GW/DT correspondence in Families

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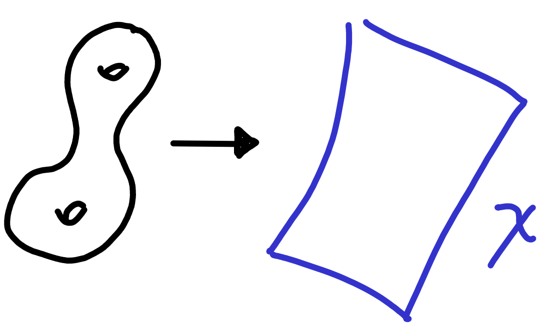
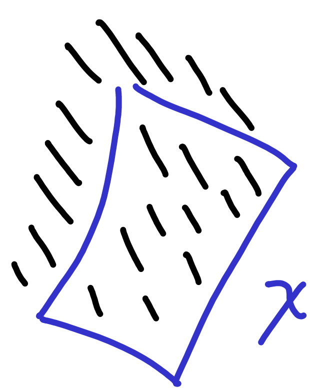
ETH ZÜRICH

Let X be a nonsingular projective algebraic variety $/\mathbb{C}$.

Gromov-Witten theory concerns the geometry of maps of curves to X.

In case $\dim_{\mathbb{C}}(X) = 3$, there is a Donaldson-Thomas theory of sheaves on X.

GW : 

, 

: DT

for X a 3-fold:

Equivalence

$$GW(X) \longleftrightarrow DT(X)$$

Long development starting with

MNOP I + II ideal sheaves ~2005

P-Thomas I, II, III stable pairs ~2010

relative / log geometries MNOP PP

P-Pixton, OOP descendants

2010-22 Moreira OOP

2022 Maulik-Ranganathan

Feyzbakhsh-Thomas higher rank CY3 2022

My goal here is to speak about the families correspondence

Moduli of Stable maps

Let X be a nonsingular projective variety $/\mathbb{C}$

We will consider maps

$$f : C \to X$$

target

algebraic morphism

Complete connected nodal curve of genus $g = 1 - \chi(\mathcal{O}_C)$

$$f_* [C] = \beta \in H_2(X, \mathbb{Z})$$

Curve class

$\overline{M}_g(X, \beta)$ is the moduli space of stable maps of genus g curves to X representing β.

- $[f : C \to X] \in \bar{\mathcal{M}}_g(X, \beta)$ is stable

  if and only if $|\mathrm{Aut}(f)| < \infty$.

- An automorphism of f is an automorphism of C which commutes with f:

$$\begin{tikzcd} C \arrow[d, "\phi"'] \arrow[dr, phantom, "\circlearrowright"] \arrow[rr, "f"] & & X \\ C \arrow[urr, "f"'] & & \end{tikzcd}$$

$$\mathrm{Aut}(f) \subset \mathrm{Aut}(C)$$

if $|\mathrm{Aut}(C)| < \infty$

then $|\mathrm{Aut}(f)| < \infty$

and f is stable

When are two stable maps

$$[f : C \to X], \quad [\hat{f} : \hat{C} \to X]$$

isomorphic? If and only if

$\exists\ \phi : C \xrightarrow[\sim]{\text{isom}} \hat{C}$ which commutes with $f, \hat{f}$:

$$\begin{tikzcd} C \arrow[dr, "f"] \arrow[d, "\phi"'] \arrow[dr, phantom, shift right=3ex, "\circlearrowright"] & \\ \hat{C} \arrow[r, "\hat{f}"'] & X \end{tikzcd}$$

parallel definitions, Auts and isoms must respect the markings

$\overline{\mathcal{M}}_g(X, \beta)$ and $\overline{\mathcal{M}}_{g,n}(X, \beta)$

are Deligne-Mumford stack, but may be reducible, non-reduced, and very singular.

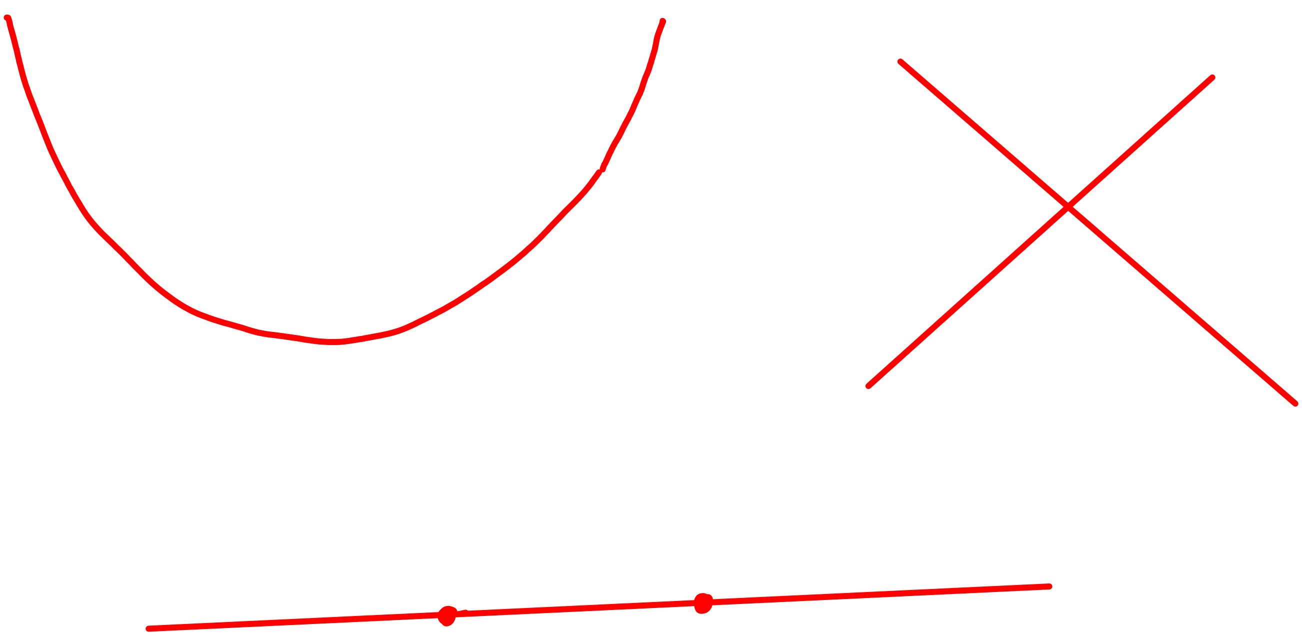
First examples:

- $\overline{\mathcal{M}}_{g,n}(X,0) = \overline{\mathcal{M}}_{g,n} \times X$ for $2g-2+n>0$

- $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^n, 1) = Gr(\mathbb{P}^1, \mathbb{P}^n)$

  class of the line $L \in H_2(\mathbb{P}^n, \mathbb{Z})$

- $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^2, 2)$ = classical space of complete conics



Obstruction theory

$\overline{M}_{g,n}(X,\beta)$ carries a Def-Obs theory with

$\chi(f^*T_X)$

$$\text{vir dim } \overline{M}_{g,n}(X,\beta) = \int_\beta c_1(X) + \dim_{\mathbb{C}} X(1-g) + 3g-3+n$$

dim of $\overline{M}_{g,n}$

The Def-Obs theory for a fixed domain curve $f: C \to X$ is

Def $\quad H^0(C, f^*T_X)$

Obs $\quad H^1(C, f^*T_X)$

higher obstructions vanish

$\mathcal{M}_C(X,\beta)$ has Def-Obs theory

fixed domain

of vir dim $\chi(C, f^*T_X)$

Then, since $\mathcal{M}_{g,n}$ is nonsingular,

Artin stack

we obtain a Def-obs theory

for $\overline{\mathcal{M}}_{g,n}(X,\beta)$.

Behrend-Fantechi

Li-Tian

$$\begin{tikzcd} & \mathcal{M}_C \times C \arrow[dl, "\pi"'] \arrow[dr, "f"] & \\ \mathcal{M}_C & & X \end{tikzcd} \qquad \begin{tikzcd} (R\pi_* f^* T_X)^\vee \arrow[d] \\ L^\bullet_{\mathcal{M}_C} \end{tikzcd}$$

Gromov witten theory

$$[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{vir} \in A_{\text{virdim}}(\overline{\mathcal{M}}_{g,n}(X,\beta))$$

In families

The whole construction is possible in families

Let $\begin{array}{c}\mathcal{X} \\ \varepsilon \downarrow \\ B\end{array}$ be a flat family of nonsingular projective varieties over B

pure dim

Then $\overline{M}_{g,n}(\varepsilon, \beta)$

fiber class

with a virtual fundamental class

$$[\overline{M}_{g,n}(\varepsilon,\beta)]^{vir} \in A_{\text{virdim} + \dim B}(\overline{M}_{g,n}(\varepsilon,\beta))$$

Example I : K3 Surfaces

Let S be a K3 surface.

The virtual dimension for the spaces of rational curves is -1 :

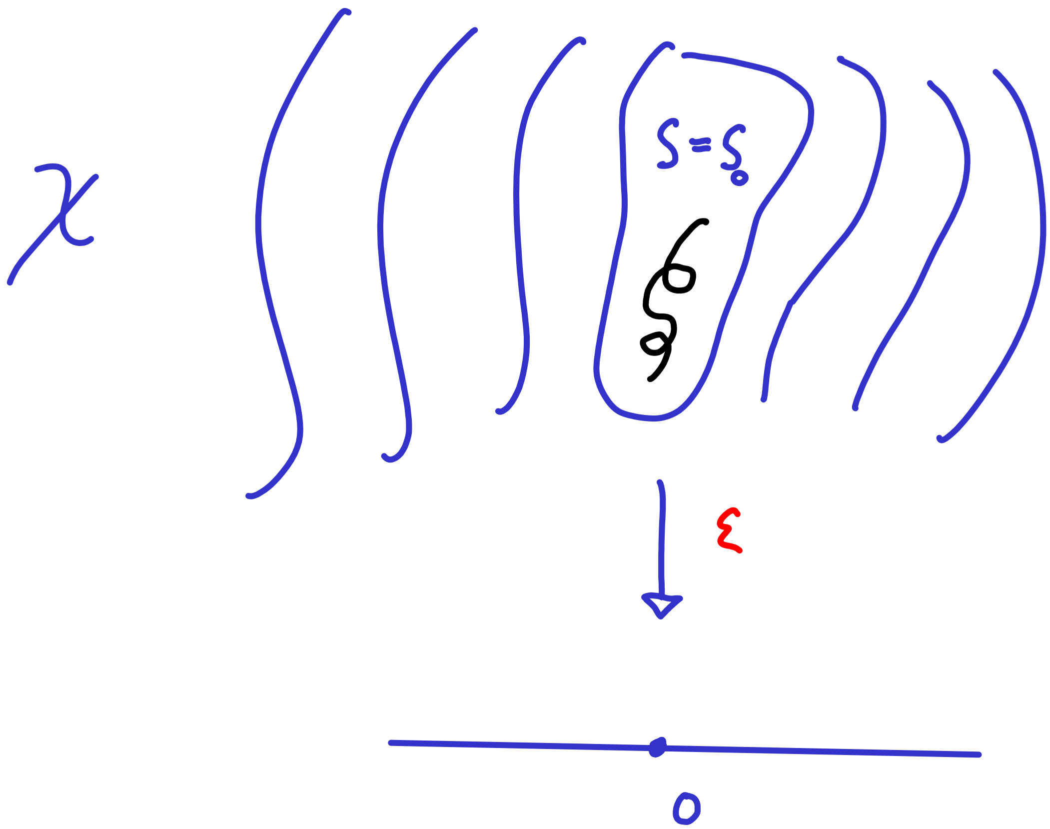
$$\text{vir dim } \overline{M}_{0,0}(S,\beta) = \int_\beta c_1(S) + \dim_{\mathbb{C}} S\,(1-0) + 3\cdot 0 - 3$$

$$= 0 + 2\cdot 1 - 3$$

$$= -1$$

But K3 surfaces have rational curves, so how can we see them?

Look at a family



Bryan, Leung

virdim of $g=0$ curves in $\mathcal{X}$ $= 0$

So we can count:

$$N_{0,h} = \int_{[\bar{M}_{g,n}(\varepsilon,\beta_h)]^{vir}} 1 \qquad \beta_h^2 = 2h-2$$

primitive

- There has been 27 years of progress starting with the Yau-Zaslow formula (1995)

$$\sum_{h \geq 0} N_{0,h}\, q^{h-1} = \frac{1}{q} \prod_{n \geq 1} \frac{1}{(1-q^n)^{24}}$$

- Families point of view $\Rightarrow$ GW/NL Correspondence Maulik-P

- See recent papers of Oberdieck 2021-2022

- With Qizheng Yin, we viewed the geometry in another direction:

Consider the moduli space of quasi polarized K3 surfaces

$$\begin{array}{c} \mathcal{S} \\ \downarrow \varepsilon \\ \mathcal{M}_h \end{array}$$

using the GW theory of reduced class

in families $[\overline{M}_{g,n}(\varepsilon, \beta_h)]^{red}$,

we proved that

P-Yin 2020

$R^*(\mathcal{M}_h)$ is generated by

Noether-Lefschetz cycles

tautological classes

Example II : Equivariant GW theory

Let $\mathbb{C}^*$ act on X nonsingular projective variety

Then we have algebraic approximations to

$$\begin{array}{ccc} \mathbb{C}^* \curvearrowright E\mathbb{C}^* & \text{given by} & \mathbb{C}^* \curvearrowright \mathbb{C}^{n+1} \setminus 0 \\ \downarrow & & \downarrow \\ B\mathbb{C}^* & & \mathbb{P}^{n+1} \end{array}$$

Equivariant GW theory is defined by the family

$$\begin{array}{c} E\mathbb{C}^* \times_{\mathbb{C}^*} X \\ \downarrow \\ B\mathbb{C}^* \end{array}$$

which is approximated by

$$\begin{array}{c} (\mathbb{C}^{n+1} \setminus 0) \times_{\mathbb{C}^*} X \\ \downarrow \\ \mathbb{P}^{n+1} \end{array}$$

which is a family $\begin{array}{c} \mathcal{X} \\ \varepsilon \downarrow \\ \mathbb{P}^{n+1} \end{array}$ with fibers isomorphic to X

Equivariant GW theory concerns

$$\varepsilon_*\left([\overline{\mathcal{M}}_{g,n}(\varepsilon, \beta)]^{vir} \cup \cdots \right) \in H^*(B\mathbb{C}^*)$$

If X is a 3-fold, then there is a GW/DT correspondence (Chow form)

<u>Gromov-Witten</u>

Let $\beta \in H_2(X, \mathbb{Z})$ be a curve class

$\overline{\mathcal{M}}_g(X, \beta)$ has virtual dim $\int_\beta c_1(X)$

independent of g

There is a map

Chow variety of curves of class β

$$CH_M : \bar{M}_g(X,\beta) \longrightarrow \mathrm{Chow}(X,\beta)$$

possibly disconnected no constant maps on connected components

Let $$Z^{GW}_{X,\beta}(u) = \sum_g u^{2g-2} \cdot CH_{M*}\left[\bar{M}^{\bullet}_g(X,\beta)\right]^{vir}$$

MOOP 2010

$$\in A_*\big(\mathrm{Chow}(X,\beta)\big) \otimes \mathbb{Q}((u))$$

Donaldson-Thomas

Let $\beta \in H_2(X,\mathbb{Z})$ be a curve class

$I_n(X,\beta)$ has virtual dim $\int_\beta c_1(X)$

independent of n

Hilbert scheme of curves

$I_n(X,\beta)$ Hilbert scheme of subschemes

$$0 \to \mathcal{I} \to \mathcal{O}_X \to \mathcal{O}_C \to 0$$

with $[\mathcal{O}_C] = \beta \in H_2(X,\mathbb{Z})$

$$\chi(\mathcal{O}_C) = n$$

$$\mathrm{Def} = \mathrm{Ext}^1_0(\mathcal{I},\mathcal{I})$$

$$\mathrm{Obs} = \mathrm{Ext}^2_0(\mathcal{I},\mathcal{I})$$

2 term obstruction theory

$I_n(X,\beta)$ carries a virtual fundamental class

of $\mathrm{vir\,dim} = \dim \mathrm{Def} - \dim \mathrm{Obs}$

$$= \int_\beta c_1(X)$$

$$[I_n(X,\beta)]^{vir} \in A_{\mathrm{virdim}}(I_n(X,\beta))$$

There is a map

Chow variety of curves of class β

$$\mathrm{CH}_I : I_n(X,\beta) \longrightarrow \mathrm{Chow}(X,\beta)$$

Let $$Z^{DT}_{X,\beta}(q) = \sum_n q^n \cdot \mathrm{CH}_{\mu *}[I_n(X,\beta)]^{vir}$$

$$\in A_*\big(\mathrm{Chow}(X,\beta)\big) \otimes \mathbb{Q}((q))$$

GW/DT Correspondence (Conjecture) MOOP

$$(-iu)^{c_\beta} Z^{GW}_{X,\beta}(u) = (-q)^{-c_\beta/2} Z^{DT}_{X,\beta}(q) \Big/ M(-q)^{\int_X c_3 - c_1 c_2}$$

after $e^{iu} = -q$, $c_\beta = \int_\beta c_1(T_X)$

There is alot to explain here.

- As formulated, the conjecture for the Chow variety is proven in very few cases

[Toric 3-fold via localization (but only after localization)]

- $M(-q)^{\int_X c_3 - c_1 c_2}$ comes from the Hilbert scheme of points of a 3-fold

$$M(q) = \prod_{n \geq 1} \frac{1}{(1-q^n)^n}$$

McMahon counts box configurations

Theorem: MNOP, Behrend-Fantechi, Li, Levine-P

$$Z^{DT}_{X,0}(q) = \sum_n q^n \cdot \int_{[I_n(X,0)]^{vir}} 1$$

$$= M(-q)^{\int_X c_3 - c_1 c_2}$$

$I_n(X,0)$ is Hilbert Scheme of n points of X

A basic step here is the

Case $X = \mathbb{C}^3$ non compact, but well defined after localization

$T = (\mathbb{C}^*)^3 \curvearrowright X$

T-fixed points are the box configurations

Sol LeWitt 1998
National Gallery DC

Wild identity (MNOP II):

$$\sum_{\pi} \frac{e\, \mathrm{Ext}^1_0(\mathcal{I}_\pi, \mathcal{I}_\pi)}{e\, \mathrm{Ext}^0_0(\mathcal{I}_\pi, \mathcal{I}_\pi)}\, q^{|\pi|} = M(-q)^{-\frac{(s_1+s_2)(s_1+s_3)(s_2+s_3)}{s_1 s_2 s_3}}$$

all 3-d box configuration

corresponding monomial ideal

s_i generator of $H_{\mathbb{C}^*_i}(pt)$

General GW/DT correspondence in families

Let $\begin{array}{c}\mathcal{X}\\ \varepsilon\downarrow\\ B\end{array}$ be a flat family of nonsingular projective 3-folds over B ← pure dimensional

Let β be a curve class on the fibers of ε

Let $\gamma_1, \gamma_2, \ldots, \gamma_k \in A^*(\mathcal{X})$

Let $Z^{GW}_{\varepsilon,\beta}\left[\gamma_1, \gamma_2, \ldots, \gamma_k\right](u)$

$$= \sum_g u^{2g-2} \cdot \mathrm{CH}_{\mu *}\left(\left[\bar{M}^{\bullet}_{g,n}(\varepsilon,\beta)\right]^{vir} \cap \prod_{i=1}^{n} ev_i^*(\gamma_i)\right)$$

$$\in A_*\left(\mathrm{Chow}(\varepsilon,\beta)\right) \otimes \mathbb{Q}((u))$$

Let $Z^{DT}_{\varepsilon,\beta}[\gamma_1, \gamma_2, \dots, \gamma_k](u)$

$$= \sum_n q^n \cdot Ch_{M*}\left([I_n(\varepsilon,\beta)]^{vir} \cap \overbrace{\prod_{i=1}^{n} \tau_0(\gamma_i)}^{\text{DT insertions}} \right)$$

$$\in A_*\big(\mathrm{Chow}(\varepsilon,\beta)\big) \otimes \mathbb{Q}((q))$$

GW/DT correspondence (Conjecture)

$$(-iu)^{c_\beta} Z^{GW}_{\varepsilon,\beta}[\gamma_1,\dots,\gamma_k](u) = (-q)^{-c_\beta/2} \frac{Z^{DT}_{\varepsilon,\beta}[\gamma_1,\dots,\gamma_k](q)}{M(-q)^{\int_X c_3 - c_1 c_2}}$$

after $e^{iu} = -q$, $\quad c_\beta = \int_\beta c_1(T_\varepsilon)$

Variations :

(i) allow fibers of $\mathcal{E}$ to have normal crossings singularities

(ii) include higher descendants

(iii) push down to B

$$A_*(\mathrm{Chow}(\mathcal{E},\beta)) \to A_*(B)$$

(iv) Study PT/DT Correspondence

Do we know any interesting example of the GW/DT Correspondence for families other than the equivariant case?

Example III

Let $B = \overline{M}_{g,n}$ with the

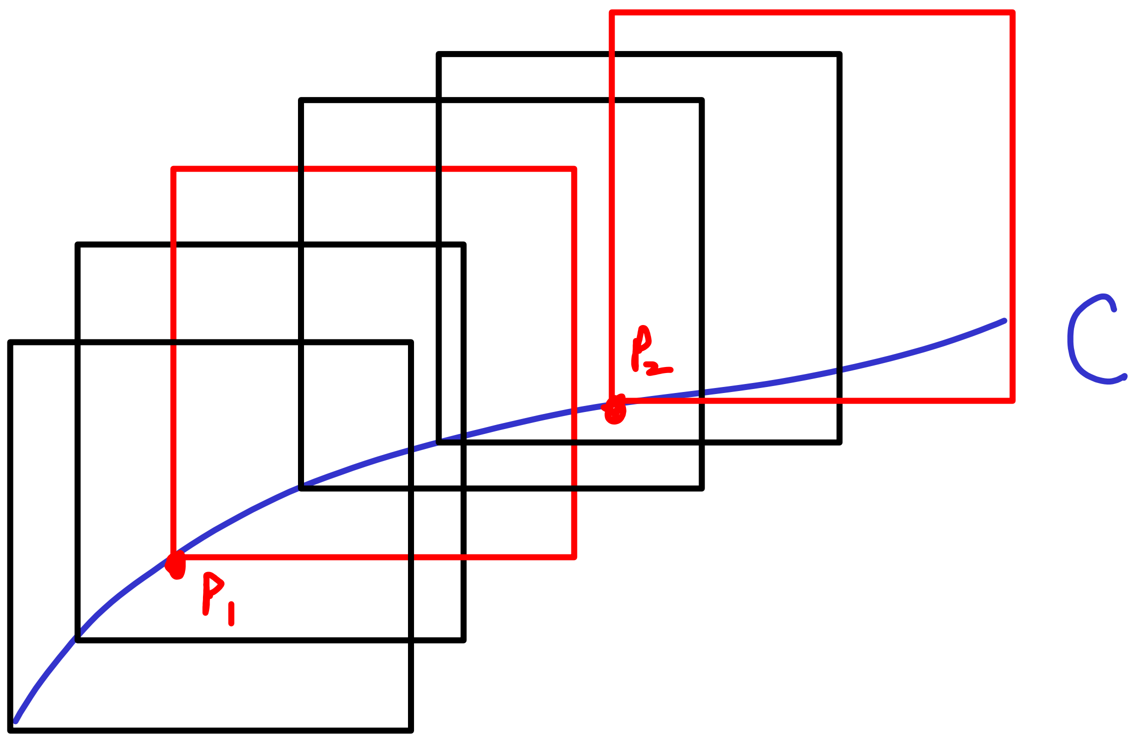
universal curve
$$\begin{tikzcd} \mathcal{C} \arrow[d] \\ \overline{M}_{g,n} \arrow[u, bend right, "p_1,\dots,p_n"'] \end{tikzcd}$$

Not a family of 3-folds!

Let
$$\begin{tikzcd} \mathcal{X} \arrow[d, "\varepsilon"'] \\ \overline{M}_{g,n} \end{tikzcd}$$
be
$$\begin{tikzcd} \mathbb{C}^2 \times \mathcal{C} \arrow[d, "\varepsilon"'] \\ \overline{M}_{g,n} \end{tikzcd}$$

We view the fibers of ε

as 3-folds relative to surfaces



Theorem (P - HH Tseng which relies on results of Okounkov - P
2020 Bryan - P)

GW/DT correspondence holds for

$$\mathcal{X} \xrightarrow{\varepsilon} \overline{M}_{g,n}$$

γ_i relative conditions given by a partition of d

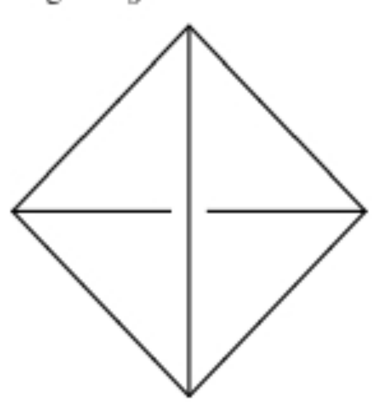
$$Z^{GW}_{\varepsilon,d}[\gamma_1,\dots,\gamma_n](u) = \Theta \; Z^{PT}_{\varepsilon,d}[\gamma_1,\dots,\gamma_n](q)$$

Comb factor

in $A^*(\overline{M}_{g,n}) \otimes \mathbb{Q}((q))$

after $e^{iu} = -q$

Proof uses Givental-Teleman CohFT theory



Jim Bryan

To show the results are not just abstract equalities, here is a specific

Consequence:

Let $\overline{\mathcal{H}}_1(\underbrace{(2),(2),\ldots,(2)}_{2n \text{ times}})$

be the space of bi-elliptics with 2n ordered branch points.

Consequence:

Hodge classes on genus $n+1$ cover

$$\sum_{n=1}^{\infty} \frac{u^{2n-1}}{(2n-1)!} \int_{\overline{H}_1((2)^{2n})} \lambda_{n+1}\lambda_{n-1} = \frac{i}{24}\,\frac{1-e^{iu}}{1+e^{iu}}$$


The
End