New results on cycles on the moduli space of curves

Princeton Algebraic Geometry Seminar

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ETHZ

The lecture has three parts:

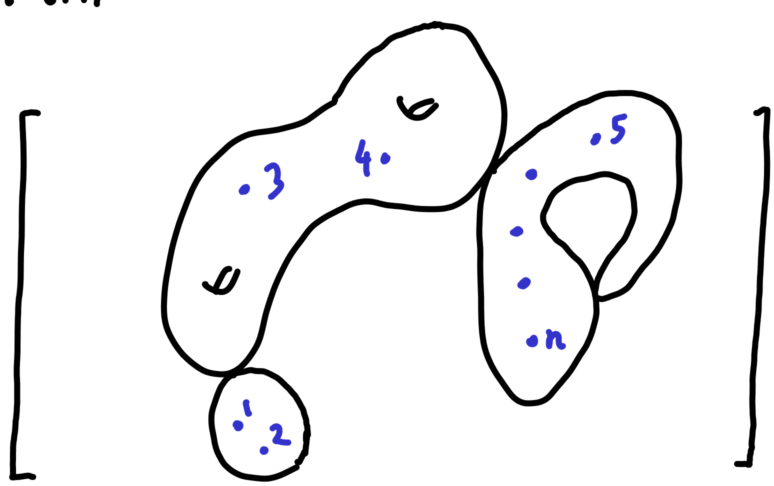
I. 0-cycles on $\overline{\mathcal{M}}_{g,n}$ w/ J. Schmitt
EPIGA 4 (2020)

II. λ_g on $\overline{\mathcal{M}}_g$ w/ S. Molcho, J. Schmitt
arXiv: 2101.08824

III. Cycles related to Abel-Jacobi theory
w/ Y. Bae, D. Holmes, J. Schmitt, R. Schwarz
arXiv: 2004.08676

All directions concern

moduli space of stable curves

$\overline{\mathcal{M}}_{g,n} \ni$ 

genus g, connected, nodal,
n marked points

I. 0-cycles on $\overline{\mathcal{M}}_{g,n}$

$\boxed{\text{w/ J. Schmitt}}$

$$R^*(\overline{\mathcal{M}}_{g,n}) \subset A^*(\overline{\mathcal{M}}_{g,n}) \qquad \mathbb{Q}\text{-coeffs}$$

subring of tautological classes

generated by strata classes $\overline{\mathcal{M}}_\Gamma \to \overline{\mathcal{M}}_{g,n}$

kappa classes κ, cotangent line classes ψ

We are interested here in $R_0(\overline{\mathcal{M}}_{g,n}) \subset A_0(\overline{\mathcal{M}}_{g,n})$

Though the birational geometry of $\overline{\mathcal{M}}_{g,n}$

is in general complicated, we have

$$\boxed{R_0(\overline{\mathcal{M}}_{g,n}) \cong \mathbb{Q}}$$

Graber-Vakil 2000

+ other approaches

Question: Let $(C, p_1, \dots, p_n) \in \overline{\mathcal{M}}_{g,n}$.

When is $[C, p_1, \dots, p_n] \in R_0(\overline{\mathcal{M}}_{g,n})$?

The answer is always if $\overline{\mathcal{M}}_{g,n}$ is rationally connected:

Benzo
Bruno
Casanati
Farkas
Fontamari
Verra

g	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$n_{\max}$	∞	10	12	14	15	12	15	11	8	9	3	10	1	0	2	0

genus 16 ?

$\overline{\mathcal{M}}_{g,n}$ is RC for $n \leq n_{max}$

But the answer is not always yes.

$CH_0(\overline{\mathcal{M}}_{1,11})$ and $CH_0(\overline{\mathcal{M}}_{2,14})$ known not to be finitely generated because of holomorphic forms, so the question is nontrivial.

What form of answer could we hope for?

- Bloch-Beilinson Conjecture $\Rightarrow$

  If $(C, p_1, \ldots, p_n)$ is defined over $\bar{\mathbb{Q}}$,

  then $[C, p_1, \ldots, p_n] \in R_0(\bar{M}_{g,n})$

  Perhaps should be proven with Belyi's Theorem, but I have nothing to report in this very interesting direction.

- Search for classical geometric conditions

  $\Downarrow$

  Suppose C lies on a surface

  $$C \subset S$$

  C: irreducible nonsingular curve of genus g

  S: nonsingular projective simply connected surface

(i) S is a rational surface.

But every curve lies on some rational surface, so we will need conditions.

$$\bar{\mathcal{M}}_g(S, [C])$$ moduli space of stable maps

$$\operatorname{vdim} \bar{\mathcal{M}}_g(S, [C]) = \int_{[C]} c_1(S) + g - 1$$

P-Schmitt

Theorem R. For $C \subset S$ (irred, ns) where S is rational and $\int_{[C]} c_1(S) > 0$, we have

$$[C, p_1, \dots, p_n] \in R_0(\bar{\mathcal{M}}_{g,n})$$

for all $p_i \neq p_j$ and $n \leq \operatorname{vdim} \bar{\mathcal{M}}_g(S, [C])$

Comments on the proof:

- Take $n = \operatorname{vdim} \overline{M}_g(S, [C])$

lower n follow by forgetting points

- Then for generic points $p_i \in C$,

$$\varepsilon_\star\left([\overline{M}_{g,n}(S,[C])]^{vir} \cap \prod_{i=1}^{n} ev_i^*[p_i] \right)$$
$$= [C, p_1, \ldots, p_n] \in A_0(\overline{M}_{g,n})$$

$$\varepsilon : \overline{M}_{g,n}(S,[C]) \to \overline{M}_{g,n}$$

The case of generic $p_i \in C$ is sufficient

- Use well-known properties of stable maps and virtual classes to conclude $\varepsilon_\star\left([\overline{M}_{g,n}(S,[C])]^{vir} \cap \prod_{i=1}^{n} [p_i] \right)$ is tautological.

Deformation, Localization

- Can the bound $n \leq \mathrm{vdim}\left(\overline{M}_g(S, [C])\right)$ be improved?

Expectation is No :

$C \subset \mathbb{P}^1 \times \mathbb{P}^1$

genus 4,
type $(3,3)$,
generic genus 4
Curve appears

vdim = 15, so all points of $\overline{M}_{4,15}$ are tautological.

But $\overline{M}_{4,16}$ is expected to carry a $(0,25)$ form, so Theorem R should fail for $n=16$

- Can the positivity $\int_{[C]} c_1(S) > 0$ be dropped?

Issue is related Harbourne-Hirschowitz Conjecture.

Conjecture: Positivity can be dropped in Theorem R

(ii) S is a $K3$ surface

In the Chow group of points of S, the is a distinguished rank 1 subspace

Beauville-Voisin points

$$BV = \mathbb{Z} \subset A_0(K3)$$

generated by points on rational curves of S

P-Schmitt

irred, ns

Theorem K3. For $C \subset S$ where S is a $K3$ surface, we have

$$[C, p_1, \ldots, p_n] \in R_0(\bar{M}_{g,n})$$

for all Beauville-Voisin points $p_i \neq p_j$ with $n \leq g = \text{genus}(C)$.

$g =$ reduced virdim $(\bar{M}_g(S, [C]))$

Pattern of the proof is similar to the rational case

- Express $[C, p_1, \dots, p_n]$ in terms of intersection with the reduced vir class

$$\varepsilon_\star \left(\left[\bar{\mathcal{M}}_{g,n}(S, [C]) \right]^{red} \cap \prod_{i=1}^{n} ev_i^* [p_i] \right)$$

BV

$$= [C, p_1, \dots, p_n] \in A_0(\bar{\mathcal{M}}_{g,n})$$

- The above cycle is defined for all pairs $(S, [C])$

- Use generic point in moduli of K3 surfaces and result by Xi Chen : There exists a nodal rational curve.

Must the points be constrained in $A_0(S)$?

Expectation is yes:

Consider genus 11 curves with 11 points on a K3 surface. By the Mukai Correspondence, we can achieve the general element of $\overline{M}_{11,11}$ by varying the K3. But $\overline{M}_{11,11}$ has Kodaira dim 19 and we expect complicated $A_0(\overline{M}_{11,11})$.

Brief excursion to the Moduli $\mathcal{M}^{K3}_{2\ell}$ of K3s:

There are now almost parallel questions for $\mathcal{M}^{K3}_{2\ell}$

$$\boxed{\begin{array}{l}\text{Question [Oprea-P]: Let } \Lambda \text{ and } \hat{\Lambda} \text{ be two rank 18 lattices with degree } 2\ell \text{ polarization classes.}\\ \text{Are the classes } [\mathcal{M}^{K3}_{\Lambda}], [\mathcal{M}^{K3}_{\hat{\Lambda}}] \in A_2(\mathcal{M}^{K3}_{2\ell}) \text{ proportional?}\\ \mathbb{Q}\text{-coeffs}\end{array}}$$

(iii) We can consider many other surfaces.

A question which I like:

Question GT. Let $C \subset S$ be an irreducible nonsingular canonical curve on a simply connected surface S of general type.

Is $[C] \in A_0(\overline{M}_{g(C)})$ a tautological class?

The point here is that for a canonical curve

$$\text{vdim} = \int_C c_1(S) + g(C) - 1$$

$$= \frac{1}{2}\int_C K(C) + g(C) - 1 = 0$$

GW invariant is sign of Θ-char $\Theta(C)|_C$

A hope: whenever an irreducible non singular curve C lies on a simply connected surface S with $\text{vdim} \geq 0$, then $[C] \in R_0(\overline{M}_{g(C)})$.

II. λ_g on $\overline{\mathcal{M}}_g$

w/ S. Molcho, J. Schmitt

The Hodge bundle:

universal curve

$$\begin{array}{c} \mathcal{C} \\ \pi \downarrow \\ \overline{\mathcal{M}}_g \end{array}$$

dualizing sheaf

$$\mathbb{E}_g = \pi_\star \omega_\pi$$

rank g vector bundle on $\overline{\mathcal{M}}_g$

Chern classes:

$$\lambda_i = c_i(\mathbb{E}_g) \begin{array}{l} \in RH^{2i}(\overline{\mathcal{M}}_g) \subset H^{2i}(\overline{\mathcal{M}}_g) \\ \in R^i(\overline{\mathcal{M}}_g) \subset CH^i(\overline{\mathcal{M}}_g) \end{array}$$

The top Chern class $\lambda_g = c_{top}(\mathbb{E}_g)$ is the most studied class on $\overline{\mathcal{M}}_g$

- Vanishing properties

$$\lambda_g^2 = 0 \quad \text{on} \quad \overline{\mathcal{M}}_g$$

$\Delta_0 \subset \overline{\mathcal{M}}_g$ divisor of curves with non-separating node

$$\lambda_g \big|_{\Delta_0} = 0 \quad \text{on} \quad \Delta_0$$

Mumford's Identity 1983

$$c(\mathbb{E}_g) \cdot c(\mathbb{E}_g^\vee) = 1$$

Trivial quotient $\mathbb{C}$ obtained from residue at the node

- Connected to Gromov-Witten theory in several ways:

λ_g Formula, Katz-Klemm-Vafa formula, Maulik-P-Thomas

Quantum tropical Vertex, ... Bousseau

λ_g Formula Faber - P 2003

$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{k_1} \cdots \psi_n^{k_n} \lambda_g = \binom{2g+n-3}{k_1, \dots, k_n} \int_{\overline{\mathcal{M}}_{g,1}} \psi_1^{2g-2} \lambda_g$$

$g \geq 1$

$\psi_i = c_1(\mathbb{L}_i)$

i^{th} cotangent line

- Connection to abelian varieties:

$(-1)^g \lambda_g$ arises as the pull - back of the universal 0 - section of the moduli space of $\overline{\text{PPAVs}}$.

Alexeev Olsson

Principally Polarized Abelian Varieties

Hain 2013

Grushevsky - Zakharov 2014

Our starting point is a beautiful formula over $\mathcal{A}_g$ ← Moduli of PPAVs of dim g

Universal abelian variety

$$\begin{tikzcd} \mathcal{X} \arrow[d, "\pi"'] \\ \mathcal{A}_g \arrow[u, bend right, "s"'] \end{tikzcd}$$

0-section

Let $Z_g \in CH^g(\mathcal{X}_g)$ be the class of the 0-section. Then:

$$\boxed{Z_g = \frac{\Theta^g}{g!} \in CH^g(\mathcal{X})}$$

via FM by Deninger, Murre

where $\Theta \in CH^1(\mathcal{X})$ is the universal symmetric theta divisor trivialized along s

Question: Does $\lambda_g \in R^*(\overline{M}_g)$ lie in the subalgebra generated by divisors?

Answer (Molcho-P-Schmitt): No

$$\lambda_g \notin \mathrm{div}\, R^*(\overline{M}_g) \quad \text{for all } g \geq 3.$$

[also λ_g does not lie in the subalgebra generated by $R^1(\overline{M}_g)$ and $R^2(\overline{M}_g)$ for all $g \geq 8$]

Proof: uses Admcycles, knowledge of $R^*(\overline{M}_g)$ and geometric arguments.

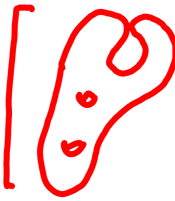
But these results are negative.

The positive result is from the log perspective

Molcho
P
Schmitt

$\boxed{\text{Theorem Log: } \lambda_g \text{ lies in the subalgebra of } \log CH^*(\bar{M}_g, \Delta_0) \text{ generated by divisors}}$

What is $\log CH^*(\bar{M}_g, \Delta_0)$?

$\Delta_0 \ni$ []

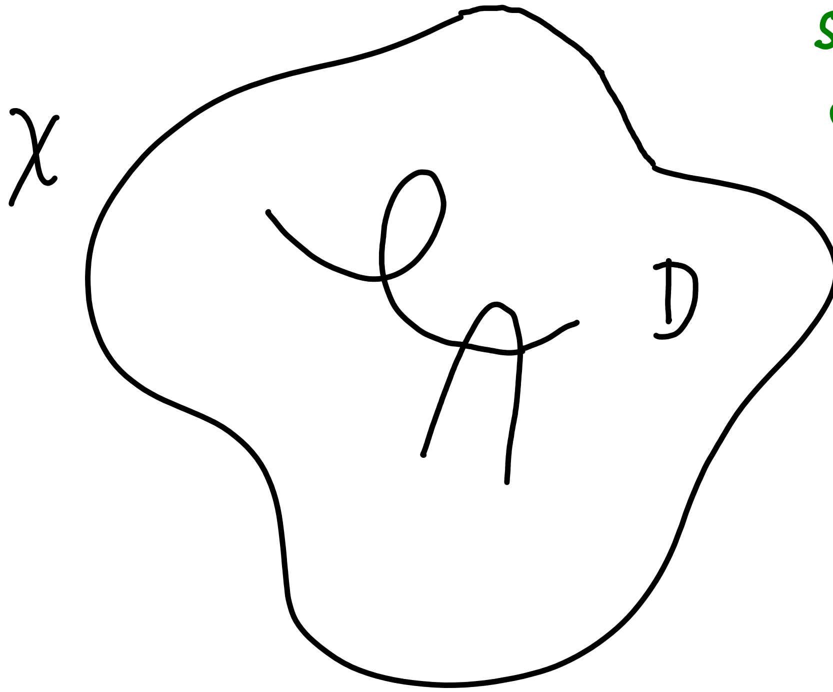
Given any nonsingular variety X with a normal crossings divisor $D \subset X$ we obtain a log scheme (X, D)

There are two related Chow construction lying over $CH^*(X)$

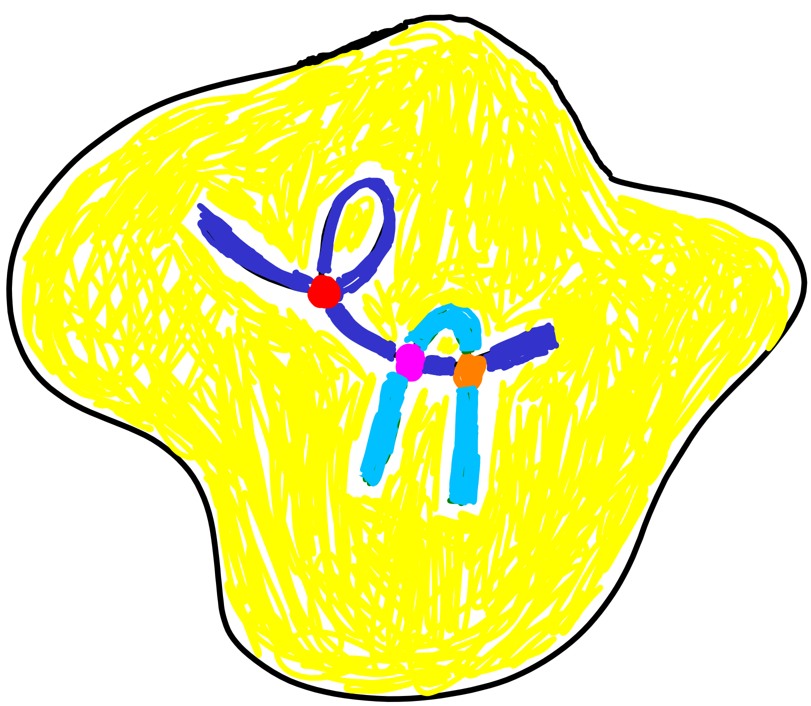
$$CH^*(X) \subset \log CH^*(X) \subset b\,CH^*(X)$$

used by D. Holmes — Shokurov

Not assumed strict normal crossings



Basic Notion of stratification



Strata indicated by colors

A stratum $S \subset X$ is nonsingular and quasiprojective

$\bar{S} \subset X$ may be singular (mildly)

A simple blow-up of (X, D) is a blow up along a nonsingular stratum closure $\bar{S} \subsetneq X$.

$$Bl : (\hat{X}, \hat{D}) \to (X, D)$$

blowup

strict transform of D union the exceptional divisor E

Define a category $\mathcal{B}(X, D)$

- Objects are $(\tilde{X}, \tilde{D}) \xrightarrow{\tilde{\phi}} (X, D)$ where $\tilde{\phi}$ is a composition of simple blowups

- Morphisms are commutative diagrams

$$\begin{tikzcd}
(\tilde{\tilde{X}}, \tilde{\tilde{D}}) \arrow[rr, "\gamma"] \arrow[dr, "\tilde{\tilde{\phi}}"'] & & (\tilde{X}, \tilde{D}) \arrow[dl, "\tilde{\phi}"] \\
& (X, D) &
\end{tikzcd}$$

γ is a composition of simple blowups

$$\boxed{\log CH^*(X, D) \overset{\text{def}}{=} \lim_{\substack{\longrightarrow \\ (\tilde{X}, \tilde{D}) \in \mathcal{B}(X, D)}} CH^*(\tilde{X})}$$

$bCH^*(X)$ has the same definition except that blowups along all nonsingular varieties are allowed.

A nice exercise: $bCH^*(X)$ is generated by divisors

But $\log CH^*(X,D)$ is much smaller.

We have $\lambda_g \in \operatorname{div} \log CH^*(\overline{M}_g, \Delta_0)$

and actually prove λ_g is generated by log divisors (components of the log boundary).

Proof: Not formal! Starts with Pixton's DR formula. Some theory of tautological classes for log schemes is needed.

Why are we interested?

Better understood → Gromov-Witten theory $\rightsquigarrow$ cycles in $CH^*(\overline{M}_g)$

less well understood → log Gromov-Witten theory $\rightsquigarrow$ cycles in $\log CH^*(\overline{M}_g, \partial\overline{M})$

In order apply the theory, we must develop facility with $\log CH^*$.

λ_g is the simplest case.

Parallel results for general DR cycles will appear in forthcoming papers by Holmes-Schwarz, Molcho-Ranganathan

Hope: eventually to prove statements such as

$$\varepsilon_* [\overline{M}_g(X,\beta)]^{vir} \in RH^*(\overline{M}_g) \text{ for every } X$$

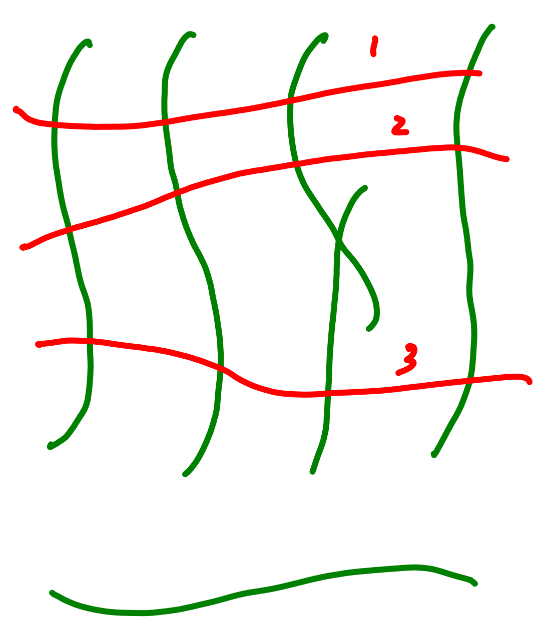
See speculation in Levine-P

III. Cycles related to Abel-Jacobi theory

$\boxed{\text{w/ Y. Bae, D. Holmes, J. Schmitt, R. Schwarz}}$

Consider a family of pointed nodal curves

$$\begin{array}{c} \mathcal{C} \\ p_1 \uparrow \cdots \uparrow p_n \;\; \downarrow \pi \\ S \end{array}$$



Curves connected, marking in smooth locus

with two additional items

- Line bundle of degree d
$$\begin{array}{c} \mathcal{L} \\ | \\ \mathcal{C} \\ \downarrow \pi \\ S \end{array}$$

- A vector of integers $A = (a_1, \dots, a_n)$ with $\sum_{i=1}^{n} a_i = d$

Codim g

There should be an Abel-Jacobi locus of points $(C, p_1, \dots, p_n) \in S$ where

$$\mathcal{O}_C\left(\sum_{i=1}^{n} a_i p_i\right) \cong \mathcal{L}_C$$

not a closed condition

But there are issues here.

Solution:

- There is a natural operational Chow class

$$AJ_{g,A} : CH_*(S) \to CH_{*-g}(S)$$

- There is a universal formula for $AJ_{g,A}$

— Definition of the AJ class is very simple:

- Picard Stack $\mathcal{P}^{d}_{g,n}$ ← Artin Stack nonsingular!

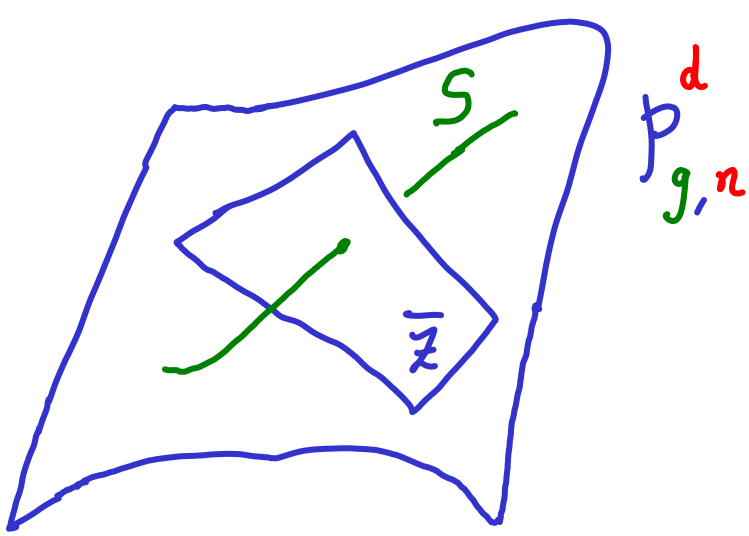
  moduli of genus g, n pointed curves with a line bundle of degree d

- $$\mathcal{Z} \underset{\text{codim } g}{\overset{\text{closed}}{\subset}} {}^{irr}\mathcal{P}^{d}_{g,n} \overset{\text{open}}{\subset} \mathcal{P}^{d}_{g,n}$$

  $\mathcal{O}_C(\sum a_i p_i) \cong \mathcal{L}$ (C irreducible)

  ${}^{irr}\mathcal{P}^{d}_{g,n}$: C irreducible

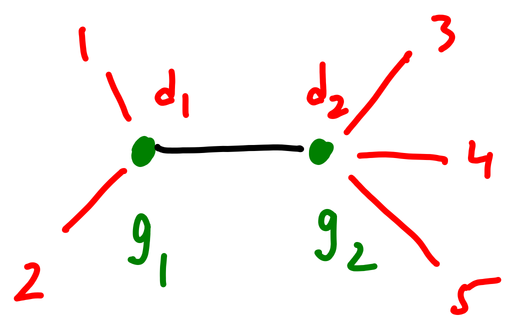
- Intersection with $\overline{\mathcal{Z}}$ (closure) defines $AJ_{g,A}$



— Formula : involves translations of the above geometric definition to Gromov-Witten theory (via log geometry)

We will write a formula in $\mathrm{CH}^{op}(\mathsf{P}^{d}_{g,n})$

- Graphs :

$\Gamma =$ 

$\in G_{g,n,d}$

No stability condition

set of all graphs ∞-many!

- Classes : ψ_i markings, ψ_j half edges

$$\xi_i = c_1(p_i^* \mathcal{L}) \leftarrow \text{marking } i$$

$$\eta(v) = \pi_*\left(c_1(\mathcal{L})^2\right) \leftarrow \text{vertex } v$$

$r \in \mathbb{N}_+$

- Weightings mod r of $\Gamma \in G_{g,n,d}$

$$W : \text{Half Edges}(\Gamma) \to \{0, 1, 2, \dots, r-1\}$$

(i) $\quad w(i) = a_i \mod r$

(ii) $\quad w(h) + w(h') = 0 \mod r$

when $\bullet\overset{h}{\text{———}}\overset{h'}{\text{———}}\bullet$ form an edge

(iii) $\quad \displaystyle\sum_{h \vdash v} w(h) = d(v) \mod r$

Let $W_{\Gamma, r}$ be the set of all weightings mod r of Γ.

$W_{\Gamma, r}$ is a finite set of cardinality $r^{h^1(\Gamma)}$

Let $P^r_{g,A}$ be the degree g part of

$$\sum_{\Gamma \in G_{g,n,d}} \sum_{w \in W_{\Gamma,r}} \frac{1}{|\mathrm{Aut}\,\Gamma|} \frac{1}{r^{h'(\Gamma)}} \cdot$$

$$i_{\Gamma *}\left[\prod_{i=1}^{n} \exp\left(\frac{a_i^2}{2}\psi_i + a_i \xi_i\right) \cdot \prod_{v \in \mathrm{Vert}(\Gamma)} \exp\left(-\frac{1}{2}\eta(v)\right) \right.$$

Version of Pixton's formula

$$\left. \cdot \prod_{\substack{e=(h,h') \\ \mathrm{Edge}(\Gamma)}} \frac{1-\exp\left(-\frac{w(h)w(h')}{2} \cdot (\psi_h + \psi_{h'})\right)}{\psi_h + \psi_{h'}} \right]$$

Claim: for $r \gg 0 \Rightarrow$ dependence upon r is polynomial

Theorem BHPSS: $AJ_{g,A} = P^{r=0}_{g,A}$ [Proof is long]

Two immediate applications

- Calculate the classes in $\overline{M}_{g,n}$ of the loci of holomorphic and meromorphic differentials.

- Yields relations in $P^r_{g,A}$ which then constrain Gromov-Witten theory Bae-Bülles

Suggests studying the operational Chow ring of the Picard stack $P^r_{g,A}$

The End

View from HG G55