Gromov-Witten theory of Complete intersections

Lothar $\chi(\mathbb{P}^{2[4]}) + \chi(\mathbb{P}^{2[2]})$

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Photo: Denys Nevozhai on Unsplash

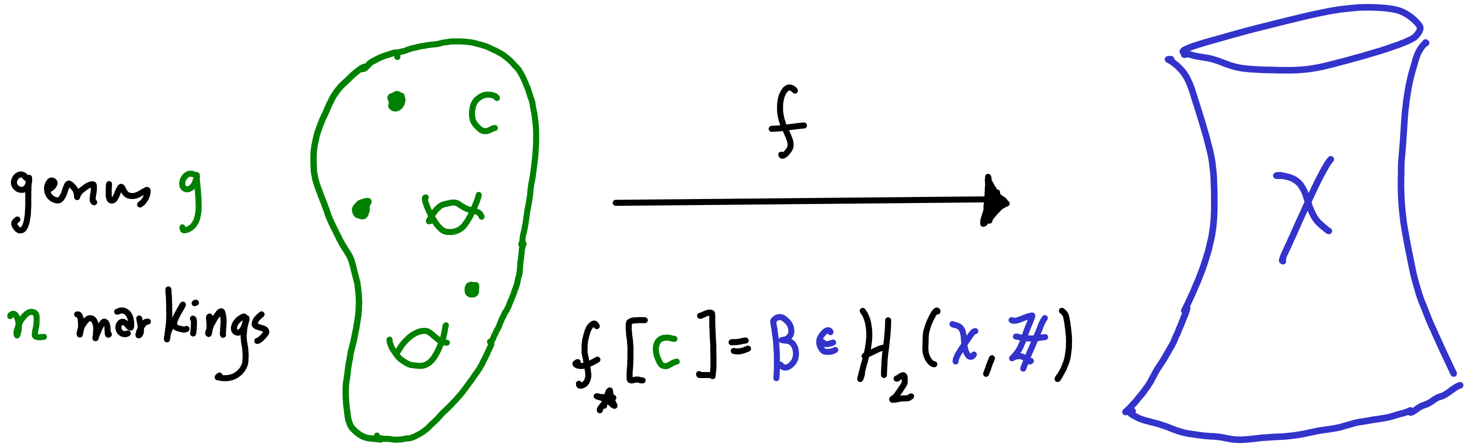
Let X be a nonsingular projective variety / $\mathbb{C}$

Questions :

(i) Is there a method for computing the Gromov-Witten theory of X ?

(ii) Is the Gromov-Witten theory of X tautological ?

$\overline{M}_{g,n}(X,\beta)$ is the moduli of stable maps :



Gromov-Witten theory of X consists of classes in $H^*(\overline{\mathcal{M}}_{g,n})$ obtained from the correspondence defined by the virtual fundamental class:

$$\overline{\mathcal{M}}_{g,n}(X,\beta) \xrightarrow{ev} X^n$$

$$\overline{\mathcal{M}}_{g,n}(X,\beta) \xrightarrow{\pi} \overline{\mathcal{M}}_{g,n}$$

$$GW(\gamma_1 \otimes \cdots \otimes \gamma_n)^X_{g,\beta} =$$

$$\pi_*\left(ev^*(\gamma_1, \ldots, \gamma_n) \cap \left[\overline{\mathcal{M}}_{g,n}(X,\beta)\right]^{vir} \right)$$

↑ result lies in $H^*(\overline{\mathcal{M}}_{g,n})$

$$GW(X) = \left\{ GW(\gamma_1 \otimes \cdots \otimes \gamma_n)^X_{g,\beta} \right\}$$

set of all GW classes on the moduli of curves for all $g, n, \beta, \gamma_i \in H^*(X)$

Also called

$CohFT(X)$

$GW(X)$ computable for

- X is a point   Kontsevich-Witten descendents $\int_{\overline{M}_{g,n}} \psi_1^{a_1} \cdots \psi_n^{a_n}$
- X is curve   Okounkov-P Janda
- X is rational surface, ruled surface   Localization
  Abelian, K3, Enriques   Maulik-P
  Reduced class   Bryan-Leung

many elliptic surfaces, No Section?

many general type surfaces $p_g = 0$?

Taubes $G = SW$

Lee - Parker

Kool - Thomas

- X is quintic 3-fold $X_5 \subset \mathbb{P}^4$

CdOGP

BCOV

Kontsevich, Givental, Lian - Liu - Yau

Gopakumar-Vafa

Zinger, Maulik - P, Pixton - P, Ionel - Parker,

Klemm ...

Chang - Guo - Li, Guo - Janda - Ruan, ...

- X is G/P or toric Graber - P

- X is a product $X_1 \times X_2$ Behrend

with both $GW(X_1)$ and $GW(X_2)$

computable

An early algorithm

arXiv: 9611012

1996: Lothar and I worked on $g=0$ for blow-ups of $\mathbb{P}^2$:

In the first table below, Gromov-Witten invariants $N_{d,\alpha}$ for $d \leq 5$ and $\alpha \geq 0$ are listed. By properties (P3), (P4), and (P5), it suffices to list the invariants for ordered sequences α satisfying $\alpha \geq 2$. Moreover, if $g_a(d,\alpha) < 0$ or if $a_i + a_j > d$, the invariant vanishes and is omitted from the table. The invariants were computed by a Maple program via the recursive algorithm of the proof of Theorem 3.6.

$d=1$	2	3	4	5	5
$N_1 = 1$	$N_2 = 1$	$N_3 = 12$	$N_4 = 620$	$N_5 = 87304$	$N_{5,(2^6)} = 1$
		$N_{3,(2)} = 1$	$N_{4,(2)} = 96$	$N_{5,(2)} = 18132$	$N_{5,(3)} = 640$
			$N_{4,(2^2)} = 12$	$N_{5,(2^2)} = 3510$	$N_{5,(3,2)} = 96$
			$N_{4,(2^3)} = 1$	$N_{5,(2^3)} = 620$	$N_{5,(3,2^2)} = 12$
			$N_{4,(3)} = 1$	$N_{5,(2^4)} = 96$	$N_{5,(3,2^3)} = 1$
				$N_{5,(2^5)} = 12$	$N_{5,(4)} = 1$

The Cremona transformation applied to the class $(5,(2,2,2))$ yields $N_{5,(2,2,2)} = N_{4,(1,1,1)}$. By Property (P5), $N_{4,(1,1,1)} = N_4 = 620$. The following table lists all the Gromov-Witten invariants for degrees 6 and 7 which are not obtained from lower degree numbers by the Cremona transformation.

$d=6$	7	7
$N_6 = 26312976$	$N_7 = 14616808192$	$N_{7,(3,2)} = 90777600$
$N_{6,(2)} = 6506400$	$N_{7,(2)} = 4059366000$	$N_{7,(3,2^2)} = 23133696$
$N_{6,(2^2)} = 1558272$	$N_{7,(2^2)} = 1108152240$	$N_{7,(3,2^3)} = 5739856$
$N_{6,(2^3)} = 359640$	$N_{7,(2^3)} = 296849546$	$N_{7,(3,2^4)} = 1380648$
$N_{6,(2^4)} = 79416$	$N_{7,(2^4)} = 77866800$	$N_{7,(3,2^5)} = 320160$
$N_{6,(2^5)} = 16608$	$N_{7,(2^5)} = 19948176$	$N_{7,(3,2^6)} = 71040$
$N_{6,(2^6)} = 3240$	$N_{7,(2^6)} = 4974460$	$N_{7,(3,2^7)} = 14928$
$N_{6,(2^7)} = 576$	$N_{7,(2^7)} = 1202355$	$N_{7,(3,2^8)} = 2928$
$N_{6,(2^8)} = 90$	$N_{7,(2^8)} = 280128$	$N_{7,(3^2)} = 6508640$
$N_{6,(3)} = 401172$	$N_{7,(2^9)} = 62450$	$N_{7,(4)} = 7492040$
$N_{6,(3,2)} = 87544$	$N_{7,(2^{10})} = 13188$	$N_{7,(4,2)} = 1763415$
$N_{6,(4)} = 3840$	$N_{7,(3)} = 347987200$	$N_{7,(5)} = 21504$

In [D-I], the Gromov-Witten invariants of X_6 are computed. Our computation $N_{6,(2^6)} = 3240$ disagrees with [D-I]. We have checked our number using different recursive strategies.

In the examples where $GW(X)$ is computable, the Gromov-Witten classes

$$GW(\gamma_1 \otimes \cdots \otimes \gamma_n)^X_{g,\beta}$$

always lie in subring of tautological classes

$$RH^*(\overline{\mathcal{M}}_{g,n}) \subset H^*(\overline{\mathcal{M}}_{g,n})$$

generated by (boundary) strata classes decorated by κ, ψ_i

kappa classes

cotangent line classes

The following proposal appears in 2005

Speculation (Levine - P):

For all nonsingular projective varieties X

and all g, n, $\beta \in H_2(X, \mathbb{Z})$, $\gamma_i \in H^*(X)$

2g-2+n>0

$$GW(\gamma_1 \otimes \cdots \otimes \gamma_n)^X_{g,\beta} \in RH^*(\overline{\mathcal{M}}_{g,n})$$

Motivation in Levine - P: degeneration/ algebraic cobordism

But there have been very few examples studied with interesting Hodge theory

Why didnt we propose

$$GW(\gamma_1 \otimes \cdots \otimes \gamma_n)^X_{g,\beta} \in R^*(\overline{\mathcal{M}}_{g,n}) \subset A^*(\overline{\mathcal{M}}_{g,n}) \,?$$

Chow theory

Claim could only make sense for $\gamma_i \in A^*(X)$,

and is known to be false.

The refined statement: $X / \overline{\mathbb{Q}}$, $\gamma_i / \overline{\mathbb{Q}}$

then the GW class is tautological in Chow

First Question (open): prove the moduli point $[C, p_1, \ldots, p_n] \in A_0(\overline{\mathcal{M}}_{g,n})$ is tautological when $(C, p_1, \ldots, p_n) / \overline{\mathbb{Q}}$

See P-Schmitt arXiv: 1905.00769

arXiv:2109.13323

Theorem A (Argüz, Bousseau, P, Zvonkine 2021)

Let $X \subset \mathbb{P}^m$ be a nonsingular complete intersection. Then, there is an algorithm to compute all of the Gromov-Witten classes

$$GW(\gamma_1 \otimes \cdots \otimes \gamma_n)^X_{g,\beta} \in H^*(\overline{M}_{g,n}).$$

Theorem B (Argüz, Bousseau, P, Zvonkine 2021)

All of the Gromov-Witten classes of nonsingular complete intersections $X \subset \mathbb{P}^m$ are tautological

$$GW(\gamma_1 \otimes \cdots \otimes \gamma_n)^X_{g,\beta} \in RH^*(\overline{M}_{g,n}).$$

The most interesting aspect of these results concerns the primitive cohomology of X:

$$H^*(X) = H_{res}(X) \oplus H_{prim}(X)$$

Im $H^*(\mathbb{P}^m) \to H^*(X)$

Annihilated by the restriction of the hyperplane class

By Lefschetz Hyperplane Theorem $\Rightarrow$

$$H_{prim}(X) \subset H^r(X) \qquad r = \dim_{\mathbb{C}}(X)$$

middle (so could be odd or even)

We must control the GW classes with primitive γ_i.

A worry you may have is about the curve class

$$\beta \in H_2(X, \mathbb{Z})$$

- $r=1$ or $r \geq 3$, $H_2(X, \mathbb{Z}) \cong \mathbb{Z}$ and measured by the degree in $\mathbb{P}^m$

Lefschetz

- $r=2$, $H_2(X, \mathbb{Z})$ can be complicated

Example $H_2(X_4, \mathbb{Z}) \cong \mathbb{Z}^{22}$ for the quartic surface $X_4 \subset \mathbb{P}^3$

But for $r=2$, we have the Noether-Lefschetz Theorem:

For $X \subset \mathbb{P}^m$ a generic complete intersection of dim 2, $\mathrm{Pic}(X) \cong \mathbb{Z}$ except in the following cases

generated by $\mathcal{O}_{\mathbb{P}^m}(1)$

- $X_2, X_3 \subset \mathbb{P}^3$
- $X_{2,2} \subset \mathbb{P}^4$

These are Del Pezzo surfaces

Thm A + B are already known

Using the deformation invariance of GW theory

(and excluding X_2, X_3, $X_{2,2}$)

we can index β by the degree in $\mathbb{P}^m$

Deformation invariance implies much more.

Consider a GW class of the form

$$GW(\gamma_1 \otimes \cdots \otimes \gamma_n \otimes H^{a_1} \otimes \cdots \otimes H^{a_m})^X_{g,\beta} \in H^*(\overline{M}_{g,n+m}).$$

$\gamma_i \in H_{prim}(X)$

Restriction of the hyperplane

The GW class is invariant under monodromy on $H_{prim}(X)$ in the family of all nonsingular complete intersections.

What is the monodromy on $V = H_{prim}(X) \otimes_{\mathbb{Q}} \mathbb{C}$?

We have a bilinear form

$$\boxed{\begin{array}{l} V \otimes V \to \mathbb{C} \\ \gamma_1 \otimes \gamma_2 \to \int_X \gamma_1 \cup \gamma_2 \end{array}}$$

Deligne 1980

- If $r = \dim_{\mathbb{C}} X$ is odd, then the Zariski closure of the monodromy on V is as large as possible : $Sp(V)$

preserves the skew symmetric intersection form

- If $r = \dim_{\mathbb{C}} X$ is even, then the Zariski closure of the monodromy on V is as large as possible : $O(V)$ preserves the symmetric form

With the following list of exceptions :

Exceptions occur only in the r even case

- $X_3 \subset \mathbb{P}^3$ cubic surface
- $X_{2,2} \subset \mathbb{P}^m$ even dimensional complete intersection of 2 quadrics

$X_3 \subset \mathbb{P}^3$ is a Del Pezzo, not a problem

$X_{2,2} \subset \mathbb{P}^m$ is much more interesting

In the complete theory of complete intersections

$X_{2,2} \subset \mathbb{P}^m$ plays a special role and requires a special study

GW theory here is a whole world

Xiaowen Hu arXiv:2109.11469

Lectures recorded,
links on my webpage

An afternoon of complete intersections

17 November 2021, ETH Zürich (Zoom)

13:30-14:45 Xiaowen Hu (Sun Yat-Sen University, Guangzhou)

On the big quantum cohomology of Fano complete intersections slides, lecture

I will report on recent progress on the quantum cohomology of Fano complete intersections in projective spaces. I will introduce the square root recursion conjecture, a puzzle in odd dimensions, and some ad hoc methods for cubic hypersurfaces. For even dimensional complete intersections of two quadrics, a kind of exceptional complete intersection, I will explain the special feature of a genus 0 invariant of odd length, and, in 4 dimension, the relationship to enumerative geometry. [arxiv:1501.03683], [arxiv:2109.11469]

15:00-16:15 Pierrick Bousseau (CNRS and University of Paris-Saclay) slides, lecture

Gromov-Witten theory of complete intersections

I will describe an inductive algorithm computing Gromov-Witten invariants in all genera with arbitrary insertions of all smooth complete intersections in projective space. The main idea is to show that invariants with insertions of primitive cohomology classes are controlled by their monodromy and by invariants defined without primitive insertions but with imposed nodes in the domain curve. To compute these nodal Gromov-Witten invariants, we introduce the new notion of nodal relative Gromov-Witten invariants. This is joint work with Hülya Argüz, Rahul Pandharipande, and Dimitri Zvonkine. [arxiv:2109.13323]

There are three main parts of the proof of Theorem A (Theorem B is a corollary)

(I) Induction on dimension of X and the degrees of the defining equations of $X \subset \mathbb{P}^m$ via degeneration

(II) Representation theory of $O(V)$, $Sp(V)$

(III) Nodal relative Gromov-Witten theory

- degeneration
- splitting formula
- descendent/relative correspondence

The three pieces fit perfectly together

(almost perfectly: There is always the issue of $X_{2,2}$)

(I) Induction via degeneration

Complete intersection case similar

- Consider a generic hypersurface $X_k \subset \mathbb{P}^m$ defined by the degree k polynomial f

- Let g and h be generic polys of degrees k_1 and k_2 with $k = k_1 + k_2$ $\quad k_i < k$

- The variety

$$\begin{array}{l} \mathcal{X} = (tf + gh) \subset \mathbb{C}_t \times \mathbb{P}^m \\ \downarrow \\ \mathbb{C}_t \end{array}$$

defines a degeneration of X_k

but $\mathcal{X}$ is singular along the locus (t, f, g, h)

- $\tilde{\mathcal{X}} \to \mathcal{X}$ small resolution by blowing-up Weil divisor (t, h)

We obtain a log smooth family $\begin{array}{c}\tilde{\mathcal{X}}\\ \downarrow\\ \mathbb{C}_t\end{array}$ which is a degeneration

$$X_k \rightsquigarrow X_{k_1} \cup_D \tilde{X}_{k_2}$$

Complete intersection of degree k_1, k_2

blow-up of X_{k_2} along a complete intersection of degree k, k_1, k_2

The point is that all the geometries here are defined by complete intesections of either lower dimension or lower defining degrees

I skip here the mechanisms for decomposing the GW theory of $X_{k_1} \cup_D \tilde{X}_{k_2}$ into the GW theories of the constituent complete intersections:

- Relative / descendent
- blow-up relations

Techniques go back to Maulik-P, Some more recent improvements on blow-ups by Honglu Fan

The difficulty is caused by the vanishing cycles in the primitive cohomology of X_k

The degeneration formula for

$$X_k \rightsquigarrow X_{k_1} \cup_D \tilde{X}_{k_2}$$

can only be applied to cohomology of $\tilde{\mathfrak{X}}$.

(II) Representation theory of $O(V)$, $Sp(V)$

I will explain the orthogonal case, the symplectic case is similar

$X_k \subset \mathbb{P}^m$ is even dimensional, $V = H_{prim}(X) \otimes_{\mathbb{Q}} \mathbb{C}$

We have already seen that

$$GW\left(\gamma_1 \otimes \cdots \otimes \gamma_n \otimes H^{a_1} \otimes \cdots \otimes H^{a_m}\right)^{X_k}_{g,\beta} \in H^*(\overline{M}_{g,n+m}).$$

$\gamma_i \in V$

interpreted as an element of

$$(V^*)^{\otimes n} \otimes H^*(\overline{M}_{g,n+m}).$$

is $O(V)$-invariant.

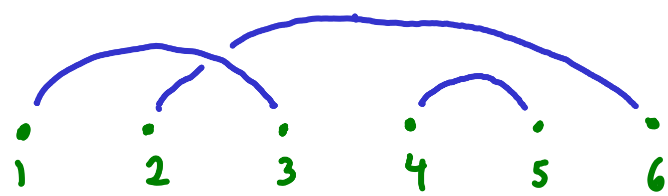
Question: What are the $O(V)$-invs of $(V^*)^{\otimes n}$?

Many Refs, see books by Procesi, Weyl

Classical solution:

- since $-\mathrm{Id} \in O(V)$, there are no invariants unless n is even, $n = 2\ell$

- an ℓ-pairing is a fixed point free involution on the set $\{1, 2, \dots, 2\ell\}$



- $\mathbb{C}\mathcal{P}_\ell$ = free $\mathbb{C}$-vector space on the set of ℓ-pairings.

  $\dim \mathbb{C}\mathcal{P}_\ell = (2\ell - 1)!!$

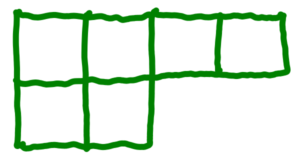
- The symmetric group $S_{2\ell}$ acts naturally on $\mathbb{C}\mathcal{P}_\ell$ by permuting labels

Decomposition as $S_{2\ell}$ representations:

$$\mathbb{C}\mathcal{B}_\ell \cong \bigoplus_{\lambda \vdash 2\ell} M_\lambda$$

irrep of $S_{2\ell}$

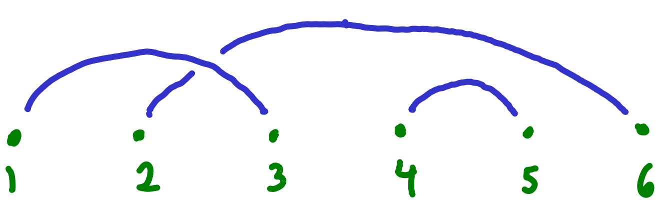
λ has even rows



- There is a canonical map of $S_{2\ell}$ reps

$$\mu : \mathbb{C}\mathcal{B}_\ell \longrightarrow (V^*)^{\otimes 2\ell}$$

defined using the product $\langle \, , \rangle$ on V

$\mu\Big($ 

$\Big)\big(v_1 \otimes v_2 \otimes \cdots \otimes v_6\big)$

$$= \langle v_1, v_3 \rangle \cdot \langle v_2, v_6 \rangle \cdot \langle v_4, v_5 \rangle$$

- Actually μ maps to the $O(V)$-invs

$$\mu : \mathbb{C}\mathcal{P}_\ell \longrightarrow \left[(V^*)^{\otimes 2\ell} \right]^{O(V)}$$

- Main Result of the classical solution:

$$\boxed{\begin{array}{l} \mu \text{ is surjective onto the } O(V)\text{-invs,} \\[1em] \operatorname{Ker}\mu = \displaystyle\bigoplus_{\substack{\lambda \vdash 2\ell \\ \lambda \text{ has even rows} \\ \operatorname{len}(\lambda) \geq \dim V + 1}} M_\lambda \end{array}}$$

The result tells us the minimum data needed to determine all

$$GW\left(\gamma_1 \otimes \cdots \otimes \gamma_n \otimes H^{a_1} \otimes \cdots \otimes H^{a_m}\right)^{X_K}_{g,\beta} \in H^*\left(\overline{M}_{g,n+m}\right).$$

$$\gamma_i \in V$$

but we still need a geometric idea to obtain any info

(III) Nodal relative Gromov-Witten theory

The idea is to use the diagonal:

$$\Delta \in H^*(X_k \times X_k)$$

Why? Two reasons

(1) Künneth decomposition

$$\Delta = \sum_{i,j} g^{ij} \gamma_i \otimes \gamma_j$$

involves all of the cohomology of X_k, including $H^*_{prim}(X_k)$

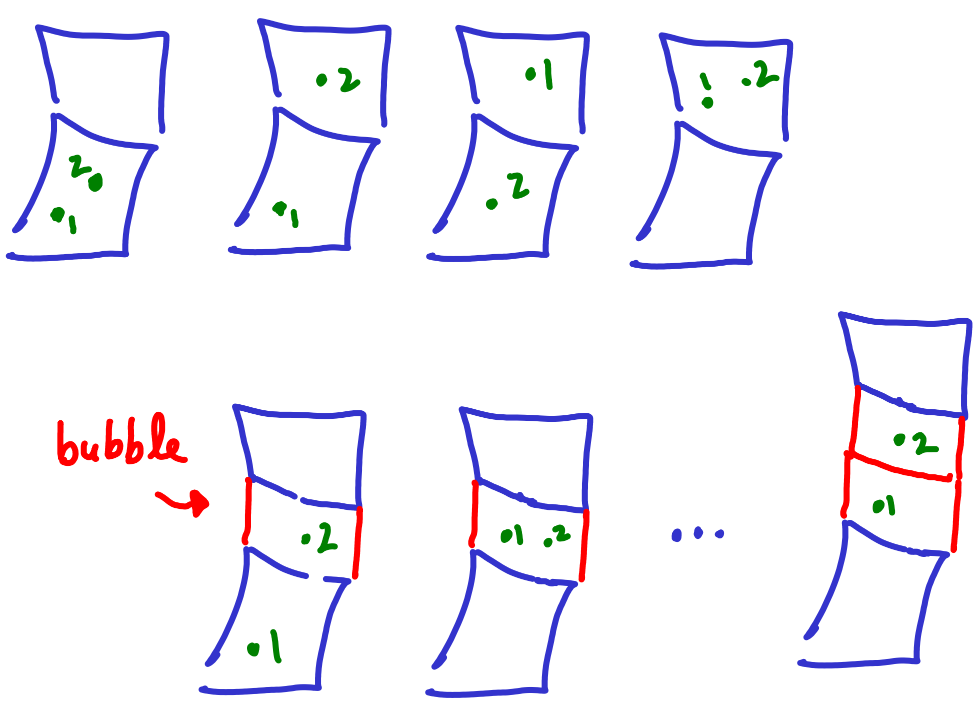
(2) The diagonal class Δ can be extended across the degeneration

Claim (2) can be seen using the log diagonal:

$$\underset{\text{diag}}{\Delta} \subset X_k[2] \longleftarrow \text{configuration space of 2 ordered points}$$

$$\begin{array}{c} \underset{\text{diag}}{\boldsymbol{\Delta}} \subset \tilde{\mathcal{X}}[2] \\ \downarrow \\ \mathbb{C}_t \end{array} \longleftarrow \text{log configuration space of 2 ordered points}$$

Hints by pictures in the special fiber



How can we bring the diagonal idea to GW theory?

A direct geometric path is to consider the GW theory of curves with imposed nodes in the domain. For standard GW theory nodes yield diagonal insertions by the splitting axiom.

Then we need the parallel theory in the relative case with all the moves: degeneration, splitting

See paper or Bousseau's lecture at ETHZ

Now we have all the pieces and it remains only to snap them together.

- What we have: $(n = 2\ell)$

  for every ℓ-pairing,

  we know inductively

$$GW(\Delta_{12} \otimes \Delta_{34} \otimes \Delta_{56} \otimes H^{a_1} \otimes \cdots \otimes H^{a_m})^{\chi_X}_{g,\beta}$$

- What we want: $O(V)$-inv GW class.

$$GW(\gamma_1 \otimes \cdots \otimes \gamma_6 \otimes H^{a_1} \otimes \cdots \otimes H^{a_m})^{\chi_X}_{g,\beta}$$

We obtain immediately now the Loop matrix :

$$\mathcal{M}^{\text{Loop}}(P, \hat{P}) = \mu_P(\Delta_{\hat{P}})$$

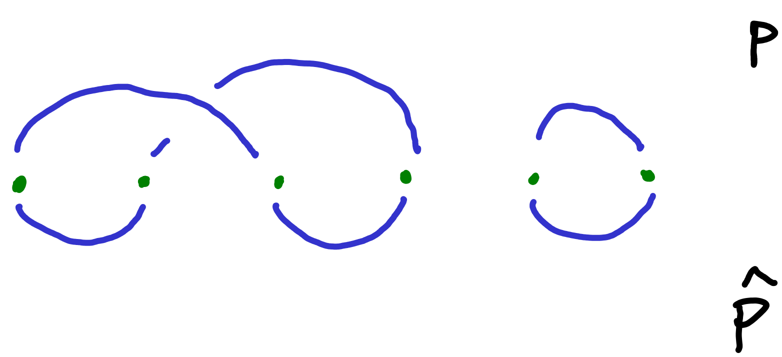
$$= (\dim V)^{\text{Loop}(P, \hat{P})}$$

both ℓ-pairings

$(2\ell-1)!! \times (2\ell-1)!!$ matrix

# of loops when $P, \hat{P}$ put together

Example :



$$\text{Loop}(P, \hat{P}) = 2$$

We view $\mathcal{M}^{\text{Loop}}(P, \hat{P})$ as a linear operator

$$\mathcal{M}^{\text{Loop}}(P, \hat{P}) : \mathbb{C}\mathcal{B}_\ell \longrightarrow \mathbb{C}\mathcal{B}_\ell$$

By study of $O(V)$-invs, it is clear that $\mathcal{M}^{\text{Loop}}(P, \hat{P})$ annihilates

$$\ker \mu = \bigoplus_{\substack{\lambda \vdash 2\ell \\ \lambda \text{ has even rows} \\ \operatorname{len}(\lambda) \geq \dim V + 1}} M_\lambda \subset \mathbb{C}\mathcal{B}_\ell$$

But we need nondegeneracy on the quotient

The winning result for us here is :

Hanlon-Wales
Macdonald
Zinn-Justin
⋮

$$\mathbb{C}P_\ell = \bigoplus_{\substack{\lambda \vdash 2\ell \\ \lambda \text{ has even rows}}} M_\lambda \quad \text{is an}$$

eigenspace decomposition for

$M^{\text{Loop}}(P, \hat{P})$ with eigenvalue

on M_λ given by

$$\prod_{(i,j) \in \frac{\lambda}{2}} \left(\dim V - i + 2j - 1 \right)$$

eigenvalues are all nonzero on M_λ

when $\operatorname{len}(\lambda) \leq \dim V$. $\square$

Further Comments:

(1) Not really the end of the proof since I skipped $\chi_{2,2}$ (see paper)

(2) Method should have several further applications, but there are interesting issues to understand.

The End