$\mathcal{A}_g$, $\mathcal{M}_g^{ct}$, and $\mathrm{Hilb}(\mathbb{C}^2, d)$

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Extended notes updated

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joint with

S. Canning

S. Molcho

D. Oprea

A. Pixton

H.H. Tseng

A. Iribar López

including results of

F. Greer – C. Lian

Photo: Steve Arrington (unsplash)

Compactifications

Deligne-Mumford

$$
\begin{tikzcd}
\mathcal{A}_g^{PC} & & \overline{\mathcal{M}}_g \arrow[ll, bend right, "\text{Extended Torelli}"'] \\
\mathcal{A}_g \arrow[u, phantom, "\cup"] & & \mathcal{M}_g \arrow[ll, bend right, "\text{Torelli}"'] \arrow[u, phantom, "\subset"] \\
Sp(2g,\mathbb{Z}) \arrow[u, Rightarrow] & & Map(g) \arrow[ll, "hom"'] \arrow[u, Rightarrow] \\
& K3(g) \arrow[d, Rightarrow] & \\
& \mathcal{M}_g^{K3} \arrow[uuur, bend right, "\text{Correspondence with linear Sections as Studied by Mukai}"'] &
\end{tikzcd}
$$

moduli of PPAV

moduli spaces as classifying spaces

moduli of Smooth Curves

Groups

moduli of quasipolarized K3 Surfaces

$\cap$

various compactifications,

Perhaps no winner yet,

but Satake is convenient

I. Moduli of abelian varieties

$Sp(2g, \mathbb{Z}) \curvearrowright \mathcal{H}_g$

Siegel upper half space (contractible)

$$\mathcal{A}_g = \mathcal{H}_g / Sp(2g, \mathbb{Z})$$

model for $B\,Sp(2g, \mathbb{Z})$ up to finite stabilizers

we have : $H^*(\mathcal{A}_g) = H^*_{Sp(2g,\mathbb{Z})}$,

All cohomology taken with $\mathbb{Q}$-coefficients .

$\mathbb{E}$ is defined by

$$\begin{tikzcd} \mathcal{X} \arrow[d, "\pi"] \\ A_g \arrow[u, bend left, "s"] \end{tikzcd}$$

zero section

universal Abelian variety of dim g

Then $\mathbb{E} = s^*(\Omega^1_\pi)$

rank g

Lambda classes: $\lambda_i = c_i(\mathbb{E})$,

$$\mathrm{ch}\, R\pi_* \mathcal{O}_{\mathcal{X}} = \mathrm{ch}\left(1 - \mathbb{E}^\vee + \Lambda^2 \mathbb{E}^\vee \cdots\right).$$

Borel 1974: Stable cohomology of $Sp(\mathbb{Z})$ generated by λ_i.

Following van der Geer, define tautological classes:

- $RH^*(A_g) \subset H^*(A_g)$ cohomology

  Subalgebra generated by all $\lambda_i = c_i(\mathbb{E})$,

- $R^*(A_g) \subset CH^*(A_g)$ algebraic cycles

  Subalgebra generated by all $\lambda_i = c_i(\mathbb{E})$.

<u>Theorem</u> (van der Geer 1996)

$RH^*(A_g) = R^*(A_g)$ with presentation

$$\frac{\mathbb{Q}[\lambda_1, \ldots, \lambda_g]}{(\lambda_g = 0,\ c(\mathbb{E} \oplus \mathbb{E}^\vee) = 1)}$$

$$(1+\lambda_1+\lambda_2+\cdots+\lambda_g)\cdot(1-\lambda_1+\lambda_2-\cdots(-1)^g\lambda_g) = 1 .$$

As a consequence, $R^*(A_g)$ is a Gorenstein ring with socle

$$R^{\binom{g}{2}}(A_g) \cong \mathbb{Q}\cdot\lambda_1^{\binom{g}{2}} \cong \mathbb{Q}\cdot\lambda_1\lambda_2\cdots\lambda_{g-1}.$$

additional argument by van der Geer provided for nonvanishing

Many open questions:

- Calculate $H^*_{Sp(2g,\mathbb{Z})}$ in unstable ranges
- Calculate $CH^*(A_g)$
- Calculate $H^*_{Sp(2g,\mathbb{Z})}$ with $\mathbb{Z}$-coefficients

all very difficult. We will go in a different direction.

II. Projection

We have $\mathcal{R}^*(A_g) \subset CH^*(A_g)$

and $\mathcal{R}^*(\mathcal{M}_g^{ct}) \subset CH^*(\mathcal{M}_g^{ct})$.

Is there a canonical projection

$$CH^*(\mathcal{M}_g^{ct}) \xrightarrow{pr_{\mathcal{M}^{ct}}} \mathcal{R}^*(\mathcal{M}_g^{ct}) \quad ?$$

There is no proposal for ,

but for $CH^*(A_g) \xrightarrow{pr_A} \mathcal{R}^*(A_g)$,

we believe pr_A exists!

The idea uses compactification:

$$\mathcal{A}_g \subset \mathcal{A}_g^{pc}$$ perfect cone compactification

Some facts:

(i) Hodge bundle extends canonically

$$\begin{array}{ccc} \mathbb{E} & \subset & \mathbb{E} \\ \downarrow & & \downarrow \\ \mathcal{A}_g & \subset & \mathcal{A}_g^{pc} \end{array}$$

(ii) $$R^*(\mathcal{A}_g^{pc}) \overset{\text{def}}{\subset} CH^*(\mathcal{A}_g^{pc})$$

Subalgebra generated by all

$$\lambda_i = c_i(\mathbb{E}).$$

<u>Theorem</u> (van der Geer 1996)

- $$R^*(A_g^{pc}) = \frac{\mathbb{Q}[\lambda_1, \dots, \lambda_g]}{\left(c(\mathbb{E} \oplus \mathbb{E}^\vee) = 1 \right)}$$

$$(1+\lambda_1+\lambda_2+\dots+\lambda_g)\cdot(1-\lambda_1+\lambda_2-\dots(-1)^g\lambda_g) = 1 .$$

- $R^*(A_g^{pc})$ is a Gorenstein ring

with socle

$$R^{\binom{g+1}{2}}(A_g^{pc}) \cong \mathbb{Q}\cdot\lambda_1^{\binom{g+1}{2}} \cong \mathbb{Q}\cdot\lambda_1\lambda_2\cdots\lambda_{g-1}\lambda_g .$$

Using integration on A_g^{pc}

and the duality of $R^*(A_g^{pc})$,

Vd Geer, Faber, Grushevsky-Hulek

We define a projection :

$$CH^*(A_g^{pc}) \xrightarrow{pr_{A^{pc}}} R^*(A_g^{pc})$$

$$pr_{A^{pc}}\left(\alpha \in CH^*(A_g^{pc})\right) = \beta \in R^*(A_g^{pc})$$

where $\forall \gamma \in R^*(A_g^{pc})$,

$$\int_{A_g^{pc}} \alpha \cdot \gamma = \int_{A_g^{pc}} \beta \cdot \gamma \,,$$

β exists and is unique.

$$\boxed{pr_{A^{pc}} \text{ is a projection}}$$

Idea for constructing

$$CH^*(\mathcal{A}_g) \xrightarrow{pr_{\mathcal{A}}} R^*(\mathcal{A}_g)$$

via $pr_{\mathcal{A}^{pc}}$:

$$pr_{\mathcal{A}}\left(\alpha \in CH^*(\mathcal{A}_g)\right) = pr_{\mathcal{A}^{pc}}\left(\bar{\alpha} \in CH^*(\mathcal{A}_g^{pc})\right)\Big|_{\mathcal{A}_g}$$

but closure
not canonical!

Not clear that $pr_{\mathcal{A}}$ is well defined.

Conjecture (Canning - Oprea - P 2023):

$$\lambda_g\Big|_{\mathcal{A}_g^{pc} \setminus \mathcal{A}_g} = 0 \in CH^*(\mathcal{A}_g^{pc} \setminus \mathcal{A}_g)$$

v d Geer
$\Downarrow$
Conjecture
true in
char p.

Conjecture $\Rightarrow$ $pr_{\mathcal{A}}$ is well defined.

Using the Conjecture, there is another path to the projection:

for $\alpha \in CH^*(A_g)$ and $\gamma \in R^*(A_g)$, define a pairing

Non canonical closure

$$\langle \alpha , \gamma \rangle_{A_g} = \int_{A_g^{PC}} \bar{\alpha} \cdot \gamma \cdot \lambda_g$$

lifting of λ classes

Conjecture $\Rightarrow$ $\langle \alpha , \gamma \rangle_{A_g}$ is well defined.

$$\boxed{\begin{array}{l} \text{Exercise: } \langle \alpha , \gamma \rangle_{A_g} = \langle Pr_A(\alpha), \gamma \rangle_{A_g} \\ \qquad \forall\, \alpha \in CH^*(A_g) \text{ and } \forall\, \gamma \in R^*(A_g) \end{array}}$$

Update (19 November 2023):

The vanishing Conjecture

$$\lambda_g \Big|_{A_g^{pc} \setminus A_g} = 0 \in CH^*(A_g^{pc} \setminus A_g)$$

is true!

- Argument by Sam Molcho constructing trivial quotients of $\mathbb{E}$ on the boundary via residue maps.
- Another path suggested by Ben Moonen using boundry geometry and rigidity results from Faltings - Chai.

Proofs work for all sufficiently fine torodial Compactifications:

Pr_A exists and is independent of choice of Compactification.

https://people.math.ethz.ch/~rahul/tautprojection.pdf

arXiv: 2401.15768

III. Noether-Lefschetz loci

The simplest NL locus to consider is

$$\mathcal{A}_1 \times \mathcal{A}_{g-1} \to \mathcal{A}_g .$$

We assume $g \geq 2$

We define a twisted generalization by the following construction:

Let $NL_d \to \mathcal{A}_g$ be the moduli of pairs:

Here $d \geq 1$.

Condition: $E \cdot \Theta = d$

theta divisor of X

$$NL_d \ni [E \hookrightarrow X]$$

1 dim subgroup, an elliptic curve

PPAV of dimension g

In case $d=1$, $\mathcal{NL}_1 = \mathcal{A}_1 \times \mathcal{A}_{g-1}$.

Easy to see:

$$\dim \mathcal{NL}_d = \dim \mathcal{A}_g - (g-1)$$

so $[\mathcal{NL}_d] \in CH^{g-1}(\mathcal{A}_g)$.

The main topic of the lecture is the computation

$$Pr_A([\mathcal{NL}_d]) \in R^*(\mathcal{A}_g).$$

There are 2 immediate issues:

- Pr_A depends on the vanishing Conjecture.
- Even if we assume the Conjecture, it is not clear how to integrate the classes of the closures in $\mathcal{A}_g^{pc}$.

Theorem (Canning–Oprea–P 2023):

If the vanishing conjecture holds,

$$\mathrm{Pr}_{A}\left([A_1 \times A_{g-1}] \right) = \frac{g}{6|B_{2g}|} \lambda_{g-1} .$$

An interesting direction:

What is the projection of

the Schottky locus,

$$\mathrm{Pr}_{A}\left(\mathrm{Tor}_*[\,\mathcal{M}_g^{ct}\,] \right) \in R^*(A_g) \;?$$

IV. A different question

We consider now a different question:

$$\boxed{\begin{array}{l}\text{If } [NL_d] \in R^*(A_g), \\ \text{what could it be ?}\end{array}}$$

- Since P_{r_A} is well-defined, the answer to the question is:

$$[NL_d] \in R^*(A_g)$$

$$\Downarrow$$

$$[NL_d] = P_{r_A}([NL_d]).$$

$$\boxed{\begin{array}{l}\text{Proposition (Canning-Oprea-P 2023):}\\ \text{If } [NL_d] \in R^*(A_g), \text{ then}\\ \qquad [NL_d] = \underset{\text{scalar in } \mathbb{Q}}{c_{g,d}} \cdot \lambda_{g-1}.\end{array}}$$

Proof: We have $[NL_d] \in R^{g-1}(A_g)$.

Moreover $\lambda_{g-1} \cdot [NL_d] = 0$,

Since

$$\begin{array}{ccc} NL_d & \ni & [E \hookrightarrow X] \\ \downarrow & & \downarrow \\ A_1 \times A_{g-1}^{Pol} & \ni & [E] \times [X/E] \end{array}$$

non principal polarization

and $\lambda_{g-1}\Big|_{NL_d}$ is pulled-back

from $\lambda_{g-1}\Big|_{A_1 \times A_{g-1}^{Pol}}$,

and $\lambda_{g-1}\Big|_{A_1 \times A_{g-1}^{Pol}} = 0$

because $c(\mathbb{E}_1)\Big|_{A_1} = 0$

and $\lambda_{g-1}\Big|_{A_{g-1}^{Pol}} = 0$.

Finally, $\mathbb{Q}\cdot\lambda_{g-1} \subset R^{g-1}(A_g)$

is the annihilator of λ_{g-1} in $R^{g-1}(A_g)$. $\square$

V. Integration

We have seen

$$[NL_d] \in R^*(\mathcal{A}_g) \Rightarrow [NL_d] = c_{g,d} \cdot \lambda_{g-1} .$$

The question is now what is the scalar $c_{g,d}$?

The idea is to pull-back via Torelli :

$$\mathrm{Tor} : \mathcal{M}_g^{ct} \longrightarrow \mathcal{A}_g ,$$

$$\mathrm{Tor}^*([NL_d]) \in R^{g-1}(\mathcal{M}_g^{ct}) ,$$

$$\mathrm{Tor}^*(\lambda_{g-1}) \in R^{g-1}(\mathcal{M}_g^{ct}) .$$

Then we can calculate $c_{g,d}$ by the λ_g-evaluation on M_g^{ct}:

any extension

$$\frac{\displaystyle\int_{\overline{M}_g} \overline{\mathrm{Tor}^*([NL_d])} \cdot \lambda_{g-2}\lambda_g}{\displaystyle\int_{\overline{M}_g} \lambda_{g-2}\lambda_{g-1}\lambda_g} = c_{g,d}$$

Computed by Faber-P (1999)

$$\int_{\overline{M}_g} \lambda_{g-2}\lambda_{g-1}\lambda_g = \frac{1}{2(2g-2)!}\frac{|B_{2g}|}{2g}\frac{|B_{2g-2}|}{2g-2}$$

But how are we going to calculate

We assume
$g \geq 2$
$d \geq 1$.

$$\int_{\overline{M}_g} \overline{\mathrm{Tor}^*([NL_d])} \cdot \lambda_{g-2}\lambda_g \ ?$$

This requires a miracle provided by stable maps and the quantum cohomology of $\mathrm{Hilb}(\mathbb{C}^2, d)$.

VI Stable maps

Consider the fiber product :

$$\begin{tikzcd}
\mathrm{Tor}^{-1}(NL_d) \arrow[r] \arrow[d] & NL_d \arrow[d] \\
\mathcal{M}_g^{ct} \arrow[r, "\mathrm{Tor}"] & \mathcal{A}_g
\end{tikzcd}$$

We will add a marked point :

$$\begin{tikzcd}
\mathrm{Tor}_1^{-1}(NL_d) \arrow[r] \arrow[d] & NL_d \arrow[d] \\
\mathcal{M}_{g,1}^{ct} \arrow[r, "\mathrm{Tor}_1"] & \mathcal{A}_g
\end{tikzcd}$$

dilaton

$$(2g-2)\cdot \int_{\overline{\mathcal{M}}_g} \overline{\mathrm{Tor}^*([NL_d])} \cdot \lambda_{g-2}\lambda_g$$

$$\|$$

cotangent line

$$\int_{\overline{\mathcal{M}}_{g,1}} \overline{\mathrm{Tor}_1^*([NL_d])} \cdot \psi_1 \cdot \lambda_{g-2}\lambda_g$$

Since on $\mathcal{M}_{g,1}^{ct}$, we have

$$\mathrm{Tor}_1^*([NL_d]) = \alpha^*\left(\mathrm{Tor}^*([NL_d])\right)$$

where $\alpha : \mathcal{M}_{g,1}^{ct} \xrightarrow{\text{forgetfull}} \mathcal{M}_g^{ct}$.

Let $\mathcal{M}_{1,1}^{ct} = \mathcal{M}_{1,1}$ be the moduli of pointed nonsingular elliptic curves:

$$\begin{tikzcd} \mathcal{E} \arrow[d, "\pi^{ct}"'] \\ \mathcal{M}_{1,1} \arrow[u, bend right, "q"'] \end{tikzcd} \qquad \text{zero section}$$

Let

$$\begin{tikzcd} & \mathcal{M}_{g,1}^{ct}(\pi^{ct}, d) \arrow[dl, "\varepsilon^{ct}"'] \arrow[d, "\mu^{ct}"] \arrow[r, "ev"] & \mathcal{E} \arrow[dl, "\pi^{ct}"'] \\ \mathcal{M}_{g,1}^{ct} & \mathcal{M}_{1,1} & \end{tikzcd}$$

be the Grothendieck π-relative space of stable maps to the fibers of π.

Connected domains

$\mathcal{M}^{ct}_{g,1}(\pi^{ct}, d)$ has a virtual class

of dimension

marked point

$$\text{virdim} = \underset{\dim \mathcal{M}_{1,1}}{1} + 1 + \underset{\substack{\text{unpointed maps} \\ \text{to a fixed} \\ \text{elliptic fiber}}}{2g-2} = 2g .$$

$\mathrm{Tor}_1^*([NL_d])$ is a intersection class

on $\mathrm{Tor}_1^{-1}(NL_d)$ of dimension

$$\text{virdim} = \underset{\dim \mathcal{M}^{ct}_{g,1}}{3g-3+1} - \underset{\text{codim of } NL_d}{(g-1)} = 2g-1 .$$

compact type

$$[f : (C, p) \to (E, q)] \in \mathcal{M}^{ct}_{g,1}(\pi^{ct}, d)$$

there is a discrete invariant :

$$f^* : \mathrm{Jac}_0(E) \to \mathrm{Jac}_0(C),$$

$$\mathrm{Jdeg}\, f = d \Big/ |\ker f^*| .$$

$\mathrm{Jdeg}_f \in \{1, 2, \dots, d\}$ must divide d

and is a discrete invariant of f,

$$\mathcal{M}^{ct}_{g,1}(\pi^{ct}, d) = \coprod_{\mathrm{Jdeg}\ \hat{d}} \mathcal{M}^{ct}_{g,1}(\pi^{ct}, d)^{\hat{d}} .$$

maps with $\mathrm{Jdeg}_f = \hat{d}$

Another way to think about Jdeg_f :

a degree d stable map

$$f : (C, p) \longrightarrow (E, q)$$

Compact type

factors uniquely as

$$(C, p) \xrightarrow{g} (\hat{E}, \hat{q}) \xrightarrow{h} (E, q)$$

where $h : \hat{E} \longrightarrow E$ is group homomorphism of elliptic curves

and $g^* : \mathrm{Jac}_0(\hat{E}) \longrightarrow \mathrm{Jac}_0(C)$

is injective.

Then $|\ker f^*| = \deg(h)$

and $d = \deg(g) \cdot \deg(h)$,

so we have

$$\mathrm{Jdeg}\, f = d \Big/ |\ker f^*| = \deg(g).$$

The disjoint decomposition

$$\mathcal{M}^{ct}_{g,1}(\pi^{ct}, d) = \coprod_{\mathrm{Jdeg}\, \hat{d}} \mathcal{M}^{ct}_{g,1}(\pi^{ct}, d)^{\hat{d}}$$

has principal part $\mathcal{M}^{ct}_{g,1}(\pi^{ct}, d)^{d}$.

The lower parts $\mathcal{M}^{ct}_{g,1}(\pi^{ct}, d)^{\hat{d}<d}$

can be studied via $\mathcal{M}^{ct}_{g,1}(\pi^{ct}, \hat{d})$.

Theorem (Canning-Oprea-P, Pixton 2023)

There is an isomorphism of DM stacks

$$\begin{array}{c} ev^{-1}(q)^d \cong \mathrm{Tor}_1^{-1}(NL_d) \; . \\ \cap \\ \mathcal{M}^{ct}_{g,1}(\pi^{ct}, d)^d \end{array}$$

Here $ev^{-1}(q)^d$ is the locus of map where the evaluation of the marking on the domain equals the zero point q of the elliptic target.

To be useful we must also match the virtual classes:

$$\mathrm{vir\,dim}\ ev^{-1}(q)^d = \mathrm{virdim}\ \mathcal{M}_{g,1}^{ct}(\pi^{ct}, d) - 1$$

$$= 2g - 1,$$

$$\left[ev^{-1}(q)^d\right]^{ct,vir} = ev^*(q) \cap \left[\mathcal{M}_{g,1}^{ct}(\pi^{ct}, d)\right]^{vir}.$$

Conjecture (Canning-Oprea-P, Pixton 2023)

under the above isomorphism,

$$\left[ev^{-1}(q)^d\right]^{ct,vir} = \mathrm{Tor}_1^*\left([NL_d]\right).$$

Update Feb 2024:

Francois Greer and Carl Lian can prove

$$\left[ev^{-1}(q)^d \right]^{ct, vir} = Tor_1^*([NL_d])$$

exactly in the required form using a matching of obstruction theories.

Update April 2024:

The Greer-Lian proof can be found here:

https://people.math.ethz.ch/~rahul/greer-lian.pdf

arXiv:2404.10826

An important property of

$$\left[ev^{-1}(q)^{d}\right]^{ct,vir} \in A_{2g-1}\left(\mathcal{M}^{ct}_{g,1}(\pi^{ct},d)^{d}\right)$$

is the existance of a canonical extension to $\overline{\mathcal{M}}_{g,1}(\pi^{ct},d)$:

$$\left[ev^{-1}(q)\right]^{vir} \in A_{2g-1}\left(\overline{\mathcal{M}}_{g,1}(\pi,d)\right)$$

where

$$\begin{tikzcd}
 & \overline{\mathcal{M}}_{g,1}(\pi,d) \arrow[dl, "\varepsilon"'] \arrow[d, "\mu"] \arrow[r, "ev"] & \mathcal{E} \arrow[dl, "\pi"] \\
\overline{\mathcal{M}}_{g,1} & \overline{\mathcal{M}}_{1,1} \arrow[ur, bend right, "q"'] &
\end{tikzcd}$$

over $[\gamma] \in \overline{\mathcal{M}}_{1,1}$ we have log stable maps

The complement

$$\overline{\mathcal{M}}_{g,1}(\pi, d) \setminus \mathcal{M}^{ct}_{g,1}(\pi^{ct}, d)$$

is mapped by ε to the complement

$$\overline{\mathcal{M}}_{g,1} \setminus \mathcal{M}^{ct}_{g,1}.$$

Over $\mathcal{M}_{1,1}$, this is by definition.

Over the point $[\gamma] \in \overline{\mathcal{M}}_{1,1}$,

the claim is more interesting:

there are no curves with compact type domains which map to γ with degree $d \geq 1$ by the definition of log maps.

We conclude:

$$\int_{\overline{\mathcal{M}}_{g,1}} \overline{\mathrm{Tor}_1^*\left([NL_d] \right)} \cdot \psi_1 \cdot \lambda_{g-2}\lambda_g$$

$$\|$$

$$\int_{\overline{\mathcal{M}}_{g,1}} \varepsilon_* \overline{\left[ev^{-1}(q)^d \right]^{vir}} \cdot \psi_1 \cdot \lambda_{g-2}\lambda_g$$

$$\|$$

$$\int_{\overline{\mathcal{M}}_{g,1}} \varepsilon_* \overline{\left[ev^{-1}(q)^d \cdot \psi_1 \right]^{vir}} \cdot \lambda_{g-2}\lambda_g$$

cotangent line now on $\overline{\mathcal{M}}_{g,1}(\pi, d)^d$,

no correction terms since there are no maps of positive degree $\mathbb{P}^1 \to E$.

We now use the extension:

$$\int_{\overline{\mathcal{M}}_{g,1}} \varepsilon_* \left[ev^{-1}(q) \cdot \psi_1 \right]^{vir} \cdot \lambda_{g-2} \lambda_g$$

$$\|$$

$$\sum_{\hat{d} \mid d} \sigma\left(\frac{d}{\hat{d}}\right) \cdot \int_{\overline{\mathcal{M}}_{g,1}} \varepsilon_* \overline{\left[ev^{-1}(q)^{\hat{d}} \cdot \psi_1 \right]}^{vir} \cdot \lambda_{g-2} \lambda_g$$

Count of $(\hat{E}, \hat{q}) \xrightarrow{\deg\left(\frac{d}{\hat{d}}\right)} (E, q)$,

$$\sigma(x) = \sum_{\ell \mid x} \ell$$

Hence the integrals

$$\int_{\overline{\mathcal{M}}_{g,1}} \varepsilon_\star \overline{\left[ev^{-1}(q)^d \cdot \psi_1 \right]}^{vir} \cdot \lambda_{g-2}\lambda_g$$

and the integrals

$$\int_{\overline{\mathcal{M}}_{g,1}} \varepsilon_\star \left[ev^{-1}(q) \cdot \psi_1 \right]^{vir} \cdot \lambda_{g-2}\lambda_g$$

are related inductively by a simple invertible transformation.

VII Gromov-Witten / Hurwitz

We will now calculate

$$\int_{\overline{M}_{g,1}} \varepsilon_* \left[ev^{-1}(q) \cdot \psi_1 \right]^{vir} \cdot \lambda_{g-2} \lambda_g$$

$$\|$$

$$\int_{\left[\overline{M}_{g,1}(\pi, d)\right]^{vir}} \tau_1(q) \, \lambda_{g-2} \lambda_g$$

using the idea of the

GW/H correspondence Okounkov-P (2006)

A new issue is the families geometry.

We assume $g \geq 2$, $d \geq 1$.

$$\int_{[\overline{M}_{g,1}(\pi, d)]^{vir}} \tau_1(q)\, \lambda_{g-2} \lambda_g$$

$$\|$$

$$\left\langle \tau_1(q)\, \lambda_{g-2} \lambda_g \right\rangle^{\pi\, \circ}_{g,d}$$

Connected GW theory

GW/H correspondence equation is found by degeneration of every fiber of

$$\begin{tikzcd} \mathcal{E} \arrow[d, "\pi"'] \\ \overline{M}_{1,1} \arrow[u, bend right, "q"'] \end{tikzcd}$$

to the normal cone of q.

The resulting equation is

$$\left\langle \tau_1(q)\, \lambda_{g-2}\lambda_g \right\rangle^{\pi \circ}_{g,d}$$

$$\|$$

$$\frac{1}{24}\,\sigma(d)\cdot(2g-2)\cdot \int_{\overline{M}_{g-1,1}} \lambda_{g-2}\lambda_{g-1}\,\frac{c(\mathbb{E}^\vee)}{1-\psi}$$

$$+ \left\langle \lambda_{g-2}\lambda_g \;\middle|\; (2) \right\rangle^{\pi \circ}_{g,d} .$$

relative condition

integral evaluated to equal

$$\frac{|B_{2g-2}|}{(2g-2)\,(2g-2)!}$$

Faber-P (1999)

VIII Quantum Cohomology of $\mathrm{Hilb}(\mathbb{C}^2, d)$

$$\frac{(t_1+t_2)^2}{t_1 t_2} \sum_{2\le e\le d} \left\langle \lambda_{g-2}\lambda_g \,\middle|\, (2) \right\rangle^{\pi\,\circ}_{g,e} \cdot \mathrm{Part}(d-e)$$

$$-\frac{1}{24}\,\frac{(t_1+t_2)}{t_1 t_2} \sum_{2\le e\le d} \left\langle (2) \right\rangle^{E\times\mathbb{C}^2\,\circ}_{g,e} \cdot \widetilde{\mathrm{Part}}(d-e)$$

invertible relation

$$\parallel$$

$$\left\langle (2) \right\rangle^{\pi\times\mathbb{C}^2\,\bullet}_{g,d}$$

Possibly disconnected

(no degree 0 connected components)

t_1, t_2 weights on $\mathbb{C}^2$

The above relation is the

Connected / disconnected equation

(together with basic Hodge identities).

There are several terms to explain:

- $\mathrm{Part}(\ell)$ = # of partions of ℓ

$$\mathrm{Part}(0) = 1$$

$$\mathrm{Part}(1) = 1$$

$$\mathrm{Part}(2) = 2$$

A well-known property is

$$\mathrm{Hur}_E^{\ell} = \mathrm{Part}(\ell) \qquad \text{for } \ell \geq 1$$

Aut-weighted count of possibly disconnected unramified covers of $E = \odot$ of degree ℓ

- $\widetilde{\mathrm{Part}}(\ell) \overset{\text{def}}{=} \widetilde{\mathrm{Hur}}_E^{\ell}$ for $\ell \geq 1$

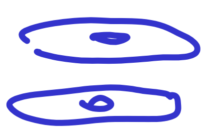
Aut-weighted count of possibly disconnected unramified covers of $E = \odot$ of degree ℓ where each cover is weighted also by the number of connected components.

$\widetilde{\mathrm{Part}}(0) \overset{\text{def}}{=} 0$

$\widetilde{\mathrm{Part}}(1) = 1$

$\widetilde{\mathrm{Part}}(2) = 1 + \frac{3}{2} = \frac{5}{2}$

disconnected cover



$\frac{1}{2} \cdot 1 \cdot 2$

Aut Count Connected components

Connected covers

$\frac{1}{2} \cdot 3 \cdot 1$

- Let $P(x) = \sum_{\ell=0}^{\infty} x^{\ell}\, \mathrm{Part}(\ell)$,

$$\tilde{P}(x) = \sum_{\ell=0}^{\infty} x^{\ell}\, \widetilde{\mathrm{Part}}(\ell).$$

$$F(x,y) = \sum_{\ell=0}^{\infty} \sum_{\kappa=0}^{\infty} x^{\ell} y^{\kappa}\, \mathrm{Hur}_{E}^{\ell,\kappa}$$

Aut-weighted count of possibly disconnected unramified covers of $E = \odot$ of degree ℓ with κ connected components

$$F(x,y) = \exp\left(y \log P(x) \right)$$

$$\tilde{P}(x) = \frac{d}{dy} F(x,y) \Big|_{y=1}$$

$$= P(x) \cdot \log P(x)$$

$$= x + \frac{5}{2} x^2 + \frac{29}{6} x^3 + \frac{109}{12} x^4 + \frac{907}{60} x^5 + \cdots$$

- $\left\langle (2) \right\rangle_{g,d}^{E\times\mathbb{C}^2\,\circ}$ denotes the connected GW theory to a fixed target $E\times\mathbb{C}^2$

The connected / disconnected calculus yields:

$$\left\langle (2) \right\rangle_{g,d}^{E\times\mathbb{C}^2\,\bullet} = \sum_{2\leq e\leq d} \left\langle (2) \right\rangle_{g,e}^{E\times\mathbb{C}^2\,\circ} \cdot \operatorname{Part}(d-e)$$

So we can easily compute $\left\langle (2) \right\rangle_{g,d}^{E\times\mathbb{C}^2\,\circ}$ from $\left\langle (2) \right\rangle_{g,d}^{E\times\mathbb{C}^2\,\bullet}$.

• By the GW/Hilb correspondence (fixed E)

Okounkov-P (2005)
Bryan-P

$$-\sum_{g\in\mathbb{Z}} u^{2g-3} \left\langle (2) \right\rangle_{g,d}^{E\times\mathbb{C}^2\bullet}$$

$$\parallel$$

$$(-i)\cdot \mathrm{Trace}\left(\mathcal{M}_{D,d}\right)$$

after $-q = e^{iu}$.

$$\parallel$$

$$(-i)\cdot \mathrm{Tr}_d \cdot (t_1+t_2)$$

Let $D = c_1(\mathcal{O}/I) \in H^2(\mathrm{Hilb}(\mathbb{C}^2, k))$

Let $M_{D,k}$ be the operator of quantum multiplication

$D = -(2)$

$$M_{D,k} = D\,\star : H^*(\mathrm{Hilb}(\mathbb{C}^2,k)) \to H^*(\mathrm{Hilb}(\mathbb{C}^2,k)).$$

computed explicitly by Okounkov - P (2010)

Let $\mathrm{Tr}_k = \dfrac{1}{t_1+t_2}\,\mathrm{Trace}\left(M_{D,k}\right)$,

$$M_D = (t_1+t_2)\sum_r \left(\frac{r}{2}\,\frac{(-q)^r+1}{(-q)^r-1} - \frac{1}{2}\,\frac{(-q)+1}{(-q)-1} \right) \alpha_{-r}\alpha_r$$

$+$ off diagonal terms.

By the GW/Hilb correspondence (for π)

Tseng-P (2019)

$$\sum_{g \geq 0} u^{2g-3} \left\langle (2) \right\rangle_{g,d}^{\pi \times \mathbb{C}^2 \bullet}$$

$$\shortparallel$$

$$(-i) \cdot \sum_{n \geq 0} q^n \left\langle (2) \right\rangle_{1,\beta_n}^{Hilb(\mathbb{C}^2,d)}$$

genus 1 GW invariants in curve class β_n of degree $n = \int_{\beta_n} c_1(\mathcal{O}/I)$

after $-q = e^{iu}$.

Tautological bundle on $Hilb(\mathbb{C}^2,d)$

The last step is to evaluate

$$\left\langle (2) \right\rangle_1^{Hilb(\mathbb{C}^2,d)} = \sum_{n \geq 0} q^n \left\langle (2) \right\rangle_{1,\beta_n}^{Hilb(\mathbb{C}^2,d)} .$$

H.-H. Tseng and I found a conjectural answer:

$$\boxed{\begin{array}{l} \text{Conjecture (H.-H. Tseng - P 2023)} \\[1em] -\left\langle (2) \right\rangle_1^{Hilb(\mathbb{C}^2,d)} = \\[1em] -\dfrac{1}{24}\dfrac{(t_1+t_2)^2}{t_1 t_2}\left(Tr_d + \displaystyle\sum_{k=2}^{d-1} \dfrac{6(d-k)}{d-k}\, Tr_k \right). \end{array}}$$

Example $d=2$:

$$\frac{(t_1+t_2)^2}{t_1 t_2}\left\langle \lambda_{g-2}\lambda_g \,\middle|\, (2)\right\rangle^{\pi\,\circ}_{g,2} = \left\langle (2)\right\rangle^{\pi\times\mathbb{C}^2\,\bullet}_{g,2}$$

$\boxed{\begin{array}{l}\text{Convention :} \\ g \text{ terms are summed as} \\ \displaystyle\sum_{g\geq 0} u^{2g-3}\cdots \\ \text{and } -q=\exp(iu)\end{array}}$

Example $d=3$:

$$\frac{(t_1+t_2)^2}{t_1 t_2}\left\langle \lambda_{g-2}\lambda_g \,\middle|\, (2)\right\rangle^{\pi\,\circ}_{g,3} + \frac{(t_1+t_2)^2}{t_1 t_2}\left\langle \lambda_{g-2}\lambda_g \,\middle|\, (2)\right\rangle^{\pi\,\circ}_{g,2}$$

$$-\left(-\frac{1}{24}\,\frac{(t_1+t_2)^2}{t_1 t_2}\,(-i)\cdot \mathrm{Tr}_2(q)\right)$$

$$\parallel$$

$$\left\langle (2)\right\rangle^{\pi\times\mathbb{C}^2\,\bullet}_{g,3}$$

Example $d=4$:

$$\frac{(t_1+t_2)^2}{t_1 t_2}\left\langle \lambda_{g-2}\lambda_g \,\middle|\, (2)\right\rangle^{\pi\,\circ}_{g,4} + \frac{(t_1+t_2)^2}{t_1 t_2}\left\langle \lambda_{g-2}\lambda_g \,\middle|\, (2)\right\rangle^{\pi\,\circ}_{g,3}$$

$$+ \frac{(t_1+t_2)^2}{t_1 t_2}\left\langle \lambda_{g-2}\lambda_g \,\middle|\, (2)\right\rangle^{\pi\,\circ}_{g,2} \cdot 2$$

$$-\frac{1}{24}\,\frac{(t_1+t_2)}{t_1 t_2}\,\left\langle (2)\right\rangle^{E\times\mathbb{C}^2\,\circ}_{g,3}\cdot \widetilde{\mathrm{Part}}(1)$$

$$-\frac{1}{24}\,\frac{(t_1+t_2)}{t_1 t_2}\,\left\langle (2)\right\rangle^{E\times\mathbb{C}^2\,\circ}_{g,2}\cdot \widetilde{\mathrm{Part}}(2)$$

$$=$$

$$\left\langle (2)\right\rangle^{\pi\times\mathbb{C}^2\,\bullet}_{g,4}$$

We simplify as

$$\frac{(t_1+t_2)^2}{t_1 t_2} \left\langle \lambda_{g-2}\lambda_g \,\middle|\, (2) \right\rangle_{g,4}^{\pi\,\circ} + \frac{(t_1+t_2)^2}{t_1 t_2} \left\langle \lambda_{g-2}\lambda_g \,\middle|\, (2) \right\rangle_{g,3}^{\pi\,\circ}$$

$$+ \frac{(t_1+t_2)^2}{t_1 t_2} \left\langle \lambda_{g-2}\lambda_g \,\middle|\, (2) \right\rangle_{g,2}^{\pi\,\circ} \cdot 2$$

$$-\left(- \frac{1}{24} \frac{(t_1+t_2)^2}{t_1 t_2} (-i) \left(Tr_3 - Tr_2 \right) \cdot 1 \right)$$

$$-\left(- \frac{1}{24} \frac{(t_1+t_2)}{t_1 t_2} (-i) \, Tr_2 \cdot \frac{5}{2} \right)$$

$$=$$

$$\left\langle (2) \right\rangle_{g,4}^{\pi \times \mathbb{C}^2 \bullet}$$

$\overline{\text{IX}}$ Projection of NL_d

By definition:

$$Pr_A([NL_d]) \in R^{g-1}(A_g).$$

Let $\delta_{g,d} \in CH^{g-1}(A_g)$,

$$\delta_{g,d} \in \operatorname{Ker}(Pr_A),$$

be the non tautological part:

$$[NL_d] = Pr_A([NL_d]) + \delta_{g,d}.$$

By definition of Pr_A,

Non canonical closure

$$\langle \delta_{g,d}, \gamma \rangle_{A_g} = \int_{A_g^{pc}} \overline{\delta}_{g,d} \cdot \gamma \cdot \lambda_g$$

lifting of λ classes

$$= 0$$

for all $\gamma \in R^{\binom{g}{2}-(g-1)}(A_g)$.

We have seen before that

$$\lambda_{g-1} \cdot [NL_d] = 0 \in R^{2g-2}(A_g).$$

So we have

$$0 = \lambda_{g-1} \cdot Pr_A([NL_d]) + \lambda_{g-1} \cdot \delta_{g,d} .$$

Certainly

$$\lambda_{g-1} \cdot Pr_A([NL_d]) \in R^{2g-2}(A_g).$$

Claim: $\lambda_{g-1} \cdot \delta_{g,d} \in \operatorname{Ker}(Pr_A)$

Proof: $\langle \lambda_{g-1} \cdot \delta_{g,d}, \gamma \rangle_{A_g} = \langle \delta_{g,d}, \underbrace{\lambda_{g-1} \cdot \gamma}_{= 0} \rangle_{A_g}$

$$\forall \gamma \in R^{\binom{g}{2} - (2g-2)}(A_g) .$$

Therefore, since

$$R^{2g-2}(A_g) \cap \ker(Pr_A) = 0,$$

$$\lambda_{g-1} \cdot Pr_A([NL_d]) = 0,$$

$$\lambda_{g-1} \cdot \delta_{g,d} = 0.$$

As before, we conclude

$$Pr_A([NL_d]) = \hat{c}_{g,d} \cdot \lambda_{g-1}.$$

If $[NL_d] \in R^*(A_g)$, then

$$\widehat{C}_{g,d} = C_{g,d}$$

defined by projection

Computed previously using $\mathrm{Hilb}(\mathbb{C}^2, k)$

$\boxed{\begin{array}{l}\text{Conjecture (Canning-Oprea-P 2023)} \\ \text{for all } g \geq 2,\ d \geq 1: \\ \qquad \widehat{C}_{g,d} = C_{g,d}.\end{array}}$

Probably $g = 1$ also works with careful definition as a degenerate case.

The $d=1$ case follows from

> **Theorem** (Canning–Oprea–P 2023):
> If the vanishing conjecture holds,
> $$Pr_A([A_1 \times A_{g-1}]) = \frac{g}{6|B_{2g}|}\lambda_{g-1}.$$

together with the calculation of $c_{g,1}$.

In general, we have

$$[NL_d] = \hat{c}_{g,d} \cdot \lambda_{g-1} + \delta_{g,d}$$

with $\delta_{g,d} \in \mathrm{Ker}(Pr_A)$

and $\alpha \cdot \delta_{g,d} = 0$

$\forall\, \alpha \in R^*(A_g)$ satisfying $\alpha \cdot \lambda_{g-1} = 0$,

$$(\alpha \in \mathrm{Ann}(\lambda_{g-1}))\,.$$

In order to prove

$$\hat{c}_{g,d} = c_{g,d}\,,$$

we must show

any extension

$$\int_{\overline{\mathcal{M}}_g} \overline{\mathrm{Tor}^*(\delta_{g,d})} \cdot \lambda_{g-2}\lambda_g = 0\,.$$

I see two possible paths to prove

I point out that

$$\mathrm{Tor}_*[\mathcal{M}_g^{ct}] \in R^*(A_g)$$

$$\Downarrow$$

$$\int_{\overline{\mathcal{M}}_g} \overline{\mathrm{Tor}^*(\delta_{g,d})} \cdot \lambda_{g-2}\lambda_g = 0 .$$

But there is not much reason to believe that $\mathrm{Tor}_*[\mathcal{M}_g^{ct}]$ is tautological.

The best reason to believe

$$\widehat{C}_{g,d} = C_{g,d}$$

is a conjecture by Aitor:

Conjecture (Iribar López 2024)

$$CH^*(\mathcal{A}_g) \xrightarrow{pr_A} R^*(\mathcal{A}_g)$$

is a ring homomorphism.

What limited evidence that we have supports this claim

(at least for the subring of $CH^*(\mathcal{A}_g)$ generated by NL and Jacobian loci.)

X Calculation of the projection of NL_d by Aitor Iribar López:

We have already proven

$$Pr_A([NL_d]) = \hat{c}_{g,d} \cdot \lambda_{g-1} \in R^*(A_g)$$

Theorem A (Iribar López 2024)

$$\hat{c}_{g,d} = d^{2g-1} \prod_{p \mid d} (1 - p^{-2g+2}) \cdot \frac{g}{6|B_{2g}|}$$

Aitor's proof uses the geometry of the moduli of abelian varieties with level structures.

Let $c_{g,d}$ be computed using the conjectural formula for

$$\left\langle (2) \right\rangle_{1}^{\mathrm{Hilb}(\mathbb{C}^2, e)}, \quad 2 \le e \le d.$$

$\boxed{\begin{array}{l} \text{Theorem } B^{\star} \text{ (Iribar López 2024)} \\ \text{for all } g \geq 2,\ d \geq 1: \\ \hat{c}_{g,d} = c_{g,d}. \end{array}}$

$\star$ here denotes the dependence on the conjectural formula for $\mathrm{Hilb}(\mathbb{C}^2)$.

Aitor's results yield the following implication

Conjecture (Iribar López 2024)

$$CH^*(\mathcal{A}_g) \xrightarrow{pr_A} R^*(\mathcal{A}_g)$$

is a ring homomorphism.

$$\Downarrow$$

Conjecture (H.-H. Tseng - P 2023)

$$-\langle (2) \rangle_1^{Hilb(\mathbb{C}^2,d)} =$$

$$-\frac{1}{24}\,\frac{(t_1+t_2)^2}{t_1 t_2}\left(Tr_d + \sum_{k=2}^{d-1} \frac{\sigma(d-k)}{d-k}\, Tr_k \right).$$

Appendix: Update March 2024

There is a new path to prove:

Conjecture (H.-H. Tseng - P 2023)

$$-\left\langle (2) \right\rangle_1^{\mathrm{Hilb}(\mathbb{C}^2,d)} =$$

$$-\frac{1}{24}\,\frac{(t_1+t_2)^2}{t_1 t_2}\left(\mathrm{Tr}_d + \sum_{k=2}^{d-1} \frac{\sigma(d-k)}{d-k}\,\mathrm{Tr}_k \right).$$

We have seen that calculating the following connected Gromov-Witten integral is sufficient:

$$\int_{[\overline{M}_{g,1}(\pi,d)]^{vir}} \tau_1(q)\,\lambda_{g-2}\lambda_g$$

$$\parallel$$

$$\left\langle \tau_1(q)\,\lambda_{g-2}\lambda_g \right\rangle_{g,d}^{\pi\,\circ}$$

Here $\overline{M}_{1,1}$ is the moduli of pointed nonsingular elliptic curves and

$$\begin{tikzcd} \mathcal{E} \arrow[d, "\pi"'] \\ \overline{M}_{1,1} \arrow[u, bend right, "q"'] \end{tikzcd} \qquad \text{zero section}$$

.

- The first idea is to switch to an elliptically fibered K3 surface:

$$\begin{tikzcd} S \arrow[d, "\pi_S"'] \\ \mathbb{P}^1 \arrow[u, bend right, "q"'] \end{tikzcd}$$

24 nodal fibers

The fibers of π_S are 1-pointed stable genus 1 curves.

The induced morphism

$$\mathbb{P}^1 \longrightarrow \overline{M}_{1,1}$$

is of degree 48.

Then we have

$$\int_{[\overline{M}_{g,1}(\pi, d)]^{vir}} T_1(q)\, \lambda_{g-2}\lambda_g$$

$$\|$$

$$\frac{1}{48} \int_{[\overline{M}_{g,1}(\pi_S, d)]^{vir}} T_1(q)\, \lambda_{g-2}\lambda_g$$

- The second idea is to use K3 vanishing.

Consider the integral:

integrand dim $= 2+g-2$
$= g$

vanishing of standard K3 virtual class

$$\int_{[\overline{M}_{g,1}(S,d)]^{vir}} \tau_1(q)\,\lambda_{g-2} = 0$$

standard virtual class

Vdim $= g-1+1$
$= g$

d times fiber class of π_S, $d>0$.

The above vanishing will give us a nontrivial relation.

Claim A:

$$\int_{[\overline{M}_{g,1}(\pi_S,\, d)]^{vir}} T_1(q)\, \lambda_{g-2}\, e\left(\mathbb{E}^\vee \otimes T_{\mathbb{P}^1}\right)$$

$$\|$$

$$\int_{[\overline{M}_{g,1}(S,\, d)]^{vir}} T_1(q)\, \lambda_{g-2} .$$

Corollary:

$$\int_{[\overline{M}_{g,1}(\pi_S,\, d)]^{vir}} T_1(q)\, \lambda_{g-2}\, e\left(\mathbb{E}^\vee \otimes T_{\mathbb{P}^1}\right) = 0 .$$

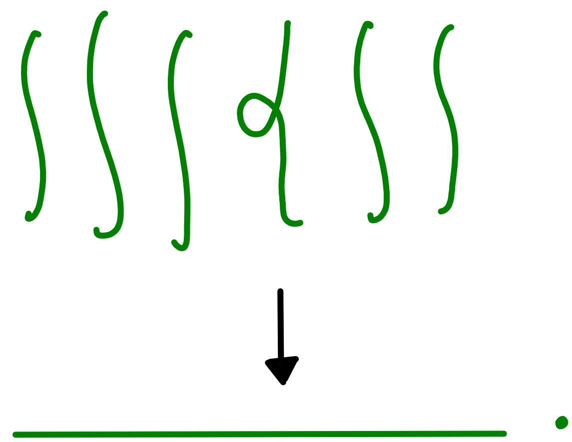
Proof: There is a morphism

$$\overline{M}_{g,1}(\pi_S, d) \longrightarrow \overline{M}_{g,1}(S, d)$$

which is an isomorphism of DM stacks away from the 24 nodal fibers of π_S. Moreover, away from the 24 nodal fibers, the obstruction theory of $\overline{M}_{g,1}(\pi_S, d)$ augmented by $\mathbb{E}^\vee \otimes \mathrm{Tan}_{\mathbb{P}^1}$ matches the standard obstruction theory of $\overline{M}_{g,1}(S, d)$.

The entire issue is about the nodal fibers

$$\begin{tikzcd} S \arrow[d, "\pi_S"'] \\ \mathbb{P}^1 \arrow[u, bend right, "q"'] \end{tikzcd}$$



We use here the degeneration to the normal cone of the divisor

$$\alpha \subset S \text{ of nodal fibers,}$$

a standard technique, but a complication here is that (S, α) requires log GW (since α is singular).

We study the normal cone

$$\begin{array}{l} \mathcal{X} = Bl(S \times \mathbb{C},\ \alpha \times 0) \\ \downarrow \\ \mathbb{C} \end{array}$$

$\mathcal{X}$ has a single singularity

(a 3-fold double point)

over each point $p \times 0$

where $p \in \alpha$ is a node.

The main observation here:

we can avoid all log complications

by studying $\mathcal{X}^\circ \subset \mathcal{X}$.

nonsingular locus

The reason that the noncompact log geometry $\mathcal{X}^\circ \subset \mathcal{X}$ can be used here is that the curve classes are fibers and have intersection 0 with α.

Said differently: the moduli spaces of log stable maps to the log degeneration $\mathcal{X}^\circ$ are compact.

Then the usual degeneration calculus of relative GW theory can be used.

After degeneration, the equality of Claim A is clear since the geometric differences of the moduli spaces vanish.

A second proof of Claim A would follow by constructing a cosection for the obstruction theory on $\overline{M}_{g,1}(\pi_S, d)$ obtained by combining the fiberwise deformation with $\mathbb{E}^\vee \otimes \mathbb{P}^1$.

- The third step is to expand

$$e\left(\mathbb{E}^\vee \otimes T_{\mathbb{P}^1}\right) = (-1)^g \lambda_g + (-1)^{g-1} \lambda_{g-1} \cdot [2pt]$$

so we obtain

$$0 = \int_{\left[\overline{M}_{g,1}(\pi_5, d)\right]^{vir}} T_1(q)\, \lambda_{g-2}\, e\left(\mathbb{E}^\vee \otimes T_{\mathbb{P}^1}\right)$$

$$= \int_{\left[\overline{M}_{g,1}(\pi_5, d)\right]^{vir}} T_1(q)\, \lambda_{g-2} \cdot (-1)^g \lambda_g$$

$$+ 2 \int_{\overline{M}_{g,1}(E, d)^{vir}} T_1(q)\, \lambda_{g-2} \cdot (-1)^{g-1} \lambda_{g-1}.$$

After rewriting, we find

$$\int_{[\overline{M}_{g,1}(\pi, d)]^{vir}} \tau_1(q)\, \lambda_{g-2}\lambda_g$$

$$\shortparallel$$

$$\frac{1}{24} \int_{[\overline{M}_{g,1}(E, d)]^{vir}} \tau_1(q)\, \lambda_{g-2}\lambda_{g-1}$$

fixed elliptic target

- The last step is the evaluation of the latter integral by Pixton (2008):

$$\sum_{d \geq 0} Q^d \int_{[\overline{M}_{g,1}(E,d)]^{vir}} \tau_1(q)\, \lambda_{g-2} \lambda_{g-1}$$

$$\|$$

$$|B_{2g-2}| \cdot \binom{2g}{2} C_{2g}(Q)$$

where

$$C_{2g}(Q) = -\frac{B_{2g}}{2g \cdot 2g!} + \frac{2}{2g!} \sum_{n \geq 1} \sigma_{2g-1}(n)\, Q^n,$$

in other words

$$C_{2g}(Q) = -\frac{B_{2g}}{2g \cdot 2g!} E_{2g}(Q) .$$

Eisenstein series

See page 32 of

https://people.math.ethz.ch/~rahul/pixton.pdf

for the results of Pixton.

Claim B: The evaluation

$$\sum_{d\geq 0} Q^d \int_{[\overline{M}_{g,1}(\pi,d)]^{vir}} T_1(q)\, \lambda_{g-2}\lambda_g$$

$$\| $$

$$\frac{1}{24}\, |B_{2g-2}| \cdot \binom{2g}{2}\, C_{2g}(Q)$$

is equivalent to the conjectured formula for $\langle (2) \rangle_1^{Hilb(\mathbb{C}^2,d)}$.

Proof by Iribar López.

The Status now is that all the claims related to

$$\mathrm{Pr}_A([NL_d]) \in R^{g-1}(A_g)$$

and the series $\langle (2) \rangle_1^{\mathrm{Hilb}(\mathbb{C}^2,d)}$

are proven:

definition

$$\mathrm{Pr}_A([NL_d]) = \hat{c}_{g,d} \cdot \lambda_{g-1},$$

$$\hat{c}_{g,d} = d^{2g-1} \prod_{p \mid d} (1 - p^{-2g+2}) \cdot \frac{g}{6 |B_{2g}|},$$

[by Iribar López]

$$-\left\langle (2) \right\rangle_1^{Hilb(\mathbb{C}^2,d)} =$$

$$-\frac{1}{24}\,\frac{(t_1+t_2)^2}{t_1t_2}\left(Tr_d + \sum_{k=2}^{d-1} \frac{\sigma(d-k)}{d-k}\, Tr_k \right),$$

[by claim A + B]

definition

$$c_{g,d} = \frac{\displaystyle\int_{\overline{M}_g} \overline{Tor^*([NL_d])}\cdot \lambda_{g-2}\lambda_g}{\displaystyle\int_{\overline{M}_g} \lambda_{g-2}\lambda_{g-1}\lambda_g},$$

$\hat{c}_{g,d} = c_{g,d}$ by calculation of

$$\left\langle (2) \right\rangle_1^{Hilb(\mathbb{C}^2,d)}.$$

Many open directions remain.

My favorites:

- **Conjecture** (Iribar López 2024)

$$CH^*(\mathcal{A}_g) \xrightarrow{pr_A} R^*(\mathcal{A}_g)$$

is a <u>ring</u> homomorphism.

- Study the extension of the diagram

$$\begin{tikzcd} \mathrm{Tor}_1^{-1}(\mathcal{NL}_d) \arrow[r] \arrow[d] & \mathcal{NL}_d \arrow[d] \\ \mathcal{M}_{g,1}^{ct} \arrow[r, "\mathrm{Tor}_1"] & \mathcal{A}_g \end{tikzcd}$$

to the perfect cone compactifications

$$\begin{tikzcd} \mathrm{Tor}_1^{-1}(\overline{NL}_d) \arrow[r] \arrow[d] & \overline{NL}_d \arrow[d] \\ \overline{\mathcal{M}}_{g,1} \arrow[r, "\overline{\mathrm{Tor}}_1"] & \overline{\mathcal{A}}_g \end{tikzcd}$$

Perfect cone

- Calculate

$$\langle \sigma_1, \sigma_2, \dots, \sigma_n \rangle_1^{\mathrm{Hilb}(\mathbb{C}^2, d)}$$

for arbitrary partion insertions σ_i.

The End

17 March 2024