Discussion HMPPS 25 Oct 2022

I. Definitions

divisor with normal crossings

We consider log targets (X, D)

nonsingular projective

or perhaps more general log targets,

but the restricted case is rich enough.

$\overline{\mathcal{M}}_{g,n}(X/D, \beta)$ moduli of stable log maps carries a log structure and Artin fan

We have
$$\begin{cases} \log CH^*(\overline{\mathcal{M}}_{g,n}(X/D, \beta)) \\ \log CH_*(\overline{\mathcal{M}}_{g,n}(X/D, \beta)) \end{cases}$$

$\log CH^*(\overline{\mathcal{M}}_{g,n}(X/D,\beta))$ defined via

$\varinjlim$ over log blowups (defined with respect to the Artin fan) of operational Chow under π^*.

$\log CH_*(\overline{\mathcal{M}}_{g,n}(X/D,\beta))$ defined via

$\varprojlim$ over log blowups (defined with respect to the Artin fan) of usual Chow under π_*.

Existence of the virtual fundamental class in push-forward compatible diagram:

$$
\begin{tikzcd}
\log CH_*\left(\overline{\mathcal{M}}_{g,n}(X/D,\beta)\right) & \log CH_*\left(\overline{\mathcal{M}}_{g,n}\right) \\
\mathrm{Vir}^{\log} \arrow[u, phantom, sloped, "\in"] \arrow[r, dash] \arrow[d, dash] & \mathrm{vir}^{\log} \arrow[u, phantom, sloped, "\in"] \arrow[d, dash] \\
\mathrm{Vir} \arrow[r, dash] & \mathrm{vir} \\
CH_*\left(\overline{\mathcal{M}}_{g,n}(X/D,\beta)\right) \arrow[u, phantom, sloped, "\ni"] & CH_*\left(\overline{\mathcal{M}}_{g,n}\right) \arrow[u, phantom, sloped, "\ni"]
\end{tikzcd}
$$

See paper of Ranganathan on expansions.

Full constructions in upcoming survey

Herr - Molcho - P - Wise

Basic DR theory is about $\overline{\mathcal{M}}_{g,A}(\mathbb{P}^1/0+\infty)^{\sim}$

$$
\begin{tikzcd}
\log \mathrm{CH}_*\left(\overline{\mathcal{M}}_{g,A}(\mathbb{P}^1/0+\infty)^{\sim}\right) & \log \mathrm{CH}_*\left(\overline{\mathcal{M}}_{g,n}\right) \\
\check{\mathrm{Vir}}^{\log} \arrow[r, dash] \arrow[d, dash] & \check{\mathrm{vir}}^{\log} = \log\mathrm{DR} \arrow[d, dash] \\
\mathrm{Vir} \arrow[r, dash] & \mathrm{vir} = \mathrm{DR} \\
\mathrm{CH}_*\left(\overline{\mathcal{M}}_{g,A}(\mathbb{P}^1/0+\infty)^{\sim}\right) & \hat{\mathrm{CH}}_*\left(\overline{\mathcal{M}}_{g,n}\right)
\end{tikzcd}
$$

Observation (J): $\check{\mathrm{vir}}^{\log} = \log\mathrm{DR}$ can be constructed without $\check{\mathrm{Vir}}^{\log}$ using the diamond space explained by (D).

II. universal log DR

$\mathcal{P}ic_{g,n,d}$ moduli of $(C, p_1, \dots p_n, L)$

genus g

Line bundle total degree d

Let $A = (a_1, \dots, a_n)$

$\sum a_i = d$

$\mathcal{P}ic^{\theta}_{g,n,d,d}$ moduli of $(C, p_1, \dots p_n, L, M)$

θ: stability condition of degree 0

genus g

Line bundles total degree d

Such that $L \otimes M^*$ is θ stable

Construction of θ using a marking (s) in quasi stable case

$$
\begin{tikzcd}
\mathrm{Pic}^{\theta}_{g,n,d,d} \\
\mathrm{Pic}_{g,n,d} \arrow[u, dashed, bend left, "AJ_A"] \arrow[u, bend right, "\Delta"']
\end{tikzcd}
$$

AJ_A rational map

$a_i, L \mapsto L, \mathcal{O}_C(\Sigma a_i p_i)$

Diagonal section Δ

$L \mapsto L, L$

We resolve AJ_A via explicit log blow up as in HMPPS:

$$
\begin{tikzcd}
 & \mathrm{Pic}^{\theta}_{g,n,d,d} \\
\widetilde{\mathrm{Pic}}^{\theta}_{g,n,d} \arrow[ur, "\widetilde{AJ}_A"] \arrow[dr] & \\
 & \mathrm{Pic}_{g,n,d} \arrow[uu, dashed, bend left, "AJ_A"] \arrow[uu, bend right, "\Delta"']
\end{tikzcd}
$$

Subdivision of Cone using stability condition θ

universal log DR $= \widetilde{AJ}_A^{-1}(\Delta)$

$\in \log CH^g(\mathrm{Pic}_{g,n,d})$

III. The main result: computation of log DR for rubber over X.

- X nonsingular projective
- $L \to X$ line bundle of
- $R/0+\infty = \mathbb{P}(L \oplus \mathcal{O}_X) / 0+\infty$ sections
$$\begin{array}{c} L \oplus \mathcal{O}_X \\ \downarrow \\ X \end{array}$$
- $\beta \in H_2(X, \mathbb{Z})$ with $\int_\beta c_1(L) = d$
- $A = (a_1, \ldots, a_n)$ with $\sum a_i = d$

We are interested in

$$
\begin{tikzcd}
\log CH_*\left(\overline{\mathcal{M}}_{g,A}(R/0+\infty)^{\sim}\right) & \log CH_*\left(\overline{\mathcal{M}}_{g,n}(X,\beta)\right) \\
\mathrm{Vir}_R^{\log} \arrow[u, phantom, sloped, "\in"] \arrow[r, dash] \arrow[d, dash] & \mathrm{vir}_X^{\log} \arrow[u, phantom, sloped, "\in"] \arrow[d, dash] \\
\mathrm{Vir}_R \arrow[r, dash] \arrow[d, phantom, sloped, "\in"] & \mathrm{vir}_X \arrow[d, phantom, sloped, "\in"] \\
CH_*\left(\overline{\mathcal{M}}_{g,A}(R/0+\infty)^{\sim}\right) & CH_*\left(\overline{\mathcal{M}}_{g,n}(X,\beta)\right)
\end{tikzcd}
$$

Since $\overline{\mathcal{M}}_{g,n}(X,\beta)$ carries a universal curve

$$
\begin{tikzcd}
\mathcal{C} \arrow[r, "f"] \arrow[d] & X \\
\overline{\mathcal{M}}_{g,n}(X,\beta)
\end{tikzcd}
$$

and a universal line bundle $f^*(L)$

We obtain a morphism

$$\overline{\mathcal{M}}_{g,n}(X,\beta) \xrightarrow{\varepsilon} \mathrm{Pic}_{g,n,d}$$

Theorem:

$$\boxed{\begin{gathered} \varepsilon^*\big(\text{universal log DR}\big) \cap \left[\overline{\mathcal{M}}_{g,n}(X,\beta)\right]^{vir} \\ \parallel \\ \mathrm{vir}_X^{\log} \\ \text{as elements of } \log CH_*\big(\overline{\mathcal{M}}_{g,n}(X,\beta)\big) \end{gathered}}$$

The statement of the Theorem is well defined by the following comments:

(i) $\overline{\mathcal{M}}_{g,n}(X,\beta) \xrightarrow{\varepsilon} \mathrm{Pic}_{g,n,d}$

have compatible log structures pulled back from the Artin stack of nodal curves.

(ii) $\varepsilon^*(\text{universal log DR})$ is naturally defined in $\log CH^*(\overline{\mathcal{M}}_{g,n}(X,\beta))$

(iii) $[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{vir}$ naturally determines an element of $\log CH_*(\overline{\mathcal{M}}_{g,n}(X,\beta))$ by gysin pullback with respect log blowups of the Artin fan.

IV. Application for target (X, D)

nonsingular projective

nonsingular divisor

We have.

$$\begin{array}{ccc}
\log CH_*(\overline{\mathcal{M}}_{g,n}(X/D,\beta)) & & \log CH_*(\overline{\mathcal{M}}_{g,n}) \\
\rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\
\mathrm{Vir}^{\log} & \text{———} & \mathrm{vir}^{\log}
\end{array}$$

Theorem: If GW theories of both X and D are tautological in $CH(\overline{\mathcal{M}}_{g,n})$, then

assumption about non-log theories

$$\boxed{\mathrm{vir}^{\log} \in \log R_+(\overline{\mathcal{M}}_{g,n})}$$

$+$ here means including ψ, kappa, etc