum_{g,d_1,d_2} Q_1^{d_1} Q_2^{d_2} u^{g-1} \frac{t^n}{n!} \sum_{d_2^i, \eta_i, \Gamma^i} \left(\int_{[M^{n+1}]} 1 \right) \frac{m(\eta_n)}{|\text{Aut}(\eta_n)|} \\ \times \prod_{i=1}^n \left[\left(\int_{[M^i]} \text{ev}_1^*(\mathbf{p}) \right) \frac{m(\eta_{i-1})}{|\text{Aut}(\eta_{i-1})|} \right] \left(\int_{[M^0]} 1 \right).$$

In the above formula, $n = 2d_1 + 2d_2 + g - 1$ as usual.

6.5. Geometry of the components.

6.5.1. The caps. We will now analyze the integrals appearing in the degeneration formula (8). Consider first the $i = 0$ term corresponding to the component S_0 . Let the relative condition be

$$\eta_0^* = \rho[\mathbf{1}] + \lambda[\mathbf{p}]$$

where ρ and λ are partitions satisfying $|\rho| + |\lambda| = d_1$.

Let R be a component of the domain curve of a map to S_0 parameterized by M^0 , and let

$$\sigma[\mathbf{1}] + \tau[\mathbf{p}]$$

be the relative condition imposed on R . Suppose the genus of R is h and $f_*[R] = \beta = (a, b)$. The dimension of the space of such maps is

$$\dim_{\mathbb{C}} \overline{M}_{h,0}(\mathbb{P}^1 \times \mathbb{P}^1, \beta) = \int_{(a,b)} c_1(T_{\mathbb{P}^1 \times \mathbb{P}^1}) + h - 1 = 2a + 2b + h - 1.$$

Meanwhile the number of conditions imposed on the map by the relative conditions is

$$\sum (\sigma_i - 1) + \sum \tau_j = a - \ell(\sigma).$$

After setting the dimension to equal the number of conditions, we obtain

$$a + 2b + h + \ell(\sigma) = 1.$$

Each term on the left hand side is nonnegative, so b must be zero. Certainly $h \leq 1$. Since R has no marked points to stabilize, a cannot also vanish. Hence, the only possible solution is

$$a = 1 \text{ and } b = h = \ell(\sigma) = 0.$$

We see R must be a genus 0 curve mapping with degree 1 onto a line in the class $(1, 0)$ with the relative condition $[\mathfrak{p}]$.

Therefore the integral over M^0 vanishes unless

$$d_2^0 = 0 \text{ and } \eta_0^* = [\mathfrak{p}] + \cdots + [\mathfrak{p}]$$

and Γ^0 is a graph on d_1 vertices with a half-edge at each vertex. If the above conditions are satisfied, the moduli space M^0 consists of a single point which parameterizes a map of d_1 disconnected rational curves to $\mathbb{P}^1 \times \mathbb{P}^1$, each with a fixed condition of multiplicity 1, mapping with degree 1 to the unique curve in the class $(1, 0)$ passing through that fixed relative condition. As such a map has no automorphisms, we find

$$\int_{[M^0]} 1 = 1.$$

The analysis for S_{n+1} is identical.

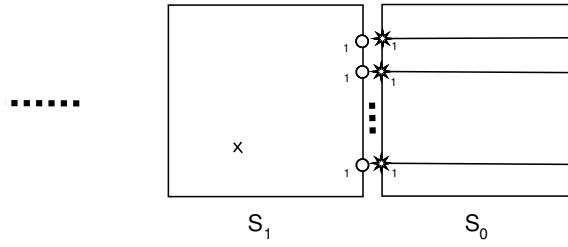


FIGURE 2. The image of the only map to the cap S_0 .

The evaluations of the integrals in (8) corresponding to the end components are

$$(9) \quad \left(\int_{[M^0]} 1 \right) = 1, \quad \left(\int_{[M^{n+1}]} 1 \right) \frac{m(\eta_n)}{|\text{Aut}(\eta_n)|} = \frac{1}{d_1!}.$$

6.5.2. *Middle components.* Next, we analyze the components S_i in case $1 \leq i \leq n$. Let

$$\eta_{i-1} = \rho[\mathbf{1}] + \lambda[\mathfrak{p}], \quad \eta_i^* = \rho'[\mathbf{1}] + \lambda'[\mathfrak{p}].$$

As before, consider a single genus h connected component R of the domain curve of a map to S_i parameterized by M^i . Let

$$\sigma[\mathbf{1}] + \tau[\mathfrak{p}] \text{ and } \sigma'[\mathbf{1}] + \tau'[\mathfrak{p}]$$

be the relative conditions imposed on R along D_{i-1} and D_i respectively. Let $f_*[R] = \beta = (a, b)$. As M^i is a moduli space of 1-pointed curves, there are two cases: either the marked point of lies on R or R is unpointed.

Consider first the case where R does not carry a marked point. Then,

$$\dim_{\mathbb{C}} \overline{M}_{h,0}(\mathbb{P}^1 \times \mathbb{P}^1, \beta) = 2a + 2b + h - 1.$$

The relative conditions impose

$$\sum (\sigma_i - 1) + \sum (\sigma'_i - 1) + \sum \tau_j + \sum \tau'_j$$

conditions. After equating the two, we obtain

$$2b + h + \ell(\sigma) + \ell(\sigma') = 1.$$

The unique solution (up to exchanging σ and σ') is easily determined to be

$$h = 0, b = 0, \ell(\sigma) = 1, \ell(\sigma') = 0.$$

We see that $\sigma' = \tau = \emptyset$ and also that σ and τ' must each be single part partitions. We then deduce that $\sigma = \tau' = a$. Then R is rational and maps with degree $|\sigma| = a$ to the line of class $(1, 0)$ through the point fixed by $\tau'_1[\mathfrak{p}]$ ramified totally over D_{i-1} and D_i . The moduli space of such maps is isomorphic as a stack to $B\mathbb{Z}/a\mathbb{Z}$.

We consider next the case where R carries the marked point,

$$\dim_{\mathbb{C}} \overline{M}_{h,1}(\mathbb{P}^1 \times \mathbb{P}^1, \beta) = 2a + 2b + h.$$

The relative conditions impose

$$\sum (\sigma_i - 1) + \sum (\sigma'_{i'} - 1) + \sum \tau_j + \sum \tau_{j'}$$

conditions on such a map. Setting their difference equal to 2, the degree of $ev_1^*(\mathfrak{p})$, we obtain

$$2b + h + \ell(\sigma) + \ell(\sigma') = 2.$$

Again, the terms on the left hand side are all nonnegative. The possible solutions are

$$\text{Type A : } h = 0, b = 0, \ell(\sigma) = \ell(\sigma') = 1,$$

$$\text{Type B : } h = 0, b = 1, \ell(\sigma) = \ell(\sigma') = 0, \ell(\tau), \ell(\tau') \neq 0,$$

$$\text{Type C : } h = 0, b = 1, \ell(\sigma) = \ell(\sigma') = 0, \ell(\tau) = \ell(\tau') = 0,$$

The arithmetically allowed solution $h = 0, b = 0, \ell(\sigma) = 2, \ell(\sigma') = 0$ and $h = 1, b = 0, \ell(\sigma) = 1, \ell(\sigma') = 0$ do not contribute by simple geometric considerations.

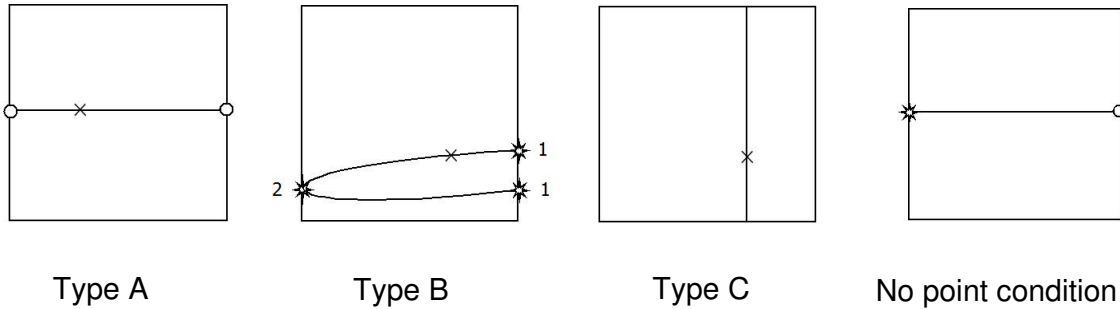


FIGURE 3. Examples of the images of maps of each type.

If R is a component of Type A, then $\sigma = \sigma' = |\sigma|$ is a partition with only one part and R is a 1-pointed rational curve mapping with degree $|\sigma|$ to a line in the class $(1, 0)$ totally ramified over the two relative divisors D_{i-1} and D_i . The moduli space M of such maps is isomorphic to \mathbb{P}^1 . Since R has a marked point and the map has two ramification points, there are no automorphisms of this map and $\int_{[M]} ev_1^*(\mathfrak{p}) = 1$.

If R is a component of Type B, then R is a rational curve mapping to $\mathbb{P}^1 \times \mathbb{P}^1$ with image the zero set of

$$x_0 p(y_0, y_1) + x_1 q(y_0, y_1)$$

where p and q are homogeneous polynomials of degree $|\tau|$. The map is degree 1 onto its image. Up to scaling, $q(y_0, y_1)$ is determined by the relative condition τ , and $p(y_0, y_1)$ is determined up to scaling by the relative condition τ' . Again the moduli space M of such maps is isomorphic to \mathbb{P}^1 . A general such map has no automorphisms, so $\int_{[M]} ev_1^*(\mathbf{p}) = 1$.

If R is a component of Type C, R is a rational curve mapping with degree 1 onto a line in the class $(0, 1)$. The moduli space M of such maps is isomorphic to \mathbb{P}^1 , and $\int_{[M]} ev_1^*(\mathbf{p}) = 1$.

In conclusion, if the partitions η_{i-1} and η_i are such that the domain curve has k components, then

$$\int_{[M^i]} ev_1^*(\mathbf{p}) = \frac{1}{m_1 \cdot \dots \cdot m_{k-1}},$$

where m_1, \dots, m_{k-1} are the degrees with which the $k-1$ unmarked components map to $\mathbb{P}^1 \times \mathbb{P}^1$.

6.6. Proof of the Main Theorem. We must show the matrix product

$$(10) \quad e^{tQ_2/u} \langle \mathbf{v} | Q_1^{|\cdot|} \exp(tM_S(u, Q_2)) | \mathbf{v} \rangle$$

is the partition function for Severi degrees of $\mathbb{P}^1 \times \mathbb{P}^1$. We begin by bringing the exponential inside and rewriting:

$$\begin{aligned} & e^{tQ_2/u} \langle \mathbf{v} | Q_1^{|\cdot|} \exp(tM_S(u, Q_2)) | \mathbf{v} \rangle \\ &= \langle \mathbf{v} | Q_1^{|\cdot|} \exp \left(t \left(\sum_{k>0} \alpha_{-k}[\mathbf{p}] \alpha_k[\mathbf{p}] + Q_2 \sum_{|\mu|=|\nu|>0} u^{\ell(\mu)-1} \alpha_{-\mu}[\mathbf{1}] \alpha_\nu[\mathbf{1}] + \frac{Q_2}{u} \right) \right) | \mathbf{v} \rangle. \end{aligned}$$

Via the operator

$$(11) \quad N_S = \sum_{k>0} \alpha_{-k}[\mathbf{p}] \alpha_k[\mathbf{p}] + \sum_{|\mu|=|\nu|>0} Q_2 u^{\ell(\mu)-1} \alpha_{-\mu}[\mathbf{1}] \alpha_\nu[\mathbf{1}] + \frac{Q_2}{u},$$

we can rewrite (10) as

$$(12) \quad \sum_{d_1} Q_1^{d_1} \sum_n \frac{t^n}{n!} \langle (1^{d_1}), \emptyset | N_S^n | (1^{d_1}), \emptyset \rangle.$$

It is a simple matter now to match equation (8) for the partition function with (12). The main point is to match the three summands of N in (11) with the three configuration types A , B , and C of Section 6.5.2 respectively. We leave the routine bookkeeping to the reader. \square

7. OTHER SURFACES

7.1. Overview. Let $B = Bl_2(\mathbb{P}^1 \times \mathbb{P}^1)$ be the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ at the (distinct) points p and \hat{p} . Let the variables E and \hat{E} correspond to the curve classes of the two exceptional divisors. As before, we define the partition function for Severi degrees as

$$Z^B = 1 + \sum_{g \in \mathbb{Z}} u^{g-1} \sum N_{g, (d_1, d_2, e, \hat{e})}^\bullet \frac{t^{2d_1+2d_2+e+\hat{e}+g-1}}{(2d_1+2d_2+e+\hat{e}+g-1)!} Q_1^{d_1} Q_2^{d_2} E^e \hat{E}^{\hat{e}}$$

where the second sum is over all tuples (d_1, d_2, e, \hat{e}) corresponding to non-zero curve classes.

We define two vectors $\mathbf{w}, \widehat{\mathbf{w}}$ in $\mathcal{F}[\mathbb{P}^1]$. First,

$$\mathbf{w} = e^{\frac{Q_2}{u} + \frac{E}{u}} \sum_{m \geq 0} \sum_{\nu} E^{-|\nu|} \prod_{j=1}^{\ell(\nu)} \frac{(-1)^{\nu_j-1}}{\nu_j} | (1^m), \nu \rangle$$

where the second sum is over all partitions ν . Second,

$$\widehat{\mathbf{w}} = e^{\frac{Q_2}{u} + \frac{\widehat{E}}{u}} \sum_{m \geq 0} \sum_{\nu} \widehat{E}^{-|\nu|} \prod_{j=1}^{\ell(\nu)} \frac{(-1)^{\nu_j-1}}{\nu_j} | (1^m), \nu \rangle.$$

Theorem 2. $Z^B = e^{tQ_2/u} \langle \mathbf{w} | Q_1^{| \cdot |} \exp(tM_S(u, Q_2)) | \widehat{\mathbf{w}} \rangle$.

The proof of Theorem 2 is given in Section 7.4. Multiple cover formulas from [4] play a crucial role. The surface $Bl_2(\mathbb{P}^1 \times \mathbb{P}^1)$ is isomorphic to $Bl_3(\mathbb{P}^2)$. The Severi degrees of the various blow-downs of $Bl_2(\mathbb{P}^1 \times \mathbb{P}^1)$ can be recovered from Z^B . Hence, Theorem 2 captures the Severi degrees of \mathbb{P}^2 — precise formulas are discussed in Section 7.5.

Similarly, for $Bl_1(\mathbb{P}^1 \times \mathbb{P}^1)$, the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ at p , we define the partition function for Severi degrees as

$$Z^{Bl_1(\mathbb{P}^1 \times \mathbb{P}^1)} = 1 + \sum_{g \in \mathbb{Z}} u^{g-1} \sum N_{g, (d_1, d_2, e)}^\bullet \frac{t^{2d_1+2d_2+e+g-1}}{(2d_1+2d_2+e+g-1)!} Q_1^{d_1} Q_2^{d_2} E^e$$

where the second sum is over all tuples (d_1, d_2, e) corresponding to non-zero curve classes. The proof of Theorem 2 also yields:

$$Z^{Bl_1(\mathbb{P}^1 \times \mathbb{P}^1)} = e^{tQ_2/u} \langle \mathbf{w} | Q_1^{| \cdot |} \exp(tM_S(u, Q_2)) | \mathbf{v} \rangle.$$

7.2. Degeneration. Consider the degeneration of $\mathbb{P}^1 \times \mathbb{P}^1$ described in Section 6.4, with the point p in the rightmost component S_0 and the point \hat{p} in the component S_{n+1} . To obtain a degeneration of B , we blow up the points p and \hat{p} .

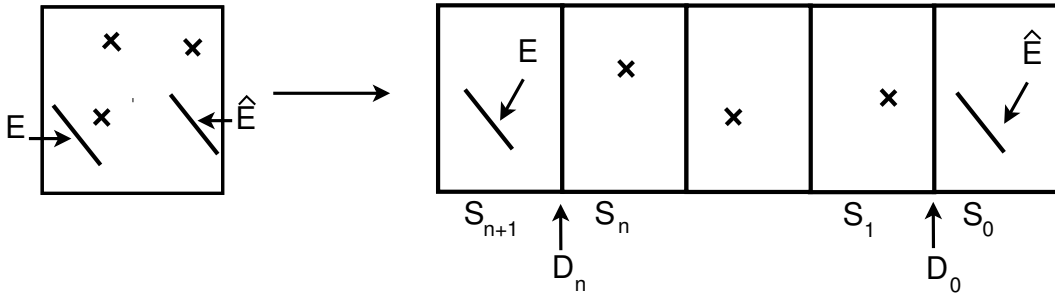


FIGURE 4. Degeneration of $Bl_2(\mathbb{P}^1 \times \mathbb{P}^1)$. As before, the first and last component carry no point conditions, and all the rest have one point condition each.

Again, by the degeneration formula of [12, 14, 15], we can write the partition function Z^B for Severi degrees in terms of integrals over moduli spaces of relative maps to the components of the degeneration. Only the moduli spaces M^0 and M^{n+1} differ from the geometry already considered in the proof Theorem 1.

7.3. The caps. We begin by analyzing the case $i = 0$ or $n + 1$. Without loss of generality take $i = 0$ and let the relative condition be

$$\eta_0^* = \rho[\mathbf{1}] + \lambda[\mathbf{p}].$$

Let R be a component of the domain curve of a maps to S_0 parameterized by M^0 , and let

$$\sigma[\mathbf{1}] + \tau[\mathbf{p}]$$

be the relative condition imposed on R . Suppose the genus of R is h and

$$f_*[R] = \beta = aL_1 + bL_2 - eE,$$

where L_1 and L_2 and the horizontal and vertical line classes. Since β is effective, we must have

$$a + b \geq e, \quad a \geq 0, \quad \text{and} \quad b \geq 0.$$

We compute the dimension of the space of maps as

$$\dim_{\mathbb{C}} \overline{M}_{h,0}(Bl_1(\mathbb{P}^1 \times \mathbb{P}^1), \beta) = \int_{\beta} c_1(T_{Bl_1(\mathbb{P}^1 \times \mathbb{P}^1)}) + h - 1 = 2a + 2b - e + h - 1.$$

Meanwhile, the number of conditions imposed on the map by the relative conditions is

$$\sum (\sigma_i - 1) + \sum \tau_j = a - \ell(\sigma).$$

After equating the dimension counts, we obtain

$$\ell(\sigma) + (a + b - e) + b + h = 1.$$

Each term on the left hand side is nonnegative so exactly one term must equal 1, and all others must vanish. We find there are four types of maps possible.

- Type I : $\ell(\sigma) = 1, a = e, b = 0,$ and $h = 0$.

Here R is a rational curve mapping as a degree a cover of the unique curve in the class $L_1 - E$.

- Type II : $\ell(\sigma) = 0, a = 1, e = 0, b = 0,$ and $h = 0$.

Here R is a rational curve mapping with degree 1 to a curve in the class L_1 .

- Type III : $\ell(\sigma) = 0, a = 0, e = -1, b = 0,$ and $h = 0$.

Here R is a rational curve mapping with degree 1 to the exceptional curve E .

- Type IV : $\ell(\sigma) = 0, a = 0, e = 1, b = 1,$ and $h = 0$.

Here R is a rational curve mapping with degree 1 to the unique curve in the class $L_2 - E$.

7.4. Proof of Theorem 2.

Proof. The geometry of the spaces of maps to S_1, \dots, S_n are the same as for the case of $\mathbb{P}^1 \times \mathbb{P}^1$, so we need only compute the contribution of the caps. Without loss of generality here we consider the cap S_0 .

By the dimension analysis of Section 7.3, only the moduli space associated to the Types I-IV need be considered. The moduli spaces $M_{II}^0, M_{III}^0,$ and M_{IV}^0 are simply points. The geometrically interesting case is Type I with moduli isomorphic to the space of degree a maps

$$\overline{M}_{0,0}(\mathbb{P}^1/\infty, a)_{(a)}$$

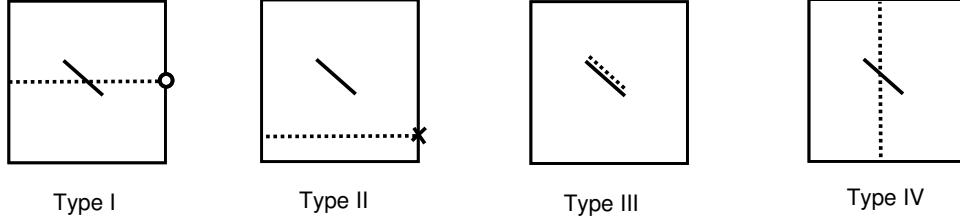


FIGURE 5. Here are the four possible types of image curves in a cap isomorphic to $Bl_1(\mathbb{P}^1 \times \mathbb{P}^1)$. The solid line represents the exceptional divisor, the dotted line represents the image curve. A map of Type I can be a multiple cover of the image curve. All other maps are degree 1.

to \mathbb{P}^1 relative to $\infty \in \mathbb{P}^1$ and fully ramified along the relative divisor. Since the normal bundle of the unique line in the class $L_1 - E$ is of degree -1 , we can write the integral against the virtual class as

$$(13) \quad \int_{[M_T^g]^{vir}} 1 = \int_{\overline{M}_{0,0}(\mathbb{P}^1/\infty, a)_{(a)}} c_{\text{top}}(R\pi_* f^*(\mathcal{O}_{\mathbb{P}^1}(-1)))$$

where π and f are the standard maps associated to the universal curve over moduli

$$\pi : \mathcal{C} \rightarrow \overline{M}_{0,0}(\mathbb{P}^1/\infty, a)_{(a)}, \quad f : \mathcal{C} \rightarrow \mathbb{P}^1.$$

Fortunately, the integral (13) has been calculated in [4],

$$\int_{\overline{M}_{0,0}(\mathbb{P}^1/\infty, a)_{(a)}} c_{\text{top}}(R\pi_* f^*(\mathcal{O}_{\mathbb{P}^1}(-1))) = \frac{(-1)^{a-1}}{a^2}.$$

The rest is bookkeeping.

The formula for $Bl_1(\mathbb{P}^1 \times \mathbb{P}^1)$ can be obtained with the same argument, beginning with the degeneration of $Bl_1(\mathbb{P}^1 \times \mathbb{P}^1)$ to a union of $n + 1$ surfaces isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ and a cap S_{n+1} , isomorphic to $Bl_1(\mathbb{P}^1 \times \mathbb{P}^1)$. Alternatively, set the coefficients \widehat{e} equal to zero in the formula for Z^B . \square

7.5. Severi Degrees of \mathbb{P}^2 . Let $H, F_1, F_2 \in \text{Pic}(Bl_2(\mathbb{P}^2))$ be the hyperplane class and the two exceptional classes respectively. Let $L_1, L_2, E \in \text{Pic}(Bl_1(\mathbb{P}^1 \times \mathbb{P}^1))$ be the pullbacks of the two generators of the Picard group of $\mathbb{P}^1 \times \mathbb{P}^1$ and the exceptional class. There is an isomorphism

$$\varphi : Bl_2(\mathbb{P}^2) \xrightarrow{\sim} Bl_1(\mathbb{P}^1 \times \mathbb{P}^1)$$

under which

$$\varphi(H) = L_1 + L_2 - E, \quad \varphi(F_1) = L_1 - E, \quad \varphi(F_2) = L_2 - E.$$

The generating function for Severi degrees of \mathbb{P}^2 is

$$Z^{\mathbb{P}^2} = 1 + \sum_{g \in \mathbb{Z}} u^{g-1} \sum_{d > 0} N_{g,d}^{\bullet} \frac{t^{3d+g-1}}{(3d+g-1)!} Q^d.$$

The terms of $Z^{\mathbb{P}^2}$ are not a subset of the terms of $Z^{Bl_2(\mathbb{P}^2)}$. For example, the coefficient of $u^{-2}Q^2t^4$ in $Z^{\mathbb{P}^2}$ is $3/24$, as there are three pairs of lines through 4 fixed points in \mathbb{P}^2 . The coefficient of $u^{-2}Q^2t^4$ in $Z^{Bl_2(\mathbb{P}^2)}$ is $5/24$, as in addition to the three pairs of lines, a smooth conic meeting the divisor F_1 , union F_1 , and a smooth conic meeting the divisor F_2 , union F_2 , also appear.

However, the *connected* Severi degrees of \mathbb{P}^2 and $Bl_2(\mathbb{P}^2)$ are directly related. For any surface S , define

$$Y^S = \log Z^S .$$

We view Y^S as the generating function for the connected Severi degrees of S .

Lemma 1. *The generating function $Y^{\mathbb{P}^2}$ appears as a summand of $Y^{Bl_2(\mathbb{P}^2)}$.*

Proof. We must show the coefficient $N_{g,d}^{\mathbb{P}^2}$ of $u^{g-1}Q^d t^n$ in $Y^{\mathbb{P}^2}$ equals the coefficient $N_{g,d}^{Bl_2(\mathbb{P}^2)}$ of the same term in $Y^{Bl_2(\mathbb{P}^2)}$. Let $p, q \in \mathbb{P}^2$ be the centers of the blow-up for $Bl_2(\mathbb{P}^2)$. The Severi degrees of \mathbb{P}^2 are enumerative. The nodal solutions are the only stable maps which pass through general point conditions, and they do *not* pass through p and q . We conclude the strict transforms of the nodal solutions are the only relevant stable maps for the connected Severi problem on $Bl_2(\mathbb{P}^2)$. \square

More precisely, under the substitutions

$$Q = \frac{Q_1 Q_2}{E}, \quad F_1 = \frac{Q_1}{E}, \quad \text{and} \quad F_2 = \frac{Q_2}{E},$$

we have $Y^{Bl_2(\mathbb{P}^2)} = Y^{Bl_1(\mathbb{P}^1 \times \mathbb{P}^1)}$. Therefore computing

$$Z^{Bl_1 \mathbb{P}^1 \times \mathbb{P}^1} = e^{tQ_2/u} \langle v \mid Q_1^{|\cdot|} \exp(tM_S(u, Q_2)) \mid \widehat{w} \rangle$$

via the Fock space formalism and then taking log of the resulting generating function computes the connected Severi degrees of $Bl_2(\mathbb{P}^2)$, and hence the connected Severi degrees of \mathbb{P}^2 .

8. THE SURFACE $E \times \mathbb{P}^1$

We consider here the Gromov-Witten invariants for the surface $E \times \mathbb{P}^1$, where E is a smooth curve of genus 1. By deformation invariance, the Gromov-Witten invariants for $E \times \mathbb{P}^1$ are equal to those for $C \times \mathbb{P}^1$ where C is a nodal rational curve. The degeneration formula immediately yields the following result in terms of the trace.

Theorem 3. $Z^{E \times \mathbb{P}^1} = e^{tQ_2/u} \text{tr} \left(Q_1^{|\cdot|} \exp(tM_S(u, Q_2)) \right)$.

A disconnected curve C of genus 1 can have connected components of arbitrary genus (since higher genus components can be balanced with genus 0 components). If all of the connected components of C have genus 1, we call C a disconnected curve of pure genus 1, and we call the pure genus 1 Gromov-Witten invariants of $E \times \mathbb{P}^1$ those which count maps from curves of pure genus 1.

The genus 1 invariants are obtained by extracting the $u = 0$ coefficient $Z^{E \times \mathbb{P}^1}$. The pure genus 1 Gromov-Witten invariants of $E \times \mathbb{P}^1$ arise from the $u = 0$ coefficient of $\text{tr} \left(Q_1^{|\cdot|} \exp(tM_S(u, Q_2)) \right)$. The prefactor $e^{tQ_2/u}$ in Theorem 3 contributes only to impure genus 1 invariants. Let $Z_1^{E \times \mathbb{P}^1}$ be the partition function for pure genus 1 invariants. We conclude

$$Z_1^{E \times \mathbb{P}^1} = \text{tr} \left(Q_1^{|\cdot|} \exp(tM_S(0, Q_2)) \right) .$$

The specialization $M_S(0, Q)$ has a simple formula,

$$M_S(0, Q) = \sum_{k>0} \alpha_k[\mathbf{p}] \alpha_{-k}[\mathbf{p}] + Q \sum_{k>0} \alpha_k[\mathbf{1}] \alpha_{-k}[\mathbf{1}].$$

Proposition 1. *The eigenvalues of $M_S(0, Q)$ on the subspace of energy s are*

$$\{ (|\mu| - |\nu|)\sqrt{Q} \}_{|\mu|+|\nu|=s}.$$

Proof. For convenience, let $M = M_S(0, Q)$. Fix any partition p , and let W_p be the subspace spanned by the vectors

$$\{ |\mu, \nu\rangle \mid \mu \cup \nu = p \},$$

corresponding to splittings of p into two subpartitions μ and ν . The subspace W_p is invariant under M . We begin by computing the eigenvalues of M on W_p .

Let us write p in the frequency representation as $p = (1^{e_1}, 2^{e_2}, \dots)$. Let $\mu = (1^{l_1}, 2^{l_2}, \dots)$ and $\nu = (1^{r_1}, 2^{r_2}, \dots)$ satisfy $\mu \cup \nu = p$, or equivalently, $l_i + r_i = e_i$.

$$\begin{aligned} & M|(1^{l_1}, 2^{l_2}, \dots), (1^{r_1}, 2^{r_2}, \dots)\rangle \\ &= \sum_k (k)(l_k)|(1^{l_1}, 2^{l_2}, \dots, k^{l_k-1}, (k+1)^{l_k+1}, \dots), (1^{r_1}, 2^{r_2}, \dots, k^{r_k+1}, (k+1)^{r_k+1}, \dots)\rangle \\ & \quad + Q(k)(r_k)|(1^{l_1}, 2^{l_2}, \dots, k^{l_k+1}, (k+1)^{l_k+1}, \dots), (1^{r_1}, 2^{r_2}, \dots, k^{r_k-1}, (k+1)^{r_k+1}, \dots)\rangle \end{aligned}$$

In fact, W_p is naturally a tensor product

$$W_p = \bigotimes_{k|e_k \neq 0} W_{p_k},$$

where $p_k = (k^{e_k})$ is the subpartition of p consisting of all parts of size k . Let

$$M_k = \alpha_k[\mathbf{p}]\alpha_{-k}[\mathbf{p}] + Q \sum_{k>0} \alpha_k[\mathbf{1}]\alpha_{-k}[\mathbf{1}].$$

M_k acts nontrivially only on W_{p_k} , and we can write $M = \sum_k M_k$.

If we choose for each k an eigenvector $v_k \in W_{p_k}$ with eigenvalue λ_k for the operator M_k acting on W_{p_k} , then $\bigotimes_{k|e_k \neq 0} v_k$ is an eigenvector of M acting on W_p with eigenvalue $\sum_{k|e_k \neq 0} \lambda_k$. In the basis of W_{p_k} given by the set of vectors $\{|\mu, \nu\rangle\}$, $M_k = kA_{e_k+1}$, where A_{e_k+1} is the matrix defined in Proposition 2. By Proposition 2 below, the eigenvalues of M_k on W_{p_k} are

$$\{k(e_k)\sqrt{Q}, k(e_k - 2)\sqrt{Q}, \dots, k(-e_k)\sqrt{Q}\}.$$

We conclude the eigenvalues of M on W_p are $\{(|\mu| - |\nu|)\sqrt{Q}\}$ where μ, ν run over all partitions satisfying $\mu + \nu = p$. Finally, the eigenvalues of M on the subspace of energy s are $\{(|\mu| - |\nu|)\sqrt{Q}\}$ where μ, ν run over all partitions satisfying $|\mu| + |\nu| = s$. \square

Proposition 2. *Let $A_n =$*

$$\begin{bmatrix} 0 & n & & & \\ Q & 0 & n-1 & & \\ & 2Q & 0 & n-2 & \\ & & \ddots & \ddots & \\ & & & & 1 \\ & & & & nQ & 0 \end{bmatrix}.$$

The eigenvalues of A_n are $\{n\sqrt{Q}, (n-2)\sqrt{Q}, \dots, -n\sqrt{Q}\}$.

where the sum is over all pairs of partitions μ and ν (of possibly different sizes). Formula (14) is the Severi analogue of (5).

9. FURTHER DIRECTIONS

9.1. Rationality. A rationality result holds for the following generating functions of Severi degrees of $\mathbb{P}^1 \times \mathbb{P}^1$. Let

$$R_0 = 1 + \sum_{g \in \mathbb{Z}} u^{g-1} \sum_{d_2 > 0} N_{g,(0,d_2)}^\bullet Q_2^{d_2},$$

and, for $a > 0$, let

$$R_a = \sum_{g \in \mathbb{Z}} u^{g-1} \sum_{d_2 \geq 0} N_{g,(a,d_2)}^\bullet Q_1^a Q_2^{d_2}.$$

Proposition 3. *For each $a \geq 0$, we have*

$$Q_1^{-a} R_a \in \mathbb{Q}(u, Q_2).$$

Proof. Recall N_S defined in (11). By Theorem 1,

$$Q_1^{-a} R_a = \langle 1^a, \emptyset \mid (1 + N_S(u, Q_2) + N_S^2(u, Q_2) + \dots) \mid 1^a, \emptyset \rangle.$$

The right hand side is $\frac{1}{a!}$ times the coefficient of $\mid \emptyset, 1^a \rangle$ in

$$(1 - N_S(u, Q_2))^{-1} \mid 1^a, \emptyset \rangle.$$

The inverse of $1 - N_S(u, Q_2)$ can be computed by the cofactor expansion. □

9.2. Further operators. A point condition in the Severi problem corresponds to the operator $M_S(Q, u)$ on $\mathcal{F}[\mathbb{P}^1]$. In fact, an algebra of commuting operators containing $M_S(Q, u)$ is determined by appropriate descendent and relative insertions. Commutation holds for geometric reasons: when we degenerate we can assign conditions to the components of the degeneration in any order.

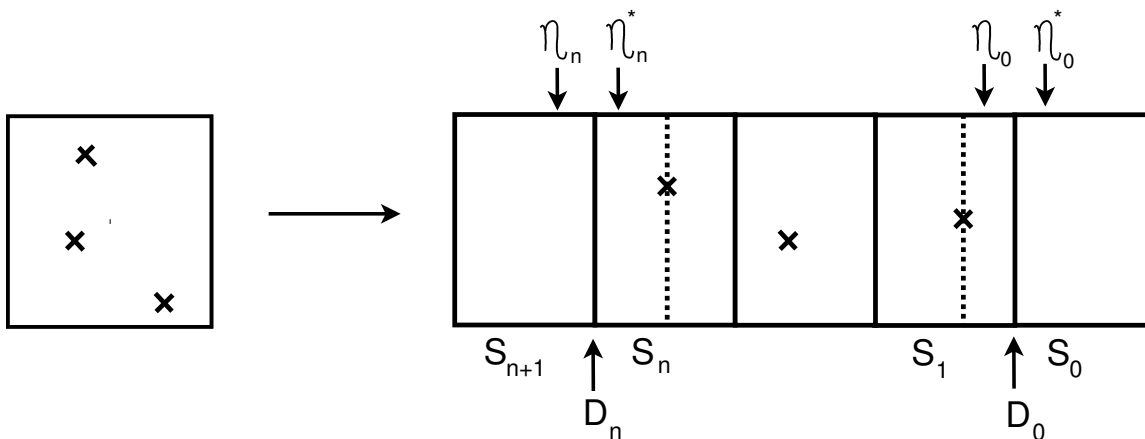


FIGURE 6. We can insert tangency conditions, as in the second and fourth components here, where we require the map to vanish to second order along the line $(0, 1)$ at a fixed point.

As an example, we consider the relative condition defined by a single tangency to a vertical line (in class $(0, 1)$) at a fixed point. The corresponding operator $M_F(Q, u)$ on the Fock space $\mathcal{F}[\mathbb{P}^1]$ is given by the following formula,

$$\begin{aligned} M_F(Q, u) &= \sum_{k=|\mu|, \ell(\mu)=2} \alpha_{-k}[\mathbf{p}] \alpha_\mu[\mathbf{p}] + u \alpha_{-\mu}[\mathbf{p}] \alpha_k[\mathbf{p}] \\ &+ \sum_{|\mu|=|\nu \cup k|, k>0, \ell(\mu)+\ell(\nu) \geq 2} Q u^{\ell(\mu)-1} \alpha_{-\mu}[\mathbf{1}] \alpha_\nu[\mathbf{1}] \alpha_k[\mathbf{p}] \\ &+ \sum_{|\mu \cup k|=|\nu|, k>0, \ell(\mu)+\ell(\nu) \geq 2} Q u^{\ell(\mu)} \alpha_{-\mu}[\mathbf{1}] \alpha_{-k}[\mathbf{p}] \alpha_\nu[\mathbf{1}] \\ &+ \sum_{k>1} \frac{k^2 - 1}{12} u \alpha_{-k}[\mathbf{p}] \alpha_k[\mathbf{p}]. \end{aligned}$$

Here, ν can be the empty partition in the second sum and μ can be empty in the third sum.

We have computed $M_F(u, Q)$, analogous to $M_S(u, Q) + \frac{Q}{u}$, by an analysis similar to the computation of Section 6.5.2. In each component, we analyze the geometries corresponding to solutions of the following equation

$$2b + g + \ell(\sigma) + \ell(\sigma') = 3.$$

The most interesting coefficient $\frac{k^2-1}{12}$ arises from a genus 1 contribution but is determined from the easier genus 0 contributions and the commutation

$$[M_S(Q, u), M_F(Q, u)] = 0.$$

The operators corresponding to higher order tangencies will involve increasingly complicated terms. The commutation relation is a considerable constraint. We leave further exploration of the algebra of descendent and relative operators for future research.

10. ACKNOWLEDGEMENTS

We thank M. Maydanskiy and A. Miller for helpful discussions. Much of the work was done during visits of Y.C. to IST Lisbon and ETH Zürich. R.P. was partially supported by the Swiss National Science Foundation grant SNF 200021143274.

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