The equivariant Gromov-Witten theory of $\mathbf{P}^1$

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0 Introduction

0.1 Overview

0.1.1

We present here the second in a sequence of three papers devoted to the Gromov-Witten theory of nonsingular target curves X. Let $\omega \in H^2(X, \mathbb{Q})$ denote the Poincaré dual of the point class. In the first paper [24], we considered the stationary sector of the Gromov-Witten theory of X formed by the descendents of ω. The stationary sector was identified in [24] with the Hurwitz theory of X with completed cycles insertions.

The target $\mathbf{P}^1$ plays a distinguished role in the Gromov-Witten theory of target curves. Since $\mathbf{P}^1$ admits a $\mathbb{C}^*$-action, equivariant localization may be used to study Gromov-Witten invariants [12]. The equivariant Poincaré duals,

$$\mathbf{0}, \boldsymbol{\infty} \in H^2_{\mathbb{C}^*}(\mathbf{P}^1, \mathbb{Q}),$$

of the $\mathbb{C}^*$-fixed points $0, \infty \in \mathbf{P}^1$ form a basis of the localized equivariant cohomology of $\mathbf{P}^1$. Therefore, the full equivariant Gromov-Witten theory of $\mathbf{P}^1$ is quite similar in spirit to the stationary non-equivariant theory. Via the non-equivariant limit, the full non-equivariant theory of $\mathbf{P}^1$ is captured by the equivariant theory.

The equivariant Gromov-Witten theory of $\mathbf{P}^1$ is the subject of the present paper. We find explicit formulas and establish connections to integrable hierarchies. The full Gromov-Witten theory of higher genus target curves will be considered in the third paper [25]. The equivariant theory of $\mathbf{P}^1$ will play a crucial role in the derivation of the Virasoro constraints for target curves in [25].

0.1.2

Our main result here is an explicit operator description of the equivariant Gromov-Witten theory of $\mathbf{P}^1$. We identify all equivariant Gromov-Witten invariants of $\mathbf{P}^1$ as vacuum matrix elements of explicit operators acting in the Fock space (in the infinite wedge realization).

The result is obtained by combining the equivariant localization formula with an operator formalism for the Hodge integrals which arise as vertex terms. The operator formalism for Hodge integrals relies crucially upon a formula due to Ekedahl, Lando, Shapiro, and Vainstein (see [6, 7, 13] and also [23]) expressing basic Hurwitz numbers as Hodge integrals.

0.1.3

As a direct and fundamental consequence of the operator formalism, we find an integrable hierarchy governs the equivariant Gromov-Witten theory of $\mathbf{P}^1$ — specifically, the 2–Toda hierarchy of Ueno and Takasaki [28]. The equations of the hierarchy, together with the string and divisor equations, uniquely determine the entire theory.

A Toda hierarchy for the non-equivariant Gromov-Witten of $\mathbf{P}^1$ was proposed in the mid 1990's in a series of papers by the physicists T. Eguchi, K. Hori, C.-S. Xiong, Y. Yamada, and S.-K. Yang on the basis of a conjectural matrix model description of the theory, see [3, 5]. The Toda conjecture was further studied in [26, 21, 10, 11] and, for the stationary sector, proved in [24].

The 2–Toda hierarchy for the equivariant Gromov-Witten theory of $\mathbf{P}^1$ obtained here is both more general and, arguably, more simple than the hierarchy obtained in the non-equivariant limit.

0.1.4

The 2–Toda hierarchy governs the equivariant theory of $\mathbf{P}^1$ just as Witten's KdV hierarchy [29] governs the Gromov-Witten theory of a point. However, while the known derivations of the KdV equations for the point require the analysis of elaborate auxiliary constructions (see [1, 14, 16, 22, 23]), the Toda equations for $\mathbf{P}^1$ follow directly, almost in textbook fashion, from the operator description of the theory.

In fact, the Gromov-Witten theory of $\mathbf{P}^1$ may be viewed as a more fundamental object than the Gromov-Witten theory of a point. Indeed, the theory of $\mathbf{P}^1$ has a simpler and more explicit structure. The theory of $\mathbf{P}^1$ is *not* based on the theory of a point. Rather, the point theory is perhaps best understood as a certain special large degree limit case of the $\mathbf{P}^1$ theory, see [23].

0.1.5

The proof of the Gromov-Witten/Hurwitz correspondence in [24] assumed a restricted case of the full result: the GW/H correspondence for the absolute stationary non-equivariant Gromov-Witten theory of $\mathbf{P}^1$. The required case is established here as a direct consequence of our operator formalism for the equivariant theory of $\mathbf{P}^1$ — completing the proof of the full GW/H correspondence.

While the present paper does not rely upon the results of [24], much of the motivation can be found in the study of the stationary theory developed there.

0.1.6

We do not know whether the Gromov-Witten theories of higher genus target curves are governed by integrable hierarchies. However, there exist conjectural Virasoro constraints for the Gromov-Witten theory of an arbitrary nonsingular projective variety X formulated in 1997 by Eguchi, Hori, and Xiong (using also ideas of S. Katz), see [4].

The results of the present paper will be used in [25] to prove the Virasoro constraints for nonsingular target curves X. Givental has recently announced a proof of the Virasoro constraints for the projective spaces $\mathbf{P}^n$. These two families of varieties both start with $\mathbf{P}^1$ but are quite different in flavor. Curves are of dimension 1, but have non-(p,p) cohomology, non-semisimple quantum cohomology, and do not, in general, carry torus actions. Projective spaces cover all target dimensions, but have algebraic cohomology, semisimple quantum cohomology, and always carry torus actions. Together, these results provide substantial evidence for the Virasoro constraints.

0.2 The equivariant Gromov-Witten theory of $\mathbf{P}^1$

0.2.1

Let $V = \mathbb{C} \oplus \mathbb{C}$. Let the algebraic torus $\mathbb{C}^*$ act on V with weights $(0,1)$:

$$\xi \cdot (v_1, v_2) = (v_1, \xi \cdot v_2)\,.$$

Let $\mathbf{P}^1$ denote the projectivization $\mathbf{P}(V)$. There is a canonically induced $\mathbb{C}^*$-action on $\mathbf{P}^1$.

The $\mathbb{C}^*$-equivariant cohomology ring of a point is $\mathbb{Q}[t]$ where t is the first Chern class of the standard representation. The $\mathbb{C}^*$-equivariant cohomology ring $H^*_{\mathbb{C}^*}(\mathbf{P}^1, \mathbb{Q})$ is canonically a $\mathbb{Q}[t]$-module.

The line bundle $\mathcal{O}_{\mathbf{P}^1}(1)$ admits a canonical $\mathbb{C}^*$-action which identifies the representation $H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(1))$ with V^*. Let $h \in H^2_{\mathbb{C}^*}(\mathbf{P}^1, \mathbb{Q})$ denote the equivariant first Chern class of $\mathcal{O}_{\mathbf{P}^1}(1)$. The equivariant cohomology ring of $\mathbf{P}^1$ is easily determined:

$$H^*_{\mathbb{C}^*}(\mathbf{P}^1, \mathbb{Q}) = \mathbb{Q}[h,t]/(h^2 + th).$$

A free $\mathbb{Q}[t]$-module basis is provided by $1, h$.

0.2.2

Let $\overline{M}_{g,n}(\mathbf{P}^1, d)$ denote the moduli space of genus g, n-pointed stable maps (with connected domains) to $\mathbf{P}^1$ of degree d. A canonical $\mathbb{C}^*$-action on $\overline{M}_{g,n}(\mathbf{P}^1, d)$ is obtained by translating maps. The virtual class is canonically defined in equivariant homology:

$$[\overline{M}_{g,n}(\mathbf{P}^1, d)]^{vir} \in H^{\mathbb{C}^*}_{2(2g+2d-2+n)}(\overline{M}_{g,n}(\mathbf{P}^1, d), \mathbb{Q}),$$

where $2g + 2d - 2 + n$ is the expected complex dimension (see, for example, [12]).

The equivariant Gromov-Witten theory of $\mathbf{P}^1$ concerns equivariant integration over the moduli space $\overline{M}_{g,n}(\mathbf{P}^1, d)$. Two types of equivariant cohomology classes are integrated. The *primary classes* are:

$$\mathrm{ev}_i^*(\gamma) \in H^*_{\mathbb{C}^*}(\overline{M}_{g,n}(\mathbf{P}^1, d), \mathbb{Q}),$$

where ev_i is the morphism defined by evaluation at the ith marked point,

$$\mathrm{ev}_i : \overline{M}_{g,n}(\mathbf{P}^1, d) \to \mathbf{P}^1 ,$$

and $\gamma \in H^*_{\mathbb{C}^*}(\mathbf{P}^1, \mathbb{Q})$. The *descendent classes* are:

$$\psi_i^k \mathrm{ev}_i^*(\gamma),$$

where $\psi_i \in H^2_{\mathbb{C}^*}(\overline{M}_{g,n}(X, d), \mathbb{Q})$ is the first Chern class of the cotangent line bundle L_i on the moduli space of maps.

Equivariant integrals of descendent classes are expressed by brackets of $\tau_k(\gamma)$ insertions:

$$\left\langle \prod_{i=1}^n \tau_{k_i}(\gamma_i) \right\rangle^{\circ}_{g,d} = \int_{[\overline{M}_{g,n}(\mathbf{P}^1,d)]^{vir}} \prod_{i=1}^n \psi_i^{k_i} \, \mathrm{ev}_i^*(\gamma_i) \, , \tag{0.1}$$

where $\gamma_i \in H^*_{\mathbb{C}^*}(\mathbf{P}^1, \mathbb{Q})$. As in [24], the superscript $\circ$ indicates the connected theory. The theory with possibly disconnected domains is denoted by $\langle \ \rangle^{\bullet}$. The equivariant integral in (0.1) denotes equivariant push-forward to a point. Hence, the bracket takes values in $\mathbb{Q}[t]$.

0.2.3

We now define the equivariant Gromov-Witten potential F of $\mathbf{P}^1$. Let z, y denote the variable sets,

$$\{z_0, z_1, z_2, \ldots\}, \quad \{y_0, y_1, y_2, \ldots\}.$$

The variables z_k, y_k correspond to the descendent insertions $\tau_k(1)$, $\tau_k(h)$ respectively. Let T denote the formal sum,

$$T = \sum_{k=0}^{\infty} z_k \tau_k(1) + y_k \tau_k(h) \, .$$

The potential is a generating series of equivariant integrals:

$$F = \sum_{g=0}^{\infty} \sum_{d=0}^{\infty} \sum_{n=0}^{\infty} u^{2g-2} q^d \left\langle \frac{T^n}{n!} \right\rangle^{\circ}_{g,d} .$$

The potential F is an element of $\mathbb{Q}[t][[z, y, u, q]]$.

0.2.4

The (localized) equivariant cohomology of $\mathbf{P}^1$ has a canonical basis provided by the classes,

$$\mathbf{0}, \boldsymbol{\infty} \in H^2_{\mathbb{C}^*}(\mathbf{P}^1)\,,$$

of Poincaré duals of the $\mathbb{C}^*$-fixed points $0, \infty \in \mathbf{P}^1$. An elementary calculation yields:

$$\mathbf{0} = t \cdot 1 + h, \quad \boldsymbol{\infty} = h. \tag{0.2}$$

Let x_i, $x_i^\star$ be the variables corresponding to the descendent insertions $\tau_k(\mathbf{0})$, $\tau_k(\boldsymbol{\infty})$, respectively. The variable sets $x, x^\star$ and z, y are related by the transform dual to (0.2),

$$x_i = \frac{1}{t} z_i, \quad x_i^\star = -\frac{1}{t} z_i + y_i.$$

The equivariant Gromov-Witten potential of $\mathbf{P}^1$ may be written in the x_i, $x_i^\star$ variables as:

$$F = \sum_{g=0}^{\infty} \sum_{d=0}^{\infty} u^{2g-2} q^d \left\langle \exp\left(\sum_{k=0}^{\infty} x_k \tau_k(\mathbf{0}) + x_k^\star \tau_k(\boldsymbol{\infty}) \right) \right\rangle_{g,d}^{\circ}.$$

0.3 The equivariant Toda equation

0.3.1

Let the classical series F^c be the genus 0, degree 0, 3-point summand of F (omitting u, q). The classical series generates the equivariant integrals of triple products in $H^*_{\mathbb{C}^*}(\mathbf{P}^1, \mathbb{Q})$. We find,

$$F^c = \frac{1}{2} z_0^2 y_0 - \frac{1}{2} t z_0 y_0^2 + \frac{1}{6} t^2 y_0^3\,.$$

The classical series does not depend upon $z_{k>0}$, $y_{k>0}$.

Let F^0 be the genus 0 summand of F (omitting u). The *small phase space* is the hypersurface defined by the conditions:

$$z_{k>0} = 0, \;\; y_{k>0} = 0\,.$$

The restriction of the genus 0 series to the small phase space is easily calculated:

$$F^0\Big|_{z_{k>0}=0,\; y_{k>0}=0} = F^c + q e^{y_0}\,.$$

The second derivatives of the restricted function F^0 are:

$$F^0_{z_0 z_0} = y_0, \quad F^0_{z_0 y_0} = z_0 - t y_0, \quad F^0_{y_0 y_0} = -t z_0 + t^2 y_0 + q e^y_0.$$

Hence, we find the equation

$$t F^0_{z_0 y_0} + F^0_{y_0 y_0} = q \exp(F^0_{z_0 z_0}) \tag{0.3}$$

is valid at least on the small phase space.

0.3.2

The equivariant Toda equation for the full equivariant potential F takes a similar form:

$$t F_{z_0 y_0} + F_{y_0 y_0} = \frac{q}{u^2} \exp(F(z_0 + u) + F(z_0 - u) - 2F), \tag{0.4}$$

where $F(z_0 \pm u) = F(z_0 \pm u, z_1, z_2, \ldots, y_0, y_1, y_2, \ldots, u, q)$. In fact, the equivariant Toda equation specializes to (0.3) when restricted to genus 0 and the small phase space.

0.3.3

In the variables x_i, $x_i^\star$, the equivariant Toda equation may be written as:

$$\frac{\partial^2}{\partial x_0 \, \partial x_0^\star} F = \frac{q}{u^2} \exp\left(\Delta F\right) . \tag{0.5}$$

Here, Δ is the difference operator,

$$\Delta = e^{u\partial} - 2 + e^{-u\partial} ,$$

and

$$\partial = \frac{\partial}{\partial z_0} = \frac{1}{t}\left(\frac{\partial}{\partial x_0} - \frac{\partial}{\partial x_0^\star}\right)$$

is the vector field creating a $\tau_0(1)$ insertion.

The equivariant Toda equation in form (0.5) is recognized as the 2–Toda equation: obtained from the standard Toda equation by replacing the second time derivative by $\frac{\partial^2}{\partial x_0 \, \partial x_0^\star}$. The 2–Toda equation is a 2-dimensional time analogue of the standard Toda equation.

0.3.4

A central result of the paper is the derivation of the 2–Toda equation for the equivariant theory of $\mathbf{P}^1$.

Theorem. The equivariant Gromov-Witten potential of $\mathbf{P}^1$ satisfies the 2–Toda equation (0.5).

The 2–Toda equation is a strong constraint. Together with the equivariant divisor and string equations, the 2–Toda determines F from the degree 0 invariants, see [26].

The 2–Toda equation arises as the lowest equation in a hierarchy of partial differential equations identified with the 2–Toda hierarchy of Ueno and Takasaki [28], see Theorem 7 in Section 4.

0.4 Operator formalism

0.4.1

The 2–Toda equation (0.5) is a direct consequence of the following operator formula for the equivariant Gromov-Witten theory of $\mathbf{P}^1$:

$$\exp F = \left\langle e^{\sum x_i \mathsf{A}_i} \; e^{\alpha_1} \left(\frac{q}{u^2}\right)^H \; e^{\alpha_{-1}} \, e^{\sum x_i^\star \mathsf{A}_i^\star} \right\rangle . \tag{0.6}$$

Here, A_i, $\mathsf{A}_i^\star$, and H are explicit operators in the Fock space. The brackets $\langle\,\rangle$ denote the vacuum matrix element. The operators A, which depend on the parameters u and t, are constructed in Sections 2 and 3. The exponential e^F of the equivariant potential is called the τ-function of the theory. The operator formalism for the 2-Toda equations was introduced in [8, 27] (see also e.g. [9]) and has since become a textbook tool for working with Toda equations.

The operator formula (0.6), stated as Theorem 4 in Section 3, is fundamentally the main result of the paper.

0.4.2

In our previous paper [24], the stationary non-equivariant Gromov-Witten theory of $\mathbf{P}^1$ was expressed as a similar vacuum expectation. The equivariant formula (0.6) specializes to the absolute case of the operator formula of

[24] when the equivariant parameter t is set to zero. Hence, the equivariant formula (0.6) completes the proof of the Gromov-Witten/Hurwitz correspondence discussed in [24].

0.5 Plan of the paper

0.5.1

In Section 1, the virtual localization formula of [12] is applied to express the equivariant $n+m$-point function as a graph sum with vertex Hodge integrals. Since $\mathbf{P}^1$ has two fixed points, the graph sum reduces to a sum over partitions.

Next, an operator formula for Hodge integrals is obtained in Section 2. A starting point here is provided by the Ekedahl-Lando-Shapiro-Vainstein formula expressing the necessary Hodge integrals as Hurwitz numbers. The main result of the section is Theorem 2 which expresses the generating function for Hodge integrals as a vacuum matrix element of a product of explicit operators $\mathcal{A}$ acting on the infinite wedge space.

Commutation relations for the operators $\mathcal{A}$ are required in the proof of Theorem 2. The technical derivation of these commutation relations is postponed to Section 5.

In Section 3, the operator formula for Hodge integrals is combined with the results of Section 1 to obtain Theorem 4, the operator formula for the equivariant Gromov-Witten theory of $\mathbf{P}^1$.

The 2–Toda equation (0.5) and the full 2–Toda hierarchy are deduced from Theorem 4 in Section 4.

0.5.2

We follow the notational conventions of [24] with one important difference. The letter H is used here to denote the generating function for Hodge integral, whereas H was used to denote Hurwitz numbers in [24].

0.6 Acknowledgments

We thank E. Getzler and A. Givental for discussions of the Gromov-Witten theory of $\mathbf{P}^1$. In particular, the explicit form of the linear change of time variables appearing in the equations of the 2–Toda hierarchy (see Theorem 7) was previously conjectured by Getzler in [11].

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1 Localization for $\mathbf{P}^1$

1.1 Hodge integrals

1.1.1

Hodge integrals of the ψ and λ classes over the moduli space of curves arise as vertex terms in the localization formula for Gromov-Witten invariants of $\mathbf{P}^1$.

Let L_i be the ith cotangent line bundle on $\overline{M}_{g,n}$. The ψ classes are defined by:

$$\psi_i = c_1(L_i) \in H^2(\overline{M}_{g,n}, \mathbb{Q})\,.$$

Let $\pi : C \to \overline{M}_{g,n}$ be the universal curve. Let ω_π be the relative dualizing sheaf. Let $\mathbb{E}$ be the rank g Hodge bundle on the moduli space $\overline{M}_{g,n}$,

$$\mathbb{E} = \pi_*(\omega_\pi).$$

The λ classes are defined by:

$$\lambda_i = c_i(\mathbb{E}) \in H^*(\overline{M}_{g,n}, \mathbb{Q}).$$

Only Hodge integrands linear in the λ classes arise in the localization formula for $\mathbf{P}^1$. Let $\mathsf{H}^\circ_g(z_1,\dots,z_n)$ be the n-point function of λ-linear Hodge integrals over the moduli space $\overline{M}_{g,n}$:

$$\mathsf{H}^\circ_g(z_1,\dots,z_n) = \prod z_i \int_{\overline{M}_{g,n}} \frac{1-\lambda_1+\lambda_2-\dots\pm\lambda_g}{\prod(1-z_i\psi_i)}\,.$$

Note the shift of indices caused by the product $\prod z_i$.

1.1.2

The function $\mathsf{H}_g^\circ(z)$ is defined for all $g, n \geq 0$. Values corresponding to unstable moduli spaces are set by definition. All 0-point functions $\mathsf{H}_g^\circ()$, both stable and unstable, vanish. The unstable 1 and 2-point functions are:

$$\mathsf{H}_0^\circ(z_1) = \frac{1}{z_1}, \quad \mathsf{H}_0^\circ(z_1, z_2) = \frac{z_1 z_2}{z_1 + z_2}. \tag{1.1}$$

1.1.3

Let $\mathsf{H}^\circ(z_1, \ldots, z_n, u)$ be the full n-point function of λ-linear Hodge integrals:

$$\mathsf{H}^\circ(z_1, \ldots, z_n, u) = \sum_{g \geq 0} u^{2g-2}\, \mathsf{H}_g^\circ(z_1, \ldots, z_n).$$

Let $\mathsf{H}(z_1, \ldots, z_n, u)$ be the corresponding disconnected n-point function. The disconnected 0-point function is defined by:

$$\mathsf{H}(u) = 1,$$

For $n > 0$, the disconnected n-point function is defined by:

$$\mathsf{H}(z_1, \ldots, z_n, u) = \sum_{P \in \mathrm{Part}[n]} \prod_{i=1}^{\ell(P)} \mathsf{H}^\circ(z_{P_i}, u),$$

where $\mathrm{Part}[n]$ is the set of partitions P of the set $\{1, \ldots, n\}$. Here, $\ell(P)$ is the length of the partition, and z_{P_i} denotes the variable set indexed by the part P_i. The genus expansion for the disconnected function,

$$\mathsf{H}(z_1, \ldots, z_n, u) = \sum_{g \in \mathbb{Z}} u^{2g-2}\, \mathsf{H}_g(z_1, \ldots, z_n), \tag{1.2}$$

contains negative genus terms.

1.2 Equivariant $n+m$-point functions

1.2.1

Let $\mathsf{G}_{g,d}^\circ(z_1, \ldots, z_n, w_1, \ldots, w_m)$ be the $n+m$-point function of genus g, degree d equivariant Gromov-Witten invariants of $\mathbf{P}^1$ in the basis determined by $\mathbf{0}$

and ∞:

$$\mathsf{G}^\circ_{g,d}(z,w) = \prod z_i \prod w_j \int_{[\overline{M}_{g,n+m}(\mathbf{P}^1,d)]^{vir}} \prod \frac{\mathrm{ev}_i^*(\mathbf{0})}{1 - z_i\,\psi_i} \prod \frac{\mathrm{ev}_j^*(\boldsymbol{\infty})}{1 - w_j\,\psi_j}\,.$$

The values corresponding to unstable moduli spaces are set by definition. The unstable 0-point functions are set to 0:

$$\mathsf{G}^\circ_{0,0}() = 0\,, \quad \mathsf{G}^\circ_{1,0}() = 0\,. \tag{1.3}$$

The unstable 1 and 2-point functions are:

$$\mathsf{G}^\circ_{0,0}(z_1) = \frac{1}{z_1}\,, \quad \mathsf{G}^\circ_{0,0}(w_1) = \frac{1}{w_1}\,, \tag{1.4}$$

$$\mathsf{G}^\circ_{0,0}(z_1,z_2) = \frac{tz_1z_2}{z_1+z_2}\,, \quad \mathsf{G}^\circ_{0,0}(z_1,w_1) = 0\,, \quad \mathsf{G}^\circ_{0,0}(w_1,w_2) = \frac{tw_1w_2}{w_1+w_2}\,.$$

These values will be seen to be compatible with the special values (1.1).

1.2.2

The $n+m$-point function $\mathsf{G}^\circ_{g,d}(z,w)$ is defined for all $g,d,n,m \geq 0$. The 0-point function $\mathsf{G}^\circ_{0,1}()$ is nontrivial since

$$\mathsf{G}^\circ_{0,1}() = \langle\rangle^\circ_{0,1} = 1.$$

In fact, $\mathsf{G}^\circ_{0,1}()$ is the *only* nonvanishing 0-point function for $\mathbf{P}^1$.

Let $\mathsf{G}^\circ_d(z,w,u)$ be the full $n+m$-point function for equivariant degree d Gromov-Witten invariants $\mathbf{P}^1$:

$$\mathsf{G}^\circ_d(z_1,\dots,z_n,w_1,\dots,w_m,u) = \sum_{g\geq 0} u^{2g-2}\,\mathsf{G}^\circ_{g,d}(z_1,\dots,z_n,w_1,\dots,w_m)\,.$$

The only nonvanishing 0-point functions is:

$$\mathsf{G}^\circ_1() = u^{-2}.$$

1.2.3

Let $\mathsf{G}_d(z, w, u)$ be the corresponding disconnected $n+m$-point function. The degree 0, 0-pointed disconnected function is defined by:

$$\mathsf{G}_0(u) = 1.$$

In all other cases,

$$\mathsf{G}_d(z_1, \ldots, z_n, w_1, \ldots, w_m, u) = \sum_{P \in \mathrm{Part}_d[n,m]} \frac{1}{|\operatorname{Aut}(P)|} \prod_{i=1}^{\ell(P)} \mathsf{G}^\circ_{d_i}(z_{P_i}, w_{P'_i}, u).$$

An element $P \in \mathrm{Part}_d[n,m]$ consists of the data

$$\{(d_1, P_1, P'_1) \ldots, (d_\ell, P_\ell, P'_\ell)\},$$

where d_i is a non-negative degree partition,

$$\sum_{i=1}^{l} d_i = d,$$

and $\{P_i\}$ and $\{P'_i\}$ are set partitions with the empty set as an allowed part,

$$\bigcup_{i=1}^{l} P_i = \{1, \ldots, n\}, \quad \bigcup_{i=1}^{l} P'_i = \{1, \ldots, m\}.$$

Because of the empty parts, an element $P \in \mathrm{Part}_d[n,m]$ may have a nontrivial group of automorphisms $\operatorname{Aut}(P)$.

1.2.4

Two remarks about the $n+m$-point function $\mathsf{G}_d(z, w, u)$ are in order. First, G_d systematically includes the unstable contributions (1.4). These contributions will later have to be removed to study the true equivariant Gromov-Witten theory. However, the inclusion of the unstable contributions here will simplify many formulas. Second, the 0-point function $\mathsf{G}^\circ_1()$ contributes to all disconnected functions G_d for positive d. For example:

$$\mathsf{G}_2(z_1) = \mathsf{G}^\circ_2(z_1) + \mathsf{G}^\circ_1(z_1)\,\mathsf{G}^\circ_1() + \mathsf{G}^\circ_0(z_1)\,\frac{\mathsf{G}^\circ_1()^2}{2}.$$

These occurrences of $\mathsf{G}^\circ_1()$ provide no difficulty.

1.3 Localization: vertex contributions

1.3.1

The localization formula for $\mathbf{P}^1$ expresses the $n+m$-point function $\mathsf{G}_d(z,w,u)$ as an automorphism-weighted sum over bipartite graphs with vertex Hodge integrals. We refer the reader to [12] for a discussion of localization in the context of virtual classes. The localization formula for $\mathbf{P}^1$ is explicitly treated in [12, 23].

1.3.2

Let Γ be a graph arising in the localization formula for the virtual class $[\overline{M}_{g,n+m}(\mathbf{P}^1,d)]^{vir}$. Let v_0 be a vertex of Γ lying over the fixed point $0 \in \mathbf{P}^1$. We will study the vertex contribution $C(v_0)$ to the equivariant integral

$$\prod z_i \prod w_j \int_{[\overline{M}_{g,n+m}(\mathbf{P}^1,d)]^{vir}} \prod \frac{\mathrm{ev}_i^*(\mathbf{0})}{1 - z_i\,\psi_i} \prod \frac{\mathrm{ev}_j^*(\infty)}{1 - w_j\,\psi_j}\,. \tag{1.5}$$

For a vertex v_∞ lying over $\infty \in \mathbf{P}^1$, the vertex contribution $C(v_\infty)$ is obtained simply by exchanging the roles of z and w and applying the transformation $t \mapsto -t$.

Each vertex v_0 of the localization graph Γ carries several additional structures:

- $g(v_0)$, a genus assignment,
- $e(v_0)$ incident edges of degrees $d_1, \ldots, d_{e(v_0)}$,
- $n(v_0)$ marked points indexed by $I(v_0) \subset \{1, \ldots, n\}$.

The data contribute factors to the vertex contribution $C(v_0)$ according to the following table:

$t^{g(v_0)-1} \left(\sum_{i=1}^{g(v_0)} (-1)^i \frac{\lambda_i}{t^i} \right)$	determined by the genus $g(v_0)$
$\frac{d_i^{d_i}\, t^{-d_i}}{d_i!} \, \frac{td_i}{t - d_i\,\psi_i}$	for each edge of degree d_i
$\frac{tz_i}{1 - z_i\,\psi_i}$	for each marking $i \in I(v_0)$

The vertex contribution $C(v_0)$ is obtained by multiplying the above factors and integrating over the moduli space $\overline{M}_{g(v_0),val(v_0)}$ where

$$val(v_0) = e(v_0) + n(v_0).$$

1.3.3

By the dimension constraint for the integrand,

$$\dim\ \overline{M}_{g(v_0),val(v_0)} = 3g(v_0) - 3 + val(v_0)\,,$$

the vertex integral is unchanged by the transformation

$$\psi_i \mapsto t\psi_i\,, \quad \lambda_i \mapsto t^i\lambda_i\,,$$

together with a division by $t^{3g(v_0)-3+val(v_0)}$. The vertex contribution $C(v_0)$ then takes the following form:

$$\frac{\prod_{i=1}^{e(v_0)} d_i^{d_i}/d_i!}{t^{2g(v_0)-2+d(v_0)+val(v_0)}} \times$$
$$\int_{\overline{M}_{g(v_0),val(v_0)}} \left(\sum_{i=1}^{g(v_0)} (-1)^i \lambda_i\right) \prod_{i=1}^{e(v_0)} \frac{d_i}{1-d_i\psi_i} \prod_{i\in I(v_0)} \frac{tz_i}{1-tz_i\psi_i},$$

where $d(v_0) = \sum_{i=1}^{e(v_0)} d_i$ is the total degree of v_0. We may rewrite $C(v_0)$ in terms of $\mathsf{H}^\circ_{g(v_0)}$:

$$C(v_0) = \frac{\prod_{i=1}^{e(v_0)} d_i^{d_i}/d_i!}{t^{2g(v_0)-2+d(v_0)+val(v_0)}}\, \mathsf{H}^\circ_{g(v_0)}(d_1,\dots,d_{e(v_0)},\dots,tz_i,\dots). \tag{1.6}$$

Since the $val(v_0)$-point function $\mathsf{H}^\circ_{g(v_0)}$ is defined for all $g(v_0), val(v_0) \geq 0$, we can define the vertex contribution $C(v_0)$ by (1.6) in case the moduli space $\overline{M}_{g(v_0),val(v_0)}$ is unstable. This convention agrees with the treatment of unstable contributions in the literature [12, 17]. We note $C(v_0)$ vanishes if $val(v_0) = 0$.

1.4 Localization: global formulas

1.4.1

Let Γ be a graph arising in the localization formula for $[\overline{M}_{g,n+m}(\mathbf{P}^1,d)]^{vir}$. Let

$$V(\Gamma) = V_0(\Gamma) \cup V_\infty(\Gamma)$$

be the vertex set divided by fixed point assignment. Let $E(\Gamma)$ be the edge set. Let d_e be the degree of an edge e. The graph Γ satisfies three global properties:

- a genus condition, $\sum_{v\in V(\Gamma)}(2g(v)-2+e(v)) = 2g-2$,
- a degree condition, $\sum_{v\in V(\Gamma)} d(v) = 2d$,
- a marking condition, $\bigcup_{v_0\in V_0(\Gamma)} I(v_0) = \{1,\dots,n\}$ (similarly for ∞).

The contribution of Γ to the integral (1.5) is:

$$\frac{1}{\prod_{e\in E(\Gamma)} d_e}\,\frac{1}{|\mathrm{Aut}(\Gamma)|}\prod_{v\in V(\Gamma)} C(v).$$

As the integral (1.5) is over the moduli space of maps with connected domains, Γ must also be connected. If disconnected domains are allowed for stable maps, the graphs Γ are also allowed to be disconnected.

1.4.2

The $n+m$-point functions G_d may be now expressed in terms of the functions H.

Proposition 1. *For $d \geq 0$, we have*

$$\begin{aligned}\mathsf{G}_d(z_1,\dots,z_n,w_1,\dots,w_m,u) = {}&\frac{1}{\mathfrak{z}(\mu)}\times \\ &\sum_{|\mu|=d}\frac{(u/t)^{\ell(\mu)}\,(-u/t)^{\ell(\mu)}}{t^{d+n}(-t)^{d+m}}\left(\prod\frac{\mu_i^{\mu_i}}{\mu_i!}\right)^2\mathsf{H}(\mu,tz,\tfrac{u}{t})\,\mathsf{H}(\mu,-tw,-\tfrac{u}{t})\,. \quad (1.7)\end{aligned}$$

The summation in (1.7) is over all partitions μ of d, $\ell(\mu)$ denotes the number of parts of μ and

$$\mathfrak{z}(\mu) = |\mathrm{Aut}(\mu)|\prod_{i=1}^{\ell(\mu)}\mu_i$$

where $\mathrm{Aut}(\mu)\cong\prod_{i\geq1}S(m_i(\mu))$ is the symmetry group permuting equal parts of the partition μ. The number $\mathfrak{z}(\mu)$ is the order of the centralizer of an element with cycle type μ in the symmetric group.

Proof. Each degree d, possibly disconnected, localization graph Γ yields a partition μ of d obtained from the edge degrees. The sum over localization graphs with a fixed edge degree partition μ can be evaluated by the vertex contribution formula (1.6) together with the global graph constraints. The result is exactly the μ summand in (1.7) (the edge and graph automorphisms are incorporated in the prefactors). The Proposition is then a restatement of the virtual localization formula: equivariant integration against the virtual class is obtained by summing over all localization graph contributions. $\square$

The degree 0 localization formula is special as the graphs are edgeless. However, with our conventions regarding 0-pointed functions, Proposition 1 holds without modification. We find, for example,

$$\mathsf{G}_0(z_1, \ldots, z_n, u) = t^{-n}\, \mathsf{H}(tz, \tfrac{u}{t})\,.$$

In particular, the definitions of the unstable contributions for G and H are compatible.

2 Operator formula for Hodge integrals

We will express Hodge integrals as matrix elements in the infinite wedge space. The basic properties of the infinite wedge space and our notational conventions are summarized in Section 2.0. A discussion can also be found in Section 2 of [24].

2.0 Review of the infinite wedge space

2.0.1

Let V be a linear space with basis $\{\underline{k}\}$ indexed by the half-integers:

$$V = \bigoplus_{k \in \mathbb{Z}+\frac{1}{2}} \mathbb{C}\, \underline{k}.$$

For each subset $S = \{s_1 > s_2 > s_3 > \ldots\} \subset \mathbb{Z} + \frac{1}{2}$ satisfying:

(i) $S_+ = S \setminus \left(\mathbb{Z}_{\leq 0} - \frac{1}{2}\right)$ is finite,

(ii) $S_- = \left(\mathbb{Z}_{\leq 0} - \frac{1}{2}\right) \setminus S$ is finite, we denote by v_S the following infinite wedge product:

$$v_S = \underline{s_1} \wedge \underline{s_2} \wedge \underline{s_3} \wedge \dots . \tag{2.1}$$

By definition,

$$\Lambda^{\frac{\infty}{2}} V = \bigoplus \mathbb{C}\, v_S$$

is the linear space with basis $\{v_S\}$. Let $(\,\cdot\,,\,\cdot\,)$ be the inner product on $\Lambda^{\frac{\infty}{2}} V$ for which $\{v_S\}$ is an orthonormal basis.

2.0.2

The fermionic operator ψ_k on $\Lambda^{\frac{\infty}{2}} V$ is defined by wedge product with the vector $\underline{k}$,

$$\psi_k \cdot v = \underline{k} \wedge v \,.$$

The operator ψ_k^* is defined as the adjoint of ψ_k with respect to the inner product $(\,\cdot\,,\,\cdot\,)$.

These operators satisfy the canonical anti-commutation relations:

$$\psi_i \psi_j^* + \psi_i^* \psi_j = \delta_{ij}\,, \tag{2.2}$$

$$\psi_i \psi_j + \psi_j \psi_1 = \psi_i^* \psi_j^* + \psi_j^* \psi_i^* = 0. \tag{2.3}$$

The *normally ordered* products are defined by:

$$:\psi_i\, \psi_j^*: = \begin{cases} \psi_i\, \psi_j^*\,, & j > 0\,, \\ -\psi_j^*\, \psi_i\,, & j < 0\,. \end{cases} \tag{2.4}$$

2.0.3

Let E_{ij}, for $i, j \in \mathbb{Z} + \frac{1}{2}$, be the standard basis of matrix units of $\mathfrak{gl}(\infty)$. The assignment

$$E_{ij} \mapsto :\psi_i\, \psi_j^*: \;\;,$$

defines a projective representation of the Lie algebra $\mathfrak{gl}(\infty) = \mathfrak{gl}(V)$ on $\Lambda^{\frac{\infty}{2}} V$.

The *charge* operator C corresponding to the identity matrix of $\mathfrak{gl}(\infty)$,

$$C = \sum_{k \in \mathbb{Z} + \frac{1}{2}} E_{kk},$$

acts on the basis v_S by:

$$C\, v_S = (|S_+| - |S_-|) v_S\,.$$

The kernel of C, the zero charge subspace, is spanned by the vectors

$$v_\lambda = \underline{\lambda_1 - \tfrac{1}{2}} \wedge \underline{\lambda_2 - \tfrac{3}{2}} \wedge \underline{\lambda_3 - \tfrac{5}{2}} \wedge \dots$$

indexed by all partitions λ. We will denote the kernel by $\Lambda_0^{\frac{\infty}{2}} V$.

The eigenvalues on $\Lambda_0^{\frac{\infty}{2}} V$ of the *energy* operator,

$$H = \sum_{k \in \mathbb{Z}+\frac{1}{2}} k\, E_{kk},$$

are easily identified:

$$H\, v_\lambda = |\lambda|\, v_\lambda\,.$$

The vacuum vector

$$v_\emptyset = \underline{-\tfrac{1}{2}} \wedge \underline{-\tfrac{3}{2}} \wedge \underline{-\tfrac{5}{2}} \wedge \dots$$

is the unique vector with the minimal (zero) eigenvalue of H.

The *vacuum expectation* $\langle A \rangle$ of an operator A on $\Lambda^{\frac{\infty}{2}} V$ is defined by the inner product:

$$\langle A \rangle = (A v_\emptyset, v_\emptyset).$$

2.0.4

For any $r \in \mathbb{Z}$, we define

$$\mathcal{E}_r(z) = \sum_{k \in \mathbb{Z}+\frac{1}{2}} e^{z(k-\frac{r}{2})}\, E_{k-r,k} + \frac{\delta_{r,0}}{\varsigma(z)}\,, \tag{2.5}$$

where the function $\varsigma(z)$ is defined by

$$\varsigma(z) = e^{z/2} - e^{-z/2}\,. \tag{2.6}$$

The exponent in (2.5) is set to satisfy:

$$\mathcal{E}_r(z)^* = \mathcal{E}_{-r}(z)\,,$$

where the adjoint is with respect to the standard inner product on $\Lambda^{\frac{\infty}{2}} V$.

Define the operators $\mathcal{P}_k$ for $k > 0$ by:

$$\frac{\mathcal{P}_k}{k!} = [z^k]\,\mathcal{E}_0(z)\,, \tag{2.7}$$

where $[z^k]$ stands for the coefficient of z^k. The operator,

$$\mathcal{F}_2 = \frac{\mathcal{P}_2}{2!} = \sum_{k\in\mathbb{Z}+\frac{1}{2}} \frac{k^2}{2} E_{k,k}\,,$$

will play a special role.

2.0.5

The operators $\mathcal{E}$ satisfy the following fundamental commutation relation:

$$[\mathcal{E}_a(z), \mathcal{E}_b(w)] = \varsigma\left(\det\left[\begin{smallmatrix} a & z \\ b & w \end{smallmatrix}\right]\right)\, \mathcal{E}_{a+b}(z+w)\,. \tag{2.8}$$

Equation (2.8) automatically incorporates the central extension of the $\mathfrak{gl}(\infty)$-action, which appears as the constant term in $\mathcal{E}_0$ when $r = -s$.

2.0.6

On setting $z = 0$, the operators $\mathcal{E}$ specialize to the standard bosonic operators on $\Lambda^{\frac{\infty}{2}} V$:

$$\alpha_k = \mathcal{E}_k(0)\,, \quad k \neq 0\,.$$

The commutation relation (2.15) specializes to the following equation

$$[\alpha_k, \mathcal{E}_l(z)] = \varsigma(kz)\,\mathcal{E}_{k+l}(z)\,. \tag{2.9}$$

When $k + l = 0$, equation (2.9) has the following constant term:

$$\frac{\varsigma(kz)}{\varsigma(z)} = \frac{e^{kz/2} - e^{-kz/2}}{e^{z/2} - e^{-z/2}}\,.$$

Letting $z \to 0$, we recover the standard relation:

$$[\alpha_k, \alpha_l] = k\,\delta_{k+l}\,.$$

2.1 Hurwitz numbers and Hodge integrals

2.1.1

Let μ be a partition of size $|\mu|$ and length $\ell(\mu)$. Let $\mu_1, \ldots, \mu_\ell$ be the parts of μ. Let $\mathsf{C}_g(\mu)$ be the Hurwitz number of genus g, degree $|\mu|$, covers of $\mathbf{P}^1$ with profile μ over $\infty \in \mathbf{P}^1$ and simple ramifications over

$$b = 2g + |\mu| + \ell(\mu) - 2$$

fixed points of $\mathbf{A}^1 \subset \mathbf{P}^1$. By definition, the Hurwitz number $\mathsf{C}_g(\mu)$ counts possibly disconnected covers with weights, where the weight of a cover is the reciprocal of the order of its automorphism group. Note that the genus of a disconnected cover may be negative.

The Ekedahl-Lando-Shapiro-Vainstein formula expresses $\mathsf{C}_g(\mu)$ in terms of λ-linear Hodge integrals:

$$\mathsf{C}_g(\mu) = \frac{b!}{\mathfrak{z}(\mu)} \left(\prod \frac{\mu_i^{\mu_i}}{\mu_i!} \right) \mathsf{H}_g(\mu_1, \ldots, \mu_\ell)\,, \tag{2.10}$$

see [6] or [7, 13] for a Gromov-Witten theoretic approach.

2.1.2

The Hurwitz numbers $\mathsf{C}_g(\mu)$ admit a standard expression in terms of the characters of the symmetric group. The character formula may be rewritten as a vacuum expectation in the infinite wedge space:

$$\mathsf{C}_g(\mu) = \frac{1}{\mathfrak{z}(\mu)} \left\langle e^{\alpha_1} \mathcal{F}_2^b \prod \alpha_{-\mu_i} \right\rangle . \tag{2.11}$$

A derivation of (2.11) can be found, for example, in [21, 24]. Using the ELSV formula (2.10), we find,

$$\mathsf{H}(\mu_1, \ldots, \mu_\ell, u) = u^{-|\mu|-\ell(\mu)} \left(\prod \frac{\mu_i!}{\mu_i^{\mu_i}} \right) \left\langle e^{\alpha_1} e^{u\mathcal{F}_2} \prod \alpha_{-\mu_i} \right\rangle .$$

2.1.3

Since the operators $e^{-\alpha_1}$ and $e^{-u\mathcal{F}_2}$ fix the vacuum vector, we may rewrite the last equation as:

$$\mathsf{H}(\mu_1, \dots, \mu_\ell, u) = \\ u^{-|\mu|-\ell(\mu)} \left(\prod \frac{\mu_i!}{\mu_i^{\mu_i}}\right) \left\langle \prod \left(e^{\alpha_1} e^{u\mathcal{F}_2} \alpha_{-\mu_i} e^{-u\mathcal{F}_2} e^{-\alpha_1}\right) \right\rangle . \quad (2.12)$$

Equation (2.12) holds, by construction, for positive integer values of μ_i. We will rewrite the right side and reinterpret (2.12) as an equality of analytic functions of μ.

2.2 The operators $\mathcal{A}$

2.2.1

The following operators will play a central role in the paper:

$$\mathcal{A}(a,b) = \mathcal{S}(b)^a \sum_{k\in\mathbb{Z}} \frac{\varsigma(b)^k}{(a+1)_k} \mathcal{E}_k(b) , \quad (2.13)$$

where a and b are parameters and

$$\varsigma(z) = e^{z/2} - e^{-z/2} , \quad \mathcal{S}(z) = \frac{\varsigma(z)}{z} = \frac{\sinh z/2}{z/2} .$$

In (2.13), we use the standard notation:

$$(a+1)_k = \frac{(a+k)!}{a!} = \begin{cases} (a+1)(a+2)\cdots(a+k) , & k \geq 0 , \\ (a(a-1)\cdots(a+k+1))^{-1} , & k \leq 0 . \end{cases}$$

If $a \neq 0, 1, 2, \dots$, the sum in (2.13) is infinite in both directions. If a is a nonnegative integer, the summands with $k \leq -a-1$ in (2.13) vanish.

2.2.2

Definition (2.13) is motivated by the following result.

Lemma 2. *For $m = 1, 2, 3, \dots$, we have*

$$e^{\alpha_1} e^{u\mathcal{F}_2} \alpha_{-m} e^{-u\mathcal{F}_2} e^{-\alpha_1} = \frac{u^m m^m}{m!} \mathcal{A}(m, um) .$$

Proof. The conjugation,

$$e^{u\mathcal{F}_2}\,\alpha_{-m}\,e^{-u\mathcal{F}_2} \;=\; \mathcal{E}_{-m}(um)\,, \tag{2.14}$$

is easily calculated from the definitions since the operator $e^{u\mathcal{F}_2}$ acts diagonally.

The operators $\mathcal{E}$ satisfy the following basic commutation relation:

$$[\mathcal{E}_a(z), \mathcal{E}_b(w)] = \varsigma\left(\det\left[\begin{smallmatrix} a & z \\ b & w \end{smallmatrix}\right]\right)\,\mathcal{E}_{a+b}(z+w)\,. \tag{2.15}$$

From (2.15), we obtain

$$[\alpha_1, \mathcal{E}_{-m}(s)] = \varsigma(s)\;\mathcal{E}_{-m+1}(s)$$

and, therefore,

$$e^{\alpha_1}\,\mathcal{E}_{-m}(s)\,e^{-\alpha_1} = \frac{\varsigma(s)^m}{m!}\sum_{k\in\mathbb{Z}}\frac{\varsigma(s)^k}{(m+1)_k}\,\mathcal{E}_k(s)\,. \tag{2.16}$$

Applying (2.16) to (2.14) completes the proof. □

2.2.3

Equation (2.12) and Lemma 2 together yield a concise formula for the evaluations of $\mathsf{H}(z_1,\dots,z_n,u)$ at the positive integers $z_i=\mu_i$:

$$\mathsf{H}(\mu_1,\dots,\mu_n,u) = u^{-n}\left\langle\prod_{i=1}^{n}\mathcal{A}(\mu_i,u\mu_i)\right\rangle\,. \tag{2.17}$$

However, we will require a stronger result. We will prove that the right side of equation (2.17) is an analytic function of the variables μ_i and that the n-point function $\mathsf{H}(z_1,\dots,z_n,u)$ is a Laurent expansion of this analytic function.

2.3 Convergence of matrix elements

2.3.1

If $a \neq 0,1,2,\dots$, the sum in (2.13) is infinite in both directions. Hence, for general values of μ_i, the matrix element on the right side of (2.17) is not à priori well-defined. By expanding the definition of $\mathcal{A}(\mu_i, u\mu_i)$, the right side of (2.17) is an n-fold series. We will prove the series converges in a suitable domain of values of μ_i.

Let Ω be the following domain in $\mathbb{C}^n$:

$$\Omega = \left\{ (z_1, \dots, z_n) \middle| |z_k| > \sum_{i=1}^{k-1} |z_i|,\ k = 1, \dots, n \right\} .$$

The constant term of the operator $\mathcal{E}_0(uz_i)$ occurring in the definition of $\mathcal{A}(z_i, uz_i)$ has a pole at $uz = 0$. For $u \neq 0$, the coordinates z_i are kept away in Ω from the poles $uz_i = 0$. We will prove the following convergence result.

Proposition 3. *Let K be a compact set,*

$$K \subset \Omega \cap \{z_i \neq -1, -2, \dots, i = 1, \dots, n\}.$$

For all partitions ν and λ, the series

$$(\mathcal{A}(z_1, uz_1) \cdots \mathcal{A}(z_n, uz_n)\, v_\nu, v_\lambda) \tag{2.18}$$

converges absolutely and uniformly on K for all sufficiently small $u \neq 0$.

2.3.2

We will require three Lemmas for the proof of Proposition 3.

Lemma 4. *Let ν be a partition of k. For any integer l, there exists at most* $\max(k, l)$ *partitions λ of l satisfying*

$$(\mathcal{A}(z, uz)\, v_\nu, v_\lambda) \neq 0 \,.$$

Proof. If $k = l$, then by the definition of $\mathcal{A}(z, uz)$, there is exactly one such partition λ, namely $\lambda = \nu$.

Next, consider the case $k > l$. If the matrix element does not vanish, then the operator $\mathcal{E}_{k-l}$ in (2.13) must act on one of the factors of

$$v_\nu = \underline{\nu_1 - \tfrac{1}{2}} \wedge \underline{\nu_2 - \tfrac{3}{2}} \wedge \underline{\nu_3 - \tfrac{5}{2}} \wedge \dots ,$$

and decrease the corresponding part of the partition ν. Since ν has at most k parts, the above action can occur in at most k ways. The argument in the $l > k$ case is similar. $\square$

Lemma 5. *For any two partitions ν and λ satisfying $|\nu| \neq |\lambda|$, we have*

$$\left|(\mathcal{E}_{|\nu|-|\lambda|}(uz)\, v_\nu, v_\lambda)\right| \leq \exp\left(\frac{|\nu|+|\lambda|}{2}\,|uz|\right).$$

If $\nu = \lambda$, then

$$\left|(\mathcal{E}_0(uz)\, v_\nu, v_\nu) - \frac{1}{\varsigma(uz)}\right| \leq |\nu|\, \exp(|\nu||uz|)\,.$$

Proof. The Lemma is obtained from the definition of $\mathcal{E}_{|\nu|-|\lambda|}(uz)$. □

Lemma 6. *For all fixed $k_0, k_n \in \mathbb{Z}$, the series*

$$\sum_{k_1,\dots,k_{n-1}\geq 0} \prod_{i=1}^{n} \frac{z_i^{k_i-k_{i-1}}}{(d_i)_{k_i-k_{i-1}}} \tag{2.19}$$

converges absolutely and uniformly on compact subsets of Ω for all values of the parameters $d_i \neq 0, -1, -2, \dots$.

By differentiating with respect to the variables z_i, we can insert in (2.19) any polynomial weight in the summation variables k_i.

Proof. Consider the factor obtain by summation with respect to k_1:

$$\sum_{k_1\geq 0} \frac{(z_1/z_2)^{k_1}}{(d_1)_{k_1-k_0}(d_2)_{k_2-k_1}}\,. \tag{2.20}$$

The above series converges absolutely and uniformly on compact sets since $|z_1/z_2| < 1$ on the domain Ω. We require a bound on (2.20) considered as a function of the parameter k_2.

The series (2.20) is bounded by a high enough derivative of the series

$$\sum_{k_1\geq 0}^{k_2} \frac{w^{k_1}}{k_1!\,(k_2-k_1)!} + \sum_{k_1>k_2} \frac{(k_1-k_2)!}{k_1!}\, w^{k_1}\,, \quad w = \left|\frac{z_1}{z_2}\right|\,. \tag{2.21}$$

The first term of (2.21) can be obviously estimated by

$$\frac{1}{k_2!}\left(\frac{|z_1|+|z_2|}{|z_2|}\right)^{k_2},$$

whereas the second term of (2.21) can be estimated by

$$\frac{1}{k_2!}\,\frac{|z_1/z_2|^{k_2+1}}{1-|z_1|/|z_2|}\,.$$

Therefore, the sum over both k_1 and k_2 behaves like the series

$$\sum_{k_2\geq 0}\frac{1}{k_2!\,(d_3)_{k_3-k_2}}\left(\frac{|z_1|+|z_2|}{|z_3|}\right)^{k_2}\,,$$

which is a sum of the form (2.20). Again, the series converges absolutely and uniformly on compact sets since $|z_1|+|z_2|<|z_3|$.

The Lemma is proved by iterating the above argument. □

2.3.3 Proof of Proposition 3

We first expand (2.18) as a sum over all intermediate vectors

$$v_\nu = v_{\mu[0]}, v_{\mu[1]}, \dots, v_{\mu[n-1]}, v_{\mu[n]} = v_\lambda.$$

Next, using Lemmas 4 and 5, we will bound the summation over all intermediate partitions μ by a summation over their sizes,

$$k_i = |\mu[i]|\ ,\quad i=0,\dots,n\,.$$

The term $\max(k_i,k_{i+1})$ of Lemma 4 can be bounded by k_i+k_{i+1} and, in any case, amounts to an irrelevant polynomial weight.

We conclude the Proposition will be established if the absolute convergence for $z\in K$ and sufficiently small u of the following series is proven:

$$\sum_{k_1,\dots,k_{n-1}\geq 0}\prod k_i^{m_i}\, e^{(k_i+k_{i-1})|uz_i|/2}\,\frac{\varsigma(uz_i)^{k_i-k_{i-1}}}{(1+z_i)_{k_i-k_{i-1}}}\,, \tag{2.22}$$

where the parameters m_i are fixed nonnegative integers. Here, we neglect the prefactors $\mathcal{S}(uz_i)^{z_i}$ of the operators $\mathcal{A}(z_i,uz_i)$ — the functions $\mathcal{S}(uz_i)^{z_i}$ are analytic and single valued (for the principal branch) on K for sufficiently small u. Also, we neglect the constant terms of $\mathcal{A}(z_i,uz_i)$ as they do not affect convergence for $u\neq 0$.

The terms raised to the power k_i in (2.22) are

$$\left(e^{(|uz_i|+|uz_{i+1}|)/2}\,\frac{\varsigma(uz_i)}{\varsigma(uz_{i+1})}\right)^{k_i}\,,\quad i=1,\dots,n-1\,.$$

Since, for $u \to 0$, we have

$$e^{(|uz_i|+|uz_{i+1}|)/2} \frac{\varsigma(uz_i)}{\varsigma(uz_{i+1})} \to \frac{z_i}{z_{i+1}} \,,$$

the convergence of the series (2.22) follows from the convergence of the series (2.19) with values

$$d_i = 1 + z_i \,, \quad i = 1, \dots, n \,.$$

$\square$

2.4 Series expansion of matrix elements

2.4.1

By Proposition 3, the vacuum matrix element

$$\langle \mathcal{A}(z_1, uz_1) \cdots \mathcal{A}(z_n, uz_n) \rangle \tag{2.23}$$

is an analytic function of the variables $z_1, \dots, z_n, u$ on a punctured open set of $\Omega \times 0$ in $\Omega \times \mathbb{C}^*$. Therefore, we may expand (2.23) in a convergent Laurent power series.

First, viewing u as a parameter, we expand in Laurent series in the variables $z_1, \dots, z_n$ in the following manner. For any point $(z_2, \dots, z_n)$ in the domain

$$\Omega' = \left\{ (z_2, \dots, z_n) \,\middle|\, |z_k| > \sum_{i=2}^{k-1} |z_i|, \; k = 2, \dots, n \right\} \,,$$

the function (2.23) is analytic and single-valued for z_1 in a sufficiently small punctured neighborhood of the origin. Hence, the function can be expanded there in a convergent Laurent series. Every coefficient of that Laurent expansion is an analytic function on the domain Ω' and, by iterating the same procedure, can be expanded completely into a Laurent power series. The coefficients of the Laurent expansion in the variables $z_1, \dots, z_n$ may be expanded as Laurent series in u.

Alternatively, we may expand the function (2.23) in the variable u first. Then, the coefficients of the expansion are analytic functions on the domain Ω.

Later, we will identify the Laurent series expansion of (2.23) with the series $u^n \, \mathsf{H}(z_1, \dots, z_n, u)$.

2.4.2

For any ring R, define the ring $R((z))$ by

$$R((z)) = \left\{ \sum_{i\in\mathbb{Z}} r_i z^i \,\middle|\, r_i \in R,\ r_n = 0,\ n \ll 0 \right\} .$$

In other words, $R((z))$ consists of formal Laurent series in z with coefficients in R and exponents bounded from below.

Proposition 7. *We have*

$$\langle \mathcal{A}(z_1, uz_1) \cdots \mathcal{A}(z_n, uz_n) \rangle \in \mathbb{Q}[u^{\pm 1}]((z_n)) \dots ((z_1)) .$$

Proof. The result follows by induction on n from the following property of the operators $\mathcal{A}$:

$$\left(\mathcal{A}(z, uz) - \frac{1}{uz} \right)^* v_\mu = O\left(z^{-|\mu|} \right) .$$

Indeed, with the exception of the term $(uz)^{-1}$ which appears in the constant term of operator $\mathcal{E}_0(uz)$, terms contributing to the coefficient

$$\left[z^{-k} \right] \left(\mathcal{A}(z, uz) - \frac{1}{uz} \right)^*$$

lower the energy by at least k and, since there are no vectors of negative energy, annihilate v_μ if $k > |\mu|$. □

2.4.3

Let $\mathcal{A}_k$ be the coefficients of the expansion of the operator $\mathcal{A}(z, uz)$ in powers of z:

$$\mathcal{A}(z, uz) = \sum_{k\in\mathbb{Z}} \mathcal{A}_k \, z^k . \tag{2.24}$$

As observed in the proof of Proposition 7, the operator $\mathcal{A}_k$ for $k \neq -1$ involves only terms of energy $\geq -k$. The same is true for $\mathcal{A}_{-1}$ with the exception of the constant term $-u^{-1}$.

In terms of the operators $\mathcal{A}_k$, the Laurent series expansion of (2.23) can be written as:

$$\langle \mathcal{A}(z_1, uz_1) \cdots \mathcal{A}(z_n, uz_n) \rangle = \sum_{k_1,\dots,k_n} \langle \mathcal{A}_{k_1} \cdots \mathcal{A}_{k_n} \rangle \ z_1^{k_1} \dots z_n^{k_n} . \tag{2.25}$$

If $k_j < -\sum_{i<j}(k_i+1)$ for some j, then the corresponding term vanishes by energy considerations.

2.5 Commutation relations and rationality

2.5.1

Consider the doubly infinite series:

$$\delta(z,-w) = \frac{1}{w}\sum_{n\in\mathbb{Z}} \left(-\frac{z}{w}\right)^n \in \mathbb{Q}((z,w)).$$

The above series is the difference between the following two expansions:

$$\frac{1}{z+w} = \frac{1}{w} - \frac{z}{w^2} + \frac{z^2}{w^3} - \dots, \quad |z| < |w|, \tag{2.26}$$

$$\frac{1}{z+w} = \frac{1}{z} - \frac{w}{z^2} + \frac{w^2}{z^3} - \dots, \quad |z| > |w|. \tag{2.27}$$

The series $\delta(z,-w)$ is a formal δ-function at $z+w=0$, in the sense that

$$(z+w)\,\delta(z,-w) = 0\,.$$

2.5.2

The following basic result will be established in Section 5.

Theorem 1. *We have*

$$[\mathcal{A}(z,uz),\mathcal{A}(w,uw)] = zw\,\delta(z,-w)\,, \tag{2.28}$$

or equivalently,

$$[\mathcal{A}_k,\mathcal{A}_l] = (-1)^l \delta_{k+l-1}\,. \tag{2.29}$$

Corollary 8. *The series*

$$\prod_{i<j}(z_i+z_j)\ \langle \mathcal{A}(z_1,uz_1)\cdots\mathcal{A}(z_n,uz_n)\rangle \in \mathbb{Q}[u^{\pm 1}]((z_n))\dots((z_1)) \tag{2.30}$$

is symmetric in $z_1,\dots,z_n$ and, hence, is an element of

$$\prod z_i^{-1}\,\mathbb{Q}[u^{\pm 1}]\,[[z_1,\dots,z_n]]\,.$$

Proof. Indeed, the exponents of z_1 in (2.30) are bounded below by -1. ◻

2.5.3

We now deduce the following result from Theorem 1:

Proposition 9. *The coefficients,*

$$[u^m]\,\langle \mathcal{A}(z_1, uz_1)\ldots\mathcal{A}(z_n, uz_n)\rangle\,, \quad m\in\mathbb{Z}\,, \tag{2.31}$$

of powers of u in the expansion (2.25) *are symmetric rational functions in $z_1,\ldots,z_n$, with at most simple poles on the divisors $z_i+z_j=0$ and $z_i=0$.*

Proof. By Corollary 8, it suffices to prove the exponents of z_n in the expansion of (2.31) are bounded from above.

The equation,

$$\langle \mathcal{E}_{k_1}(uz_1)\ldots\mathcal{E}_{k_n}(uz_n)\rangle = \left\langle \frac{\mathcal{E}_{k_1}(uz_1)}{u^{k_1}}\ldots\frac{\mathcal{E}_{k_n}(uz_n)}{u^{k_n}}\right\rangle\,, \tag{2.32}$$

holds since the vacuum expectation vanishes unless $\sum k_i = 0$. The transformation $\mathcal{E}_k \to u^{-k}\mathcal{E}_k$ applied to the operator $\mathcal{A}(z,uz)$ acts as the substitution

$$\varsigma(uz)^k \mapsto \frac{\varsigma(uz)^k}{u^k}\,,$$

which makes all terms regular and nonvanishing at $u=0$, except for the simple pole in the constant term $\varsigma(uz)^{-1}$.

Since (2.32) vanishes if $k_n>0$, the vacuum expectation

$$\langle \mathcal{A}(z_1, uz_1)\ldots\mathcal{A}(z_n, uz_n)\rangle$$

depends on z_n only through terms of the form

$$\mathcal{S}(uz_n)^{z_n}\,,\quad e^{auz_n}\,,\quad a\in\tfrac{1}{2}\mathbb{Z}\,,$$

as well as

$$z_n(z_n-1)\ldots(z_n-k+1)\,\frac{u^k}{\varsigma(uz_n)^k} = \\ \left(1-\frac{1}{z_n}\right)\cdots\left(1-\frac{k-1}{z_n}\right)\,\mathcal{S}(uz_n)^{-k}\,,\quad k=1,2,\ldots\,.$$

Because these terms are multiplied by a function of u with a bounded order of pole at $u=0$, the required boundedness of degree in z_n for fixed powers of u is now immediate. ∎

2.6 Identification of $\mathsf{H}(z,u)$

2.6.1

By definition (1.2), $\mathsf{H}(z_1,\dots,z_n,u)$ is a Laurent series in u with coefficients given by rational functions of $z_1,\dots,z_n$ which have at most first order poles at the divisors $z_i+z_j=0$ and $z_i=0$.

By Proposition 9, the expansion (2.25) has the exact same form. We can now state the main result of the present section.

Theorem 2. *We have*

$$\mathsf{H}(z_1,\dots,z_n,u)=u^{-n}\left\langle \mathcal{A}(z_1,uz_1)\dots\mathcal{A}(z_n,uz_n)\right\rangle\,. \tag{2.33}$$

Proof. By Proposition 3, the coefficients (2.31) are analytic functions on the domain Ω. Moreover, by Proposition 9, these functions are rational. By (2.17), for positive integer values of z_i in the domain Ω, these functions take the same values as the corresponding coefficients of H. Since positive integer values of z_i inside Ω form a Zariski dense set, the Theorem follows. □

2.6.2

As an illustration of Theorem 2, we obtain the following result.

Proposition 10. *The connected* 2*-point generating function* $\mathsf{H}^\circ(z_1,z_2,u)$ *for Hodge integrals is given by*

$$\begin{aligned}\mathsf{H}^\circ(z_1,z_2,u)=\;&\frac{\mathcal{S}(uz_1)^{z_1}\,\mathcal{S}(uz_2)^{z_2}}{\varsigma(u(z_1+z_2))}\times\\ &\left[{}_2F_1\left(\begin{matrix}-z_2,1\\1+z_1\end{matrix};\frac{1-e^{uz_1}}{1-e^{-uz_2}}\right)-{}_2F_1\left(\begin{matrix}-z_2,1\\1+z_1\end{matrix};\frac{1-e^{-uz_1}}{1-e^{uz_2}}\right)\right],\end{aligned} \tag{2.34}$$

where ${}_2F_1$ *the Gauss hypergeometric function* (5.2).

Proof. We first calculate:

$$\begin{aligned}&\left\langle \mathcal{E}_{k_1}(uz_1)\,\mathcal{E}_{k_2}(uz_2)\right\rangle-\left\langle \mathcal{E}_{k_1}(uz_1)\right\rangle\,\left\langle \mathcal{E}_{k_2}(uz_2)\right\rangle=\\ &\qquad\begin{cases}\dfrac{\varsigma(k_1u(z_1+z_2))}{\varsigma(u(z_1+z_2))}\,, & 0<k_1=-k_2\,,\\ 0\,, & \text{otherwise.}\end{cases}\end{aligned} \tag{2.35}$$

The nonzero term in (2.35) arises from the constant term in the commutator $[\mathcal{E}_{k_1}(uz_1), \mathcal{E}_{-k_1}(uz_2)]$. Then, by formula (2.33), we obtain

$$\mathsf{H}^\circ(z_1, z_2, u) = \frac{\mathcal{S}(uz_1)^{z_1}\,\mathcal{S}(uz_2)^{z_2}}{\varsigma(u(z_1+z_2))} \sum_{k>0} \varsigma(ku(z_1+z_2))\, \frac{\varsigma(uz_1)^k\,\varsigma(uz_2)^{-k}}{(1+z_1)_k\,(1+z_2)_{-k}}\,,$$

which is equivalent to (2.34) . □

The symmetry in z_1 and z_2 is not at all obvious from formula (2.34).

3 Operator formula for Gromov-Witten invariants

3.1 Localization revisited

3.1.1

Propositions 1 and 2 together yield the following localization formula in terms of vacuum expectations:

$$\mathsf{G}_d(z_1, \dots, z_n, w_1, \dots, w_m, u) = \sum_{|\mu|=d} \frac{1}{\mathfrak{z}(\mu)}\, \mathsf{J}(z, \mu, u, t)\, \mathsf{J}(w, \mu, u, -t)\,, \quad (3.1)$$

where the function $\mathsf{J}(z, \mu, u, t)$ is defined by:

$$\mathsf{J}(z_1, \dots, z_n, \mu_1, \dots, \mu_\ell, u, t) = t^{-d} u^{-n} \left(\prod \frac{\mu_i^{\mu_i}}{\mu_i!}\right) \left\langle \prod \mathcal{A}(tz_i, uz_i) \prod \mathcal{A}(\mu_i, \tfrac{u}{t}\mu_i) \right\rangle = u^{-d-n} \left\langle \prod \mathcal{A}(tz_i, uz_i)\, e^{\alpha_1}\, e^{\frac{u}{t}\mathcal{F}_2} \prod \alpha_{-\mu_i} \right\rangle. \quad (3.2)$$

3.1.2

For each partition μ, define the vector $\chi_\mu \in \Lambda^{\frac{\infty}{2}} V$ by:

$$\chi_\mu = \prod_{i=1}^{\ell(\mu)} \alpha_{-\mu_i}\, v_\emptyset\,.$$

The expansion of χ_μ in the standard basis v_ν is given by the values of the symmetric group characters χ^ν on the conjugacy class determined by μ:

$$\chi_\mu = \sum_{|\nu|=|\mu|} \chi^\nu_\mu \, v_\nu \,.$$

¿From the commutation relations

$$[\alpha_k, \alpha_l] = k \, \delta_{k+l} \,, \tag{3.3}$$

or from the orthogonality relation for characters, we find

$$(\chi_\mu, \chi_\nu) = \mathfrak{z}(\mu) \, \delta_{\mu,\nu}.$$

Let $P_\emptyset$ denote the orthogonal projection onto the vector $v_\emptyset$. Since the vectors $\{\chi_\mu\}_{|\mu|=d}$ span the eigenspace of H with eigenvalue d, the operator

$$P_d = \sum_{|\mu|=d} \frac{1}{\mathfrak{z}(\mu)} \prod \alpha_{-\mu_i} \, P_\emptyset \prod \alpha_{\mu_i}$$

is the orthogonal projection onto the d-eigenspace of H.

3.1.3

Using definition (3.2) and the projection $P_\emptyset$, we can write

$$u^{2d+n+m} \mathsf{J}(z, \mu, u, t) \, \mathsf{J}(w, \mu, u, -t) = \\ \Big\langle \prod \mathcal{A}(tz_i, uz_i) \, e^{\alpha_1} \, e^{\frac{u}{t}\mathcal{F}_2} \prod \alpha_{-\mu_i} \, P_\emptyset \, \times \\ \prod \alpha_{\mu_i} \; e^{-\frac{u}{t}\mathcal{F}_2} \, e^{\alpha_{-1}} \prod \mathcal{A}(-tw_j, uw_j)^* \Big\rangle \,.$$

Since $\mathcal{F}_2$ commutes with H, $\mathcal{F}_2$ also commutes with P_d. Therefore,

$$\sum_{|\mu|=d} \frac{1}{\mathfrak{z}(\mu)} \, e^{\frac{u}{t}\mathcal{F}_2} \prod \alpha_{-\mu_i} \, P_\emptyset \prod \alpha_{\mu_i} e^{-\frac{u}{t}\mathcal{F}_2} = P_d \tag{3.4}$$

After summing (3.1) using (3.4), we find:

$$\mathsf{G}_d(z_1, \ldots, z_n, w_1, \ldots, w_m, u) = \\ u^{-2d-n-m} \Big\langle \prod \mathcal{A}(tz_i, uz_i) \, e^{\alpha_1} \, P_d \, e^{\alpha_{-1}} \prod \mathcal{A}(-tw_j, uw_j)^* \Big\rangle \,. \tag{3.5}$$

3.1.4

Define the $n+m$-point function $\mathsf{G}(z,w,u)$ of equivariant Gromov-Witten invariants of all degrees by:

$$\mathsf{G}(z,w,u) = \sum_{d\geq 0} q^d\, \mathsf{G}_d(z,w,u)\,.$$

Since $H = \sum_d d\, P_d$, we find:

$$\mathsf{G}(z_1,\dots,z_n,w_1,\dots,w_m,u) = \\ u^{-n-m} \left\langle \prod \mathcal{A}(tz_i,uz_i)\, e^{\alpha_1} \left(\frac{q}{u^2}\right)^H e^{\alpha_{-1}} \prod \mathcal{A}(-tw_j,uw_j)^* \right\rangle . \quad (3.6)$$

Introduce the following operators:

$$\begin{aligned} \mathsf{A}(z) &= \frac{1}{u}\,\mathcal{A}(tz,uz), \\ \mathsf{A}^\star(w) &= \frac{1}{u}\,\mathcal{A}(-tw,uw)^*\,. \end{aligned} \quad (3.7)$$

Recall, by definition,

$$\mathsf{A}(z) = u^{-1}\, \mathcal{S}(uz)^{tz} \sum_{k\in\mathbb{Z}} \frac{\varsigma(uz)^k}{(1+tz)_k}\, \mathcal{E}_k(uz)\,. \quad (3.8)$$

We obtain the following result by substituting the operators $\mathsf{A}(z)$, $\mathsf{A}^\star(w)$ in equation (3.6).

Theorem 3. *The function* $\mathsf{G}(z,w,u)$ *is the following vacuum expectation:*

$$\mathsf{G}(z_1,\dots,z_n,w_1,\dots,w_m,u) = \\ \left\langle \prod \mathsf{A}(z_i)\; e^{\alpha_1} \left(\frac{q}{u^2}\right)^H e^{\alpha_{-1}} \prod \mathsf{A}^\star(w_j) \right\rangle . \quad (3.9)$$

In particular, for the 0-point function, Theorem 3 yields the following correct evaluation:

$$\mathsf{G}() = \left\langle e^{\alpha_1} \left(\frac{q}{u^2}\right)^H e^{\alpha_{-1}} \right\rangle = e^{q/u^2}\,.$$

3.2 The τ-function

3.2.1

By definition, $\mathsf{G}(z,w,u)$ includes unstable contributions obtained from (1.4). We will now introduce the τ-function: a generating function for the true equivariant Gromov-Witten invariants of $\mathbf{P}^1$. The τ-function does not include unstable contributions. In Theorems 5 and 7, we will show the τ-function of the equivariant theory of $\mathbf{P}^1$ is a τ-function of an integrable hierarchy, namely, the 2–Toda hierarchy of Ueno and Takasaki.

3.2.2

Let A_k denote the coefficient of z^{k+1} in the expansion of A:

$$\mathsf{A}_k = [z^{k+1}]\,\mathsf{A}\,, \quad \mathsf{A}^\star_k = [z^{k+1}]\,\mathsf{A}^\star\,, \quad k \in \mathbb{Z}\,.$$

Then, by Theorem 3,

$$\sum_{g\in\mathbb{Z}}\sum_{d\geq 0} u^{2g-2}q^d \left\langle \prod \tau_{k_i}(\mathbf{0}) \prod \tau_{l_j}(\infty)\right\rangle^{\bullet}_{g,d} = \\ \left\langle \prod \mathsf{A}_{k_i}\, e^{\alpha_1} \left(\frac{q}{u^2}\right)^H e^{\alpha_{-1}} \prod \mathsf{A}^\star_{l_j} \right\rangle\,, \tag{3.10}$$

where, the left side consists of true equivariant Gromov-Witten invariant (with no unstable contributions). The unstable contributions (1.4) produce terms of degrees at most 0 in their variables and, therefore, do not contribute to (3.10).

3.2.3

Let the variable sets $x_i, x_i^\star$ correspond to the descendents $\tau_i(\mathbf{0}), \tau_i(\infty)$ respectively. Define the equivariant τ-function by:

$$\tau(x,x^\star,u) = \sum_{g\in\mathbb{Z}}\sum_{d\geq 0} u^{2g-2}\, q^d \left\langle \exp\left(\sum_{i\geq 0} x_i\, \tau_i(\mathbf{0}) + x_i^\star\, \tau_i(\infty)\right)\right\rangle^{\bullet}_{g,d}\,.$$

Theorem 4. *The equivariant τ-function is a vacuum expectation in $\Lambda^{\frac{\infty}{2}}V$:*

$$\tau(x,x^\star,u) = \left\langle e^{\sum x_i \mathsf{A}_i}\, e^{\alpha_1} \left(\frac{q}{u^2}\right)^H e^{\alpha_{-1}}\, e^{\sum x_i^\star \mathsf{A}_i^\star}\right\rangle\,. \tag{3.11}$$

Proof. The formula is a restatement of (3.10). ∎

3.3 The GW/H correspondence

The generating function for the absolute stationary non-equivariant Gromov-Witten theory of $\mathbf{P}^1$ is obtained from the generating function (3.9) by taking

$$m = 0\,, \quad t = 0\,, \quad u = 1\,.$$

The operator formula (3.9) then specializes to

$$\mathsf{G}(z, \emptyset, 1)\big|_{t=0} = \left\langle \prod \mathcal{A}(0, z_i)\ e^{\alpha_1}\, q^H\, e^{\alpha_{-1}} \right\rangle .$$

We have

$$\begin{aligned} \mathcal{A}(0, z) &= \sum_{k \geq 0} \frac{\varsigma(z)^k}{k!}\, \mathcal{E}_k(z) \\ &= e^{\alpha_1}\, \mathcal{E}_0(z)\, e^{-\alpha_1}\,, \end{aligned} \tag{3.12}$$

where the second equality follows from (2.16). We obtain the following result.

Proposition 11. *The n-point function of absolute stationary non-equivariant Gromov-Witten invariants of $\mathbf{P}^1$ is given by:*

$$\mathsf{G}(z, \emptyset, 1)\big|_{t=0} = \left\langle e^{\alpha_1}\, q^H \prod \mathcal{E}_0(z_i)\, e^{\alpha_{-1}} \right\rangle . \tag{3.13}$$

Extracting the coefficient of q^d in (3.13), we obtain the following equivalent formula:

$$\mathsf{G}_d(z, \emptyset, 1)\big|_{t=0} = \frac{1}{(d!)^2} \left\langle \alpha_1^d \prod \mathcal{E}_0(z_i)\, \alpha_{-1}^d \right\rangle . \tag{3.14}$$

This is precisely the special case of the GW/H correspondence [24] required for the proof of the general GW/H correspondence given in [24].

4 The 2–Toda hierarchy

4.1 Preliminaries on the 2–Toda hierarchy

4.1.1

Let M be an element of the group $GL(\infty)$ acting in the $GL(\infty)$-module $\Lambda^{\frac{\infty}{2}}V$. The matrix elements of the operator M,

$$(Mv, w)\,, \quad v, w \in \Lambda^{\frac{\infty}{2}}V\,,$$

can be viewed as, suitably regularized, $\frac{\infty}{2} \times \frac{\infty}{2}$-minors of the matrix M. In particular, the matrix elements satisfy quadratic Plücker relations.

A concise way to write all the Plücker relations is the following, see for example [15, 18]. Introduce the following operator on $\Lambda^{\frac{\infty}{2}}V \otimes \Lambda^{\frac{\infty}{2}}V$:

$$\Omega = \sum_{k\in\mathbb{Z}+\frac{1}{2}} \psi_k \otimes \psi_k^* .$$

The operator Ω operator can be defined $GL(\infty)$-invariantly by taking, instead of $\{\psi_k\}$ and $\{\psi_k^*\}$, any linear basis of the space V of creation operators and the corresponding dual basis of the space of annihilation operators. The $GL(\infty)$-invariance implies

$$[M \otimes M, \Omega] = 0 \tag{4.1}$$

for any operator M in the closure of the image of $GL(\infty)$ in the endomorphisms of $\Lambda^{\frac{\infty}{2}}V$.

Concretely, for any $v, v', w, w' \in \Lambda^{\frac{\infty}{2}}V$, we obtain the following quadratic relation between the matrix coefficients of M

$$([M \otimes M, \Omega]\ v \otimes v', w \otimes w') = 0 . \tag{4.2}$$

4.1.2

For example, consider the following vectors in (4.2):

$$\begin{aligned}
v = v_\emptyset &= \underline{-\tfrac{1}{2}} \wedge \underline{-\tfrac{3}{2}} \wedge \underline{-\tfrac{5}{2}} \wedge \underline{-\tfrac{7}{2}} \wedge \dots , \\
v' = v_\square &= \underline{\tfrac{1}{2}} \wedge \underline{-\tfrac{3}{2}} \wedge \underline{-\tfrac{5}{2}} \wedge \underline{-\tfrac{7}{2}} \wedge \dots , \\
w = v_1 &= \underline{\tfrac{1}{2}} \wedge \underline{-\tfrac{1}{2}} \wedge \underline{-\tfrac{3}{2}} \wedge \underline{-\tfrac{5}{2}} \wedge \dots , \\
w' = v_{-1} &= \underline{-\tfrac{3}{2}} \wedge \underline{-\tfrac{5}{2}} \wedge \underline{-\tfrac{7}{2}} \wedge \underline{-\tfrac{9}{2}} \wedge \dots ,
\end{aligned}$$

where $v_\emptyset, v_1, v_{-1}$ are the vacua in subspaces of charge 0, 1, and -1, respectively, and $v_\square$ is the unique charge 0 vector of energy 1, corresponding to the partition $\lambda = (1)$.

We find from the definitions,

$$\begin{aligned}
\Omega\, v_\emptyset \otimes v_\square &= v_1 \otimes v_{-1} , \\
\Omega^*\, v_1 \otimes v_{-1} &= v_\emptyset \otimes v_\square - v_\square \otimes v_\emptyset .
\end{aligned}$$

Hence, (4.2) yields the following identity:

$$(M\,v_1, v_1)\,(M\,v_{-1}, v_{-1}) = \\ (M\,v_\emptyset, v_\emptyset)\,(M\,v_\Box, v_\Box) - (M\,v_\emptyset, v_\Box)\,(M\,v_\Box, v_\emptyset)\,. \quad (4.3)$$

The above identity, which remains valid for matrices of finite size, is often associated with Lewis Carroll [2], but was first established by P. Desnanot in 1819 (see [19]).

Another way to write identity (4.3) is the following:

$$\left\langle T^{-1} M T \right\rangle \left\langle T M T^{-1} \right\rangle = \langle M \rangle \, \langle \alpha_1\, M\, \alpha_{-1} \rangle - \langle \alpha_1\, M \rangle \, \langle M\, \alpha_{-1} \rangle \ , \qquad (4.4)$$

where T is the translation operator on the in infinite wedge space

$$T \cdot \bigwedge \underline{s_i} = \bigwedge \underline{s_i + 1}\,.$$

4.1.3

Using the *vertex operators*

$$\Gamma_\pm(t) = \exp\left(\sum_{k>0} t_k \, \frac{\alpha_{\pm k}}{k} \right)\,,$$

we define a sequence of *τ-functions* corresponding to the operator M,

$$\tau_n^M(t,s) = \left\langle T^{-n}\, \widehat{M}\, T^n \right\rangle\,, \quad \widehat{M} = \Gamma_+(t)\, M\, \Gamma_-(s)\,, \quad n \in \mathbb{Z}\,.$$

The derivatives of τ_n^M with respect to the variables t and s are nothing but matrix elements of the matrix $\widehat{M} \in GL(\infty)$. Hence, the functions τ_n^M satisfy a collection of bilinear partial differential equations. This collection is known as the 2–Toda hierarchy of Ueno and Takasaki, see [28] and also, for example, the Appendix to [20] for a brief exposition.

In particular, the lowest equation of the hierarchy is a restatement of the equation (4.4):

$$\tau_n \frac{\partial^2}{\partial t_1 \partial s_1} \tau_n - \frac{\partial}{\partial s_1} \tau_n \, \frac{\partial}{\partial t_1} \tau_n = \tau_{n+1}\, \tau_{n-1}\,, \quad n \in \mathbb{Z}. \qquad (4.5)$$

We may rewrite (4.5) as:

$$\frac{\partial^2}{\partial t_1 \partial s_1} \log \tau_n = \frac{\tau_{n+1}\, \tau_{n-1}}{\tau_n^2}\,. \qquad (4.6)$$

4.2 String and divisor equations

4.2.1

The equivariant divisor equations describes the effects of insertions of $\tau_0(\mathbf{0})$ and $\tau_0(\infty)$. In terms of the disconnected $(n+m)$-point generating function $\mathsf{G}_d(z_1,\dots,z_n,w_1,\dots,w_m,u)$, the divisor equation for $\tau_0(\mathbf{0})$ insertion takes the following form.

Proposition 12. *We have*

$$\left[z_0^1\right]\ \mathsf{G}_d(z_0,z_1,\dots,z_n,w_1,\dots,w_m,u) = \\ \left(d-\frac{1}{24}+t\sum_{i=1}^{n} z_i\right)\ \mathsf{G}_d(z_1,\dots,z_n,w_1,\dots,w_m,u)\,. \quad (4.7)$$

Recall, by construction, the function G_d includes contributions from unstable moduli spaces. Therefore, the usual geometric proof of the divisor equation requires a modification. Instead, we will prove the formula (4.7) using the operator formalism.

The presence of the disconnected and unstable contributions in G_d actually simplifies the form of the divisor equation — special handling of the exceptional cases is no longer required.

Proof. Equation (3.9) states:

$$\mathsf{G}_d(z,w,u) = u^{-2d}\left\langle \prod \mathsf{A}(z_i)\, e^{\alpha_1}\, P_d\, e^{\alpha_{-1}} \prod \mathsf{A}(w_i)^\star \right\rangle\,,$$

and hence

$$\left[z_0^1\right]\ \mathsf{G}_d(z_0,z_1,\dots,z_n,w,u) = \left\langle \mathsf{A}_0 \prod \mathsf{A}(z_i)\, e^{\alpha_1}\, P_d\, e^{\alpha_{-1}} \prod \mathsf{A}(w_i)^\star \right\rangle\,.$$

The operator A_0 has the following form

$$\mathsf{A}_0 = \alpha_1 - \frac{1}{24} + \dots\,, \quad (4.8)$$

where the dots stand for terms for which the adjoint annihilates the vacuum. Since the energy operator H also annihilates the vacuum, we can write:

$$\left[z_0^1\right]\ \mathsf{G}_d(z_0,z_1,\dots,z_n,w,u) = \\ \left\langle \left(-\tfrac{1}{24}+\alpha_1+H\right) \prod \mathsf{A}(z_i)\, e^{\alpha_1}\, P_d\, e^{\alpha_{-1}} \prod \mathsf{A}(w_i)^\star \right\rangle\,. \quad (4.9)$$

¿From definition (3.8), we find:

$$[\alpha_1 + H, \mathsf{A}(z)] = tz\,\mathsf{A}(z)\,. \tag{4.10}$$

Also, we have $[H, \alpha_1] = -\alpha_1$ and $H\,P_d = d\,P_d$. Therefore,

$$(\alpha_1 + H)\, e^{\alpha_1}\, P_d = e^{\alpha_1}\, H\, P_d = d\, e^{\alpha_1}\, P_d\,.$$

Hence, commuting the operator $\alpha_1 + H$ in (4.9) to the middle, we obtain formula (4.7). □

4.2.2

The string equation describes the effect of the insertion of $\tau_0(1)$, where 1 is the identity class in the equivariant cohomology of $\mathbf{P}^1$. Since

$$1 = \frac{\mathbf{0} - \boldsymbol{\infty}}{t}$$

in the localized equivariant cohomology of $\mathbf{P}^1$, the string equation is a linear combination of the divisor equations associated to two torus fixed points. The effect of an arbitrary number of the $\tau_0(1)$-insertions can be conveniently described in the following form.

Proposition 13. *We have*

$$\left\langle e^{\tau_0(1)} \prod \tau_{k_i}(\mathbf{0}) \prod \tau_{l_i}(\infty) \right\rangle^{\bullet}_{g,d} = \left[\prod z_i^{k_i+1} \prod w_i^{l_i+1}\right] e^{\sum z_i + \sum w_i}\, \mathsf{G}_{g,d}(z, w, u)\,. \tag{4.11}$$

4.3 The 2–Toda equation

4.3.1

Let M be the matrix appearing in (3.11),

$$\mathsf{M} = e^{\sum x_i \mathsf{A}_i}\, e^{\alpha_1} \left(\frac{q}{u^2}\right)^H e^{\alpha_{-1}}\, e^{\sum x_i^\star \mathsf{A}_i^\star}\,. \tag{4.12}$$

In Section 4.4, we will see that for a suitable matrix M, one can conjugate M to the canonical form $\Gamma_+(t)\, M\, \Gamma_-(s)$ required of the 2-Toda hierarchy. Here, the time variables $\{t_i\}$ and $\{s_i\}$ are related to the variables $\{x_i\}$ and $\{x_i^\star\}$ by an explicit linear transformation.

The 2–Toda equation, the lowest equation of the Ueno-Takasaki hierarchy, is then a consequence of the results Section 4.4. However, a direct derivation of the 2–Toda equation, without the full hierarchy, is presented here first.

4.3.2

From (4.8) we obtain

$$\frac{\partial}{\partial x_0}\,\tau(x,x^\star,u) = \left\langle (\alpha_1 - \tfrac{1}{24})\,\mathsf{M} \right\rangle \,,$$

and, similarly,

$$\frac{\partial}{\partial x_0^\star}\,\tau(x,x^\star,u) = \left\langle \mathsf{M}\,(\alpha_{-1} - \tfrac{1}{24}) \right\rangle \,.$$

We therefore find

$$\begin{aligned} \tau\,\frac{\partial^2}{\partial x_0\,\partial x_0^\star}\,\tau - \frac{\partial}{\partial x_0}\,\tau\,\frac{\partial}{\partial x_0^\star}\,\tau = \\ \langle \mathsf{M} \rangle\,\langle \alpha_1\,\mathsf{M}\,\alpha_{-1} \rangle - \langle \alpha_1\,\mathsf{M} \rangle\,\langle \mathsf{M}\,\alpha_{-1} \rangle = \\ \left\langle T^{-1}\,\mathsf{M}\,T \right\rangle \left\langle T\,\mathsf{M}\,T^{-1} \right\rangle \,, \end{aligned} \tag{4.13}$$

where the second equality follows from (4.4).

4.3.3

We will now study the conjugation of M by the translation operator T. The result combined with (4.13) will yield the 2–Toda equation.

We first examine the T conjugation of the constituent operators of M. The conjugation of the operators A_k is best summarized by the equation

$$T^{-1}\,\mathsf{A}(z)\,T = e^{uz}\,\mathsf{A}(z)\,, \tag{4.14}$$

which follows directly from definitions. The conjugation equations for $\mathsf{A}_k^\star$ are identical.

Since T commutes with $\alpha_{\pm 1}$, the only other conjugation we require is:

$$T^{-n}\,H\,T^n = H + nC + \frac{n^2}{2}\,, \tag{4.15}$$

where C is the charge operator (see Section 2.2.3 of [24]). Since C commutes with the remaining operators $\mathsf{A}_k, \mathsf{A}_k^\star, \alpha_{\pm 1}$ and annihilates the vacuum, we may ignore C.

We now observe the evolution of the operators $\mathsf{A}_k, \mathsf{A}_k^\star$ under the string equation in (4.11) has exactly same form as (4.14). Introduce, the following differential operator

$$\partial = \frac{1}{t}\left(\frac{\partial}{\partial x_0} - \frac{\partial}{\partial x_0^\star}\right),$$

the action of which on τ corresponds to the insertion of $\tau_0(1)$.

Combining (4.14), (4.15), and (4.11), we obtain

$$\left\langle T^{-n}\,\mathsf{M}\,T^n\right\rangle = \frac{q^{n^2/2}}{u^{n^2}}\,e^{nu\partial}\tau\,, \tag{4.16}$$

and therefore,

$$\left\langle T^{-1}\,\mathsf{M}\,T\right\rangle\left\langle T\,\mathsf{M}\,T^{-1}\right\rangle = \frac{q}{u^2}\,e^{u\partial}\tau\;e^{-u\partial}\tau\,.$$

Thus, we have established the following version of the 2–Toda equation for the function $\tau(x, x^\star, u)$.

Theorem 5. *The function $\tau(x, x^\star, u)$ satisfies the following form of the 2–Toda equation:*

$$\frac{\partial^2}{\partial x_0\,\partial x_0^\star}\log\tau = \frac{q}{u^2}\,\frac{e^{u\partial}\tau\;e^{-u\partial}\tau}{\tau^2}\,. \tag{4.17}$$

Since the degree variable q appears as a factor on the right side of (4.17), the equation (4.17) determines all positive degree Gromov-Witten invariants of $\mathbf{P}^1$ from the degree 0 invariants.

4.4 The 2–Toda hierarchy

4.4.1

Our goal now is to prove that there exists a upper unitriangular matrix W such that

$$W^{-1}\,\exp\left(\sum x_i\,\mathsf{A}_i\right)\,W = \Gamma_+(t)\,, \tag{4.18}$$

where the time variables $\{t_i\}$ are obtained from the variables $\{x_i\}$ by certain explicit linear transformation which will be described below.

Once (4.18) is established, one deduces the 2-Toda hierarchy for the τ-function (3.11) as follows. First, taking the adjoint of the equation (4.18) and reversing the sign of the equivariant parameter t, we obtain

$$W^\star\,\exp\left(\sum x_i^\star\,\mathsf{A}_i^\star\right)\,(W^\star)^{-1} = \Gamma_-(s) \tag{4.19}$$

where

$$W^\star = W^*\big|_{t\mapsto -t}\,.$$

The linear transformation

$$\{x_i^\star\} \mapsto \{s_i\}$$

is obtained from the linear transformation $\{x_i\} \mapsto \{t_i\}$ by reversing the sign of the equivariant parameter t.

Together, the equations (4.18) and (4.19), give the following formula for the matrix (4.12)

$$\mathsf{M} = W\,\Gamma_+(t)\,M\,\Gamma_-(s)\,W^\star\,, \tag{4.20}$$

where

$$M = W^{-1}\,e^{\alpha_1}\,\left(\frac{q}{u^2}\right)^H\,e^{\alpha_{-1}}\,(W^\star)^{-1}\,.$$

The unitriangularity of W implies

$$W^*\,v_\emptyset = W^\star\,v_\emptyset = v_\emptyset\,,$$

and, more generally,

$$W^*\,T^n\,v_\emptyset = W^\star\,T^n\,v_\emptyset = T^n\,v_\emptyset\,,\quad n\in\mathbb{Z}\,.$$

Therefore, we obtain

$$\begin{aligned}\frac{q^{n^2/2}}{u^{n^2}}\,e^{nu\partial}\tau &= \left\langle T^{-n}\,\mathsf{M}\,T^n\right\rangle \\ &= \left\langle T^{-n}\,\Gamma_+(t)\,M\,\Gamma_-(s)\,T^n\right\rangle\,,\end{aligned} \tag{4.21}$$

where the first equation is copied from (4.16). It then follows that the sequence (4.21) is a sequence of τ-functions for the full 2–Toda hierarchy of Ueno and Takasaki.

4.4.2

We now proceed with the realization of the above plan.

We will now view the operators A_k as matrices in the associative algebra $\mathrm{End}(\infty)$. All multiplication operations in Sections 4.4.2 – 4.4.9 should be interpreted as multiplication in $\mathrm{End}(\infty)$, and *not* in $\mathrm{End}(\Lambda^{\frac{\infty}{2}}V)$.

For $k\geq 0$, the matrices A_k commute by Theorem 1 and have the form

$$\mathsf{A}_k = \frac{u^k}{(k+1)!}\,\alpha_{k+1} + \dots\,, \tag{4.22}$$

where the dots stand for term of energy larger than $-k-1$.

Since the matrix A_0 has form (4.22), there exists an upper unitriangular matrix $W \in GL(\infty)$ conjugating A_0 to α_1:

$$W^{-1}\,\mathsf{A}_0\,W = \alpha_1 \,.$$

We call the matrix W the *dressing operator*. The explicit form of W is rather complicated, unique only up to left multiplication by a element of the centralizer of α_1, and will not be required.

However, the dressed matrices

$$\widetilde{\mathsf{A}}_k = W^{-1}\,\mathsf{A}_k\,W, \quad k \geq 0,$$

are uniquely defined and can be identified explicitly.

Because the matrices $\widetilde{\mathsf{A}}_k$ commute with the matrix $\widetilde{\mathsf{A}}_0 = \alpha_1$, the matrices have the following form:

$$\widetilde{\mathsf{A}}_k = \sum_{l \leq k+1} c_{k,l}(u,t)\,\alpha_l\,, \quad k = 0, 1, \dots\,, \tag{4.23}$$

where

$$c_{k,k+1} = \frac{u^k}{(k+1)!}\,.$$

The other coefficients of the expansion are determined by the following result.

Theorem 6. *The dressed operators $\widetilde{\mathsf{A}}_k$ are determined by a generating function identity:*

$$\sum_{k\geq 0} z^{k+1}\,\widetilde{\mathsf{A}}_k = \sum_{n\geq 1} \frac{u^{n-1}\,z^n}{(1+tz)\cdots(n+tz)}\,\alpha_n\,. \tag{4.24}$$

As an immediate consequence of Theorem 6, we see $c_{k,l}(u,t) = 0$ unless $l > 0$.

4.4.3

Equation (4.23) is equivalent to the equation

$$\mathsf{A}_k = \sum_{l \leq k+1} c_{k,l}(u,t)\,\mathsf{A}_0^l\,, \quad k = 0, 1, \dots\,, \tag{4.25}$$

where the powers of A_0 are taken in the associative algebra $\mathrm{End}(\infty)$.

The operator $\mathsf{A}(z)$ is homogeneous of degree -1 with respect to the following grading:

$$\deg u = \deg t = -\deg z = 1\,.$$

Therefore, the operator A_k has degree k with respect to the grading. Therefore, by (4.25), we see

$$\deg\ c_{k,l}(u,t)\ = k\,. \tag{4.26}$$

Theorem 6 implies $c_{k,l}(u,t)$ is a *monomial*:

$$c_{k,l}(u,t) = c_{k,l}\, u^{l-1}\, t^{k-l+1}\,, \quad c_{k,l} \in \mathbb{Q}\,, \tag{4.27}$$

a nontrivial fact which will play an important role in the proof.

Because of the homogeneity property (4.26), we may set $u = 1$ in order to simplify our computations.

4.4.4

Taking the adjoint of equation (4.24) and reversing the sign of t, we find:

$$\sum_{k\geq 0} z^{k+1}\, \widetilde{\mathsf{A}}^\star_k = \sum_{n\geq 1} \frac{u^{n-1}\, z^n}{(1-tz)\cdots(n-tz)}\, \alpha_{-n}\,. \tag{4.28}$$

where

$$\widetilde{\mathsf{A}}^\star_k = W^\star\, \mathsf{A}^\star_k\, (W^\star)^{-1}\,,$$

and $W^\star = W^*(u,-t)$.

Following the discussion of Section 4.4.1, we immediately obtain the following result.

Theorem 7. *The triangular linear change of time variables given by* (4.24) *and* (4.28) *makes the sequence of functions,*

$$\frac{q^{n^2/2}}{u^{n^2}}\, e^{nu\partial}\, \tau(x,x^\star,u)\,, \quad n\in\mathbb{Z}\,,$$

a sequence of τ-functions for the full 2–Toda hierarchy of Ueno and Takasaki.

Our derivation has neglected a minor point: the operators $\mathsf{A}_k, \widetilde{\mathsf{A}}_k$ have constant terms when acting on $\Lambda^{\frac{\infty}{2}}V$ (and similarly for $\mathsf{A}^\star_k, \widetilde{\mathsf{A}}^\star_k$). However, these constants can be removed by further conjugation by operators α_n in $\Lambda^{\frac{\infty}{2}} V$. The constants do not affect Theorem 7.

The explicit form of the linear change of variables from the Gromov-Witten times to the standard times of the 2–Toda hierarchy was conjectured by Getzler, see [11].

4.4.5

We now proceed with the proof of Theorem 6 starting with the following result.

Proposition 14. *For $k \geq 0$ and $l > 0$, the coefficient $c_{k,l}(u,t)$ is a monomial in t of degree $k - l + 1$.*

Proof. We set $u = 1$. By (4.25), we may equivalently prove the coefficient of A_0^l in the expansion of A_k is a monomial in t of degree $k - l + 1$. Further, by induction, it suffices to prove the coefficients $b_{k,l}(t)$ in the expansion

$$\mathsf{A}_0 \, \mathsf{A}_k = \sum_{l \leq k+1} b_{k,l}(t) \, \mathsf{A}_l \tag{4.29}$$

are monomials in t of degree $k + 1 - l$ for $l \geq 0$.

The coefficients $b_{k,l}(t)$ with $l \geq 0$ can be determined from the negative energy matrix elements of the product $\mathsf{A}_0 \, \mathsf{A}_k$. The matrix elements of $\mathsf{A}_0 \, \mathsf{A}_k$ are obtained as the $z \, w^{k+1}$ coefficient of the expansion of $\mathsf{A}(z) \, \mathsf{A}(w)$. Since

$$\mathcal{E}_a(z) \, \mathcal{E}_b(w) = e^{(aw-bz)/2} \, \mathcal{E}_{a+b}(z+w) \,,$$

we compute

$$\begin{aligned} \mathsf{A}(z) \, \mathsf{A}(w) \; &= \mathcal{S}(z)^{tz} \, \mathcal{S}(w)^{tw} \sum_{m \in \mathbb{Z}} \mathcal{E}_m(z+w) \times \\ & \frac{\varsigma(z)^m \, e^{mw/2}}{(1+tz)_m} \left(\sum_{n \in \mathbb{Z}} \frac{(-tz-m)_n}{(1+tw)_n} \left(\frac{1-e^{-w}}{1-e^z} \right)^n \right) . \end{aligned} \tag{4.30}$$

The summation over n in (4.30) is formally infinite, but only finitely many terms actually contribute to the $z \, w^{k+1}$ coefficient. Indeed, the coefficient of z vanishes if $m > n + 1$, while the coefficient of w^{k+1} vanishes if $n > k + 1$.

The sum over n in (4.30) can be written as:

$$
{}_2F_1\left(\begin{matrix}-tz-m,1\\1+tw\end{matrix};\frac{1-e^{-w}}{1-e^{z}}\right)+{}_2F_1\left(\begin{matrix}-tw,1\\1+m+tz\end{matrix};\frac{1-e^{z}}{1-e^{-w}}\right)-1\,, \tag{4.31}
$$

where the hypergeometric function is defined by (5.2). The two series in (4.31) converge for $|w|<|z|\ll 1$ and $|z|<|w|\ll 1$, respectively. Therefore, we can write the coefficient of $z\,w^{k+1}$ as a sum of two contour integrals in two different domains.

We may now deform these contour integrals to integrals over

$$
|z|=|w|=\epsilon\ll 1\,.
$$

The condition $m>0$ is needed for the hypergeometric function to remain continuous in this limit. On the new contour, which is now common to both integrals, we can use formula (5.8). After some simplifications, we find:

$$
\begin{aligned}
\mathsf{A}_0\,\mathsf{A}_k = \frac{1}{(2\pi i)^2}\iint_{|z|=|w|=\varepsilon} \frac{dz\,dw}{z^2\,w^{k+2}}\times \\
\frac{\left(1+\frac{w}{z}\right)^{tz+tw}}{\left(\frac{w}{z}\right)^{tw}}\,\frac{\Gamma(1+tz)\,\Gamma(1+tw)}{\Gamma(1+tz+tw)}\,\mathsf{A}(z+w)+\dots
\end{aligned} \tag{4.32}
$$

where the dots denote terms of non-negative energy.

The meaning of formula (4.32) is the following. First, the multivalued function

$$
\frac{\left(1+\frac{w}{z}\right)^{tz+tw}}{\left(\frac{w}{z}\right)^{tw}} \tag{4.33}
$$

is defined using the cut

$$
\frac{w}{z}\neq(-\infty,0]\,.
$$

Because both z and w are small, the function (4.33) is integrable in the neighborhood of the singularity $w=-z$ on the contour of integration. Second, the negative energy terms in $\mathsf{A}(z+w)$ are nonsingular at $z+w=0$ and, hence, their expansion in powers of z and w is unambiguous. Also, these terms do not spoil the convergence of the integral at $w=-z$.

From formula (4.32), we deduce, for $l \geq 0$,

$$b_{k,l}(t) = \frac{1}{(2\pi i)^2} \sum_{a=0}^{l+1} \binom{l+1}{a} \iint_{|z|=|w|=\varepsilon} \frac{dz\, dw}{z^{2-a}\, w^{k+a+1-l}} \times \\ \frac{\left(1+\frac{w}{z}\right)^{tz+tw}}{\left(\frac{w}{z}\right)^{tw}} \frac{\Gamma(1+tz)\,\Gamma(1+tw)}{\Gamma(1+tz+tw)}\,. \quad (4.34)$$

After replacing tz and tw by new variables, we see (4.34) is indeed a monomial in t of degree $k-l+1$. $\square$

4.4.6

From Lemma 2, we expect the following heuristic result:

$$\mathsf{A}(z)\,\mathsf{A}(w) \text{ “=” } \frac{(z+w)^{tz+tw}}{z^{tz}\, w^{tw}} \frac{(tz)!\,(tw)!}{(tz+tw)!} \mathsf{A}(z+w)\,,$$

which becomes a true equality when both tz and tw are positive integers. Equation (4.32) is a way to make sense of of the heuristic formula.

4.4.7

The next step in the proof of Theorem 6 is the following result.

Proposition 15. *For all l, the coefficient $c_{k,l}(u,t)$ is a monomial in t of degree $k-l+1$.*

Proof. By Proposition 14, we need only consider $l \leq 0$. Define the operator D by:

$$\mathsf{D} = W^{-1}\,(\alpha_1 + H)\,W\,.$$

Equation (4.10) implies:

$$\left[\mathsf{D}, \widetilde{\mathsf{A}}_k\right] = t\,\widetilde{\mathsf{A}}_{k-1}\,. \quad (4.35)$$

Also, since $\mathsf{A}_0 = \alpha_1 + H + \dots$, we see

$$\mathsf{D} = \alpha_1 + \dots,$$

where, in both cases, the dots stand for terms with positive energy.

Since the matrix $[\mathsf{D}, \alpha_1] = t\,\widetilde{\mathsf{A}}_{-1}$ commutes with α_1, the matrix D has the form

$$\mathsf{D} = \alpha_1 + \sum_{n>0} d_n(u,t)\,\alpha_{-n}\,H + \dots\,,$$

where the dots stand for terms that commute with α_1 and whose precise form depends on the ambiguity in the choice of the dressing matrix W. Here, H is the energy operator and the product is taken in the algebra $\mathrm{End}(V)$.

It is easy to see equation (4.10) uniquely determines all the coefficients d_n in terms of $c_{k,l}(u,t)$ with $l > 0$. The coefficients d_n, in turn, determine all remaining coefficients $c_{k,l}(u,t)$. In fact,

$$d_n(u,t) = -\frac{t^n}{u^n}\,.$$

However, for the proof of the Proposition, we need only observe the uniqueness forces d_n to have degree n in t. Then, the coefficients $c_{k,l}(u,t)$ must have degrees $k - l + 1$ in t. $\square$

4.4.8

From the proof of Proposition 15, we see the matrices $\widetilde{\mathsf{A}}_k$ can be uniquely characterized by the two following conditions:

(i) $\widetilde{\mathsf{A}}_0 = \alpha_1$ and $\widetilde{\mathsf{A}}_k$ is a linear combination of $\alpha_1, \dots, \alpha_{k+1}$.

(ii) There exists a matrix of the form

$$\mathsf{D} = \alpha_1 + \sum_{n>0} d_n\,\alpha_{-n}\,H\,, \quad d_1 = -\frac{t}{u}\,,$$

such that $\left[\mathsf{D}, \widetilde{\mathsf{A}}_k\right] = t\,\mathsf{A}_{k-1}$ for $k > 0$.

4.4.9

We can now complete the proof of Theorem 6. Since the coefficients $c_{k,l}(u,t)$ are monomials in t, the coefficients are identical to their leading order asymptotics as $u \to 0$. Hence, the operators $\widetilde{\mathsf{A}}_k$ can be determined by studying the

$u \to 0$ asymptotics of the operators $\mathsf{A}(z)$. In the $u \to 0$ limit, we have

$$\mathsf{A}(z) \sim \sum_{n\geq 0} \frac{u^{n-1}\, z^n}{(1+tz)\cdots(n+tz)}\, \alpha_n + \\ \sum_{n>0} \frac{t}{u^{n+1}} \left(t - \frac{1}{z}\right) \cdots \left(t - \frac{n-1}{z}\right) \alpha_{-n}\,. \quad (4.36)$$

In the $u \to 0$ limit, the dressing matrix W becomes trivial and the statement of Theorem 6 can be read off directly from (4.36).

Formula (4.36) also contains the description of the dressed operators $\widetilde{\mathsf{A}}_k$ for $k < 0$.

5 Commutation relations for operators $\mathcal{A}$

Our goal here is to prove Theorem 1:

$$[\mathcal{A}(z,uz), \mathcal{A}(w,uw)] = zw\, \delta(z,-w)\,,$$

5.1 Formula for the commutators

5.1.1

We may calculate $[\mathcal{A}(z,uz), \mathcal{A}(w,uw)]$ by the commutation relation (2.15). We find,

$$[\mathcal{A}(z,uz), \mathcal{A}(w,uw)] = \mathcal{S}(uz)^z\, \mathcal{S}(uw)^w \sum_{m\in\mathbb{Z}} c_m(z,w)\, \mathcal{E}_m(u(z+w)) \quad (5.1)$$

where the functions $c_m(z,w)$ are defined by:

$$c_m(z,w) = \begin{cases} \dfrac{\varsigma(uz)^s\, \varsigma(uw)^s}{(z+1)_s(w+1)_s} \big[f_{s,u}(z,w) - f_{s,u}(w,z)\big], & m = 2s\,, \\[2ex] \dfrac{\varsigma(uz)^s\, \varsigma(uw)^s}{(z+1)_s(w+1)_s} \big[g_{s,u}(z,w) - g_{s,u}(w,z)\big], & m = 2s-1\,. \end{cases}$$

Here, $f_{s,u}(z,w)$ and $g_{s,u}(z,w)$ are hypergeometric series which are explicitly defined below.

We recall the definition of the hypergeometric series which we require:

$$ {}_2F_1\left(\begin{matrix}-\nu, 1\\ \mu+1\end{matrix}; z\right) = \sum_{k=0}^{\infty} \frac{\nu(\nu-1)\cdots(\nu-k+1)}{(\mu+1)\cdots(\mu+k)}(-z)^k, \quad |z|<1. \tag{5.2} $$

Define $f_{s,u}(\mu,\nu)$ and $g_{s,u}(\mu,\nu)$ by:

$$ f_{s,u}(\mu,\nu) = e^{-su\mu}{}_2F_1\left(\begin{matrix}-\nu-s, 1\\ \mu+1+s\end{matrix}; \frac{1-e^{u\mu}}{1-e^{-u\nu}}\right) - \\ e^{-su\nu}{}_2F_1\left(\begin{matrix}-\nu-s, 1\\ \mu+1+s\end{matrix}; \frac{1-e^{-u\mu}}{1-e^{u\nu}}\right) + \frac{e^{-su\nu}-e^{-su\mu}}{2} \tag{5.3} $$

and,

$$ g_{s,u}(\mu,\nu) = \frac{\nu+s}{\varsigma(u\nu)}\left[e^{(1-s)u\mu}{}_2F_1\left(\begin{matrix}-\nu-s+1, 1\\ \mu+1+s\end{matrix}; \frac{1-e^{u\mu}}{1-e^{-u\nu}}\right) - \right. \\ \left. e^{-su\nu}{}_2F_1\left(\begin{matrix}-\nu-s+1, 1\\ \mu+1+s\end{matrix}; \frac{1-e^{-u\mu}}{1-e^{u\nu}}\right)\right]. \tag{5.4} $$

5.1.2

The series $f_{s,u}(z,w)$ and $g_{s,u}(z,w)$ in formula (5.1) are to be expanded in the ring $\mathbb{Q}[u^{\pm}]((w))((z))$, that is, expanded in Laurent series of z with coefficients given by Laurent series in w. Since, for example, the kth term in

$$ {}_2F_1\left(\begin{matrix}-w-s, 1\\ z+s+1\end{matrix}; \frac{1-e^{uz}}{1-e^{-uw}}\right) = \\ \sum_{k=0}^{\infty} \frac{(w+s)\cdots(w+s-k+1)}{(z+s+1)\cdots(z+s+k)}\left(\frac{e^{uz}-1}{1-e^{-uw}}\right)^k \tag{5.5} $$

is of order z^k, the extraction of any given term in these expansions is, in principle, a finite computation. Similarly, the series $f_{s,u}(w,z)$ and $g_{s,u}(w,z)$ in formula (5.1) are to be expanded in the ring $\mathbb{Q}[u^{\pm}]((z))((w))$.

5.1.3

The constant term of $\mathcal{E}_0(u(z+w))$ plays a special role in formula (5.1). The expansion rules for the constant term,

$$ \frac{f_{0,u}(z,w)}{\varsigma(u(z+w))} - \frac{f_{0,u}(w,z)}{\varsigma(u(z+w))}, \tag{5.6} $$

are the following. The first summand is to be expanded in ascending powers of z whereas the second summand is to be expanded in ascending powers of w.

5.1.4

We will show the expansions of the two terms of $c_m(z, w)$ exactly cancel each other. The commutator is therefore obtained entirely from the constant term. We will show the expansions of the two terms of (5.6) cancel except for the two different expansions of the simple pole at $z + w = 0$.

5.2 Some properties of the hypergeometric series

5.2.1

To proceed, several properties of the hypergeometric series (5.2) are required. Define the analytic continuation of (5.2) to the complex plane with a cut along $[1, +\infty)$ by the following integral:

$$ {}_2F_1\left(\begin{matrix}-\nu, 1\\ \mu+1\end{matrix}; z\right) = \mu \int_0^1 (1-x)^{\mu-1}(1-zx)^{\nu}\, dx\,, \qquad \Re\mu > 0\,. \tag{5.7} $$

The above hypergeometric function is degenerate since the elementary function,

$$ z^{-\mu}\,(1-z)^{\mu+\nu}\,, $$

is a second solution to the hypergeometric equation and, in addition, is an eigenfunction of monodromy at $\{0, 1, \infty\}$. As a consequence, the analytic continuation of the function (5.7) through the cuts $[1, +\infty)$ leads only to the appearance of elementary terms. In fact, the analytic continuation of (5.7) through the cut $[1, +\infty)$ is given explicitly by the formula (5.8) below.

5.2.2

Lemma 16. *For $z \notin [0, +\infty)$ we have:*

$$ {}_2F_1\left(\begin{matrix}-\nu, 1\\ \mu+1\end{matrix}; z\right) = \\ 1 - {}_2F_1\left(\begin{matrix}-\mu, 1\\ \nu+1\end{matrix}; \frac{1}{z}\right) + \frac{(1-z)^{\mu+\nu}}{(-z)^{\mu}}\, \frac{\Gamma(\mu+1)\,\Gamma(\nu+1)}{\Gamma(\mu+\nu+1)}\,. \tag{5.8} $$

Here and in what follows we use the principal branches of the functions $\ln w$ and w^a for $w \notin (-\infty, 0]$.

Proof. Integrating by parts and setting $y = zx$, the integral (5.7) is transformed to the following form:

$$1 - \nu \int_0^1 \left(1 - \frac{y}{z}\right)^{\mu} (1-y)^{\nu-1}\, dy + \nu \int_z^1 \left(1 - \frac{y}{z}\right)^{\mu} (1-y)^{\nu-1}\, dy\,.$$

The last integral here is a standard beta-function integral and, thus, the three terms in the above formula correspond precisely to the three terms on the right side of (5.8) . □

5.2.3

A similar argument proves the following result.

Lemma 17. *For $z \notin [0, +\infty)$ we have*

$$\nu\, {}_2F_1\left(\begin{matrix} -\nu+1, 1 \\ \mu+1 \end{matrix}; z\right) = \\ \frac{\mu}{z}\, {}_2F_1\left(\begin{matrix} -\mu+1, 1 \\ \nu+1 \end{matrix}; \frac{1}{z}\right) + \frac{(1-z)^{\mu+\nu-1}}{(-z)^{\mu}} \frac{\Gamma(\mu+1)\,\Gamma(\nu+1)}{\Gamma(\mu+\nu)}\,.$$

5.3 Conclusion of the proof of Theorem 1

5.3.1

Lemma 18. *The functions $f_{s,u}(\mu,\nu)$ and $g_{s,u}(\mu,\nu)$ are analytic in a neighborhood of the origin $(\mu,\nu) = (0,0)$ and symmetric in μ and ν .*

Proof. We will prove the Lemma for $f_{s,u}(\mu,\nu)$. The argument for $g_{s,u}(\mu,\nu)$ is parallel with Lemma 17 replacing Lemma 16. The proof will show the neighborhood can be chosen to be independent of the parameter s.

For simplicity, we will first assume s is not a negative integer. The assumption will be removed at the end of the proof. Using relation (5.8), we find,

$$f_{s,u}(\mu,\nu) = f_{s,u}(\nu,\mu) \tag{5.9}$$

on the intersection of the domains of applicability of (5.8).

The possible singularities of $f_{s,u}(\mu,\nu)$ near the origin are at $\nu = 0$ and $\mu + \nu = 0$, corresponding to the singularities $z = \infty$ and $z = 1$ of the hypergeometric function (5.7), respectively. The hypergeometric function is analytic and single-valued in the complex plane with a cut from 1 to ∞. The function $f_{s,u}(\mu,\nu)$ is well-defined if the arguments,

$$\frac{1-e^{u\mu}}{1-e^{-u\nu}}, \frac{1-e^{-u\mu}}{1-e^{u\nu}} \approx -\frac{\mu}{\nu},$$

do not fall on the cut $[1,+\infty)$. Similarly, the function $f_{s,u}(\nu,\mu)$ is well-defined if the arguments,

$$\frac{1-e^{u\nu}}{1-e^{-u\mu}}, \frac{1-e^{-u\nu}}{1-e^{u\mu}} \approx -\frac{\nu}{\mu},$$

do not fall on the cut $[1,+\infty)$. By (5.9), the two functions above agree on the region where both are defined. It follows that $f_{s,u}(\mu,\nu)$ is single-valued and analytic near the origin in the complement of the divisor $\mu+\nu = 0$. By Lemma 19 below, $f_{s,u}(\mu,\nu)$ remains bounded as $\nu \to -\mu$ and hence the singularity at $\mu+\nu = 0$ is removable. We conclude $f_{s,u}(\mu,\nu)$ is analytic and symmetric near the origin.

Finally, consider the case when $s \to -n$, where n is positive integer. The apparent simple pole of $f_{s,u}(\mu,\nu)$ at $\mu = -s-n$ is, in fact, removable. The removability follows either from symmetry (because there is no such singularity in ν) or else can be checked directly using the formula

$$\operatorname{Res}_{\mu=-n} {}_2F_1\left(\begin{matrix}-\nu, 1\\ \mu\end{matrix}; z\right) = (-1)^{n-1}\frac{(-\nu)_n}{(n-1)!} z^n (1-z)^{\nu-n}.$$

$\square$

5.3.2

Lemma 19. *We have*

$$f_{s,u}(\mu,-\mu) = -\frac{\mu}{s}\sinh(us\mu) \tag{5.10}$$

and, in particular,

$$f_{0,u}(\mu,-\mu) = -u\mu^2. \tag{5.11}$$

Similarly,

$$g_{s,u}(\mu,-\mu) = \frac{s^2-\mu^2}{2s-1}\frac{\sinh\frac{(2s-1)u\mu}{2}}{\sinh\frac{u\mu}{2}}.$$

Proof. For $\Re s > 0$ we can use the formula

$$ {}_2F_1\left(\begin{matrix}\mu - s, 1 \\ \mu + 1 + s\end{matrix}; 1\right) = \frac{\Gamma(\mu + 1 + s)\,\Gamma(2s)}{\Gamma(\mu + s)\Gamma(2s + 1)} = \frac{\mu + s}{2s}\,, $$

from which (5.10) follows. By analytic continuation, (5.10) holds for all s. The computation of $g_{s,u}(\mu, -\mu)$ is identical. □

5.3.3

We may now complete the proof of Theorem 1. Since the functions $f_{s,u}(\mu, \nu)$ and $g_{s,u}(\mu, \nu)$ are analytic near the origin and symmetric in μ and ν, the nonconstant terms of formula (5.1) cancel.

The summands of the constant term (5.6) of formula (5.1) can be analyzed using (5.11):

$$ \frac{f_{0,u}(z, w)}{\varsigma(u(z + w))} = \frac{zw}{z + w} + \dots\,, $$

where the dots represent a function analytic at the origin and symmetric in z and w. Observe the prefactor in formula (5.1) is identically equal to 1 on the divisor $z + w = 0$ and does not affect the singularity. The proof of Theorem 1 is complete.

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