

ON THE MOTIVIC STABLE PAIRS INVARIANTS OF $K3$ SURFACES

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WITH AN APPENDIX BY R. P. THOMAS

ABSTRACT. For a $K3$ surface S and a class $\beta \in \text{Pic}(S)$, we study motivic invariants of stable pairs moduli spaces associated to 3-fold thickenings of S . We conjecture suitable deformation and divisibility invariances for the Betti realization. Our conjectures, together with earlier calculations of Kawai-Yoshioka, imply a full determination of the theory in terms of the Hodge numbers of the Hilbert schemes of points of S . The work may be viewed as the third in a sequence of formulas starting with Yau-Zaslow and Katz-Klemm-Vafa (each recovering the former). Numerical data suggest the motivic invariants are linked to the Mathieu M_{24} moonshine phenomena.

The KKV formula and the Pairs/Noether-Lefschetz correspondence together determine the BPS counts of $K3$ -fibered Calabi-Yau 3-folds in fiber classes in terms of modular forms. We propose a framework for a refined P/NL correspondence for the motivic invariants of $K3$ -fibered CY 3-folds. For the STU model, a complete conjecture is presented.

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0. INTRODUCTION

A beautiful connection between curve counting on $K3$ surfaces and modular forms was conjectured in 1995 by Yau and Zaslow [39]: the generating series of the counts of rational curves in primitive classes was conjectured to equal the inverse of the discriminant

$$\Delta(q) = q \prod_{n=1}^{\infty} (1 - q^n)^{24} .$$

By work of Göttsche [13], $\Delta(q)^{-1}$ was already known to arise as the generating series of the Euler characteristics of Hilbert schemes of points of $K3$ surfaces S ,

$$\sum_{n \geq 0} \chi(\text{Hilb}^n(S)) q^{n-1} = \frac{1}{\Delta(q)} .$$

An argument by Beauville [2] in 1997 provided a geometric link between curve counting in primitive classes and the Euler characteristics of $\text{Hilb}^n(S)$.

A connection between the higher genus curve counts on $K3$ surfaces and the generating series of χ_y genera of $\text{Hilb}^n(S)$,

$$\sum_{n \geq 0} \chi_y(\text{Hilb}^n(S)) q^{n-1} = \frac{1}{q \prod_{n=1}^{\infty} (1 - yq^n)^2 (1 - q^n)^{20} (1 - y^{-1}q^n)^2} ,$$

was proposed in 1999 [17]. The conjectures of [17] govern *all* classes on $K3$ surfaces via a subtle divisibility invariance for multiple classes. In the genus 0 primitive case, the Yau-Zaslow conjecture is recovered. A proof of the KKV conjecture was recently found [31]. The moduli of sheaves (via stable pairs [28]) on $K3$ surfaces play a central role.

We propose here a third step in the sequence of conjectures starting with Yau-Zaslow and KKV. We conjecture the Betti realization of the motivic stable pairs theory of $K3$ surfaces is connected to the

generating series of Hodge numbers of $\text{Hilb}^n(S)$,

$$\sum_{n \geq 0} \chi_{\text{Hodge}}(\text{Hilb}^n(S)) q^{n-1} = \frac{1}{q \prod_{n=1}^{\infty} (1 - u^{-1}y^{-1}q^n)(1 - u^{-1}yq^n)(1 - q^n)^{20}(1 - uy^{-1}q^n)(1 - uyq^n)},$$

where the variables u and y keep track of the Hodge grading. Our conjecture governs all curve classes and specializes to the KKV conjecture after taking Euler characteristics.

In addition to the surprising divisibility invariance already present in the KKV conjecture [17], we propose a new deformation invariance of the Betti realization of the motivic invariants of $K3$ geometries. To support our conjectures, we provide a few basic calculations.

The KKV formula and the Pairs/Noether-Lefschetz correspondence [25, 31] together determine the BPS counts of $K3$ -fibered Calabi-Yau 3-folds in fiber classes in terms of modular forms. We propose a framework for a refined P/NL correspondence for the motivic invariants of $K3$ -fibered Calabi-Yau 3-folds. For the STU model, a complete conjecture is provided.

In the Appendix by R. Thomas, the Gopakumar-Vafa perspective on the motivic invariants of $K3$ surfaces is discussed. The relevant moduli spaces [18] are shown to be nonsingular even in the imprimitive case. The results of the Appendix may be viewed as supporting our motivic stable pairs conjectures in the larger framework of the conjectural Pairs/Gopakumar-Vafa correspondence.¹

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¹An elementary overview of the various correspondences for curve counts on Calabi-Yau 3-folds can be found in [30].

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1. CURVE CLASSES ON $K3$ SURFACES

Let S be a nonsingular projective $K3$ surface. The second cohomology of S is a rank 22 lattice with intersection form

$$(1.1) \quad H^2(S, \mathbb{Z}) \cong U \oplus U \oplus U \oplus E_8(-1) \oplus E_8(-1),$$

where

$$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and $E_8(-1)$ is the (negative) Cartan matrix. The intersection form (1.1) is even.

The *divisibility* m_β is the maximal positive integer dividing the lattice element $\beta \in H^2(S, \mathbb{Z})$. If the divisibility is 1, β is *primitive*. Elements with equal divisibility and norm square $\langle \beta, \beta \rangle$ are equivalent up to orthogonal transformation [38].

The *Picard* lattice of S is the intersection

$$\text{Pic}(S) = H^2(S, \mathbb{Z}) \cap H^{1,1}(S, \mathbb{C}).$$

For a family of nonsingular $K3$ surfaces

$$\pi : \mathcal{X} \rightarrow (\Delta, 0)$$

with special fiber $\mathcal{X}_0 \cong S$ and trivial local system $R^2\pi_*\mathbb{Z}$, the Noether-Lefschetz locus associated to $\gamma \in H^2(S, \mathbb{Z})$ is

$$NL_\gamma = \{p \in \Delta \mid \gamma \in \text{Pic}(\mathcal{X}_p)\} .$$

The Noether-Lefschetz locus is naturally a subscheme $NL_\gamma \subset \Delta$.

2. STABLE PAIRS MOTIVIC INVARIANTS

Let S be a nonsingular projective $K3$ surface. Curve counting on S may be approached via the *reduced* virtual fundamental class of the moduli space of stable maps to S or the stable pairs theory of Calabi-Yau 3-fold thickenings of S . An equivalence relating these two counts is proven in [31] essentially using [25, 27]. We are interested here in the motivic invariants associated to S . Since no motivic theory is available on the Gromov-Witten side, we will consider here the moduli spaces of stable pairs.

BPS counts for S via stable pairs were defined in [31]. The construction uses $K3$ -fibrations sufficiently transverse to Noether-Lefschetz loci. We follow the geometric perspective of [31] to define a motivic theory associated to S .

Let $\alpha \in \text{Pic}(S)$ be a nonzero class which is both positive (with respect to any ample polarization of S) and primitive. Let T be a nonsingular 3-dimensional quasi-projective variety,

$$\epsilon : T \rightarrow (\Delta, 0) ,$$

fibered in $K3$ surfaces over a pointed curve $(\Delta, 0)$ satisfying:

- (i) Δ is a nonsingular quasi-projective curve with trivial canonical class,
- (ii) ϵ is smooth, projective, and $T_0 \cong S$,

The class $\alpha \in \text{Pic}(S)$ is *m-rigid* with respect to the family ϵ if the following further condition is satisfied:

(\star) for every *effective* decomposition²

$$m\alpha = \sum_{i=1}^l \gamma_i \in \text{Pic}(S),$$

the local Noether-Lefschetz locus $\text{NL}(\gamma_i) \subset \Delta$ corresponding to each class $\gamma_i \in \text{Pic}(S)$ is the *reduced* point $0 \in \Delta$.

Let $\text{Eff}(m\alpha) \subset \text{Pic}(S)$ denote the subset of effective summands of $m\alpha$. The existence of m -rigid families is easy to see [31, Section 6.2].

Assume α is m -rigid with respect to the family ϵ . By property (\star), there is a compact, open, and closed component

$$P_n^\star(T, \gamma) \subset P_n(T, \gamma)$$

parameterizing stable pairs³ supported set-theoretically over the point $0 \in \Delta$ for *every* effective summand $\gamma \in \text{Eff}(m\alpha)$. We define

$$(2.1) \quad \mathbf{W}_{n,\gamma}^\star(T) \in \mathbf{K}_{\text{var}}^{\widehat{\mu}}[\mathbb{L}^{-1}]$$

to be the motivic⁴ stable pairs invariant associated to the component $P_n^\star(T, \gamma)$ following Joyce and collaborators [4].

The motivic invariant (2.1) takes values in the Grothendieck ring of varieties carrying actions of groups of n^{th} roots of unity $\mathbf{K}_{\text{var}}^{\widehat{\mu}}$ extended by the inverse of the Tate class,

$$\mathbb{L} = [\mathbb{A}^1].$$

The product in $\mathbf{K}_{\text{var}}^{\widehat{\mu}}$ is *not* induced by the ordinary product of varieties, but rather defined explicitly by motivic convolution with a Fermat curve [9, 24].

²An effective decomposition requires all parts γ_i to be effective divisors.

³For any class $\gamma \in \text{Pic}(S)$, we denote the push-forward to $H_2(T, \mathbb{Z})$ also by γ . Let $P_n(T, \gamma)$ be the moduli space of stable pairs of Euler characteristic n and class $\gamma \in H_2(T, \mathbb{Z})$.

⁴At the moment, $\mathbf{W}_{n,\gamma}^\star(T)$ is defined only after a choice of orientation data is made. Our discussion implicitly assumes *either* that there is a canonical choice *or* that the choice does not affect the motivic class for our $K3$ geometry (or, at the very least, does not affect the Poincaré polynomials of the motivic classes here). Certainly $P_n^\star(T, \gamma)$ is often simply connected. If π_1 is trivial, then the orientation is unique. Perhaps $P_n^\star(T, \gamma)$ is always simply connected?

Let ρ denote the canonical action of the finite group scheme

$$\mu_2 = \{\pm 1\}$$

on itself. We obtain an element

$$[\mu_2, \rho] \in \mathbf{K}_{\text{var}}^{\widehat{\mu}}.$$

Straightforward calculation using the definition of the product or the motivic Thom-Sebastiani formula yields the relation

$$(2.2) \quad (1 - [\mu_2, \rho])^2 = \mathbf{L},$$

so we define $\mathbf{L}^{\frac{1}{2}}$ by

$$(2.3) \quad \mathbf{L}^{\frac{1}{2}} = 1 - [\mu_2, \rho],$$

see [23, Remark 19]. The ring $\mathbf{K}_{\text{var}}^{\widehat{\mu}}[\mathbf{L}^{-1}]$ therefore contains all powers of $\mathbf{L}^{\pm 1/2}$.

Definition. *Let $\alpha \in \text{Pic}(S)$ be a primitive, positive class. Given a family $\epsilon : T \rightarrow (\Delta, 0)$ satisfying conditions (i), (ii), and (\star) for $m\alpha$, let*

$$\sum_{n \in \mathbb{Z}} \mathbf{V}_{n, m\alpha}^{\epsilon}(S) q^n = \text{Coeff}_{v, m\alpha} \left[\log \left(1 + \sum_{n \in \mathbb{Z}} \sum_{\gamma \in \text{Eff}(m\alpha)} q^n v^\gamma \mathbf{W}_{n, \gamma}^{\star}(T) \right) \right].$$

The motivic invariant $\mathbf{V}_{n, m\alpha}^{\epsilon}(S)$ is the main topic of the paper. The superscript ϵ records the family

$$\epsilon : T \rightarrow (\Delta, 0)$$

used in the definition.

For positive $\beta \in \text{Pic}(S)$, we may write $\beta = m\alpha$ where $\alpha \in \text{Pic}(S)$ is positive and primitive and $m = m_\beta$ is the divisibility of β . Hence,

$$\mathbf{V}_{n, \beta}^{\epsilon}(S) = \mathbf{V}_{n, m\alpha}^{\epsilon}(S)$$

is defined.

3. CONJECTURES A AND B

3.1. Poincaré polynomial. We formulate here several conjectures and speculations concerning the motivic stable pairs invariants $V_{n,\beta}^\epsilon(S)$ introduced in Section 2. Let

$$H_{n,\beta}^\epsilon(S) \in \mathbb{Q}[u]$$

denote the virtual Poincaré polynomial of the motivic invariant $V_{n,\beta}^\epsilon(S)$.

Conjecture A. *The virtual Poincaré polynomial $H_{n,\beta}^\epsilon(S)$ is independent of the family*

$$\epsilon : T \rightarrow (\Delta, 0)$$

satisfying conditions (i), (ii), and (\star) for $\alpha = \frac{1}{m_\beta}\beta$ and $m = m_\beta$.

Assuming the validity of Conjecture A, we may drop the ϵ superscript and write $H_{n,\beta}(S)$ for the virtual Poincaré polynomial.

Conjecture B. *The virtual Poincaré polynomial $H_{n,\beta}(S)$ is invariant under deformations of S for which β remains algebraic.*

The divisibility m_β and the norm square

$$\langle \beta, \beta \rangle = 2h - 2$$

are the only deformation invariants of the pair (S, β) with $\beta \in \text{Pic}(S)$. Assuming the validity of Conjecture B, we write

$$H_{n,\beta}(S) = H_{n,m_\beta,h} ,$$

replacing (S, β) by m_β and h .

3.2. Motives. Conjecture A for the motivic invariant $V_{n,\beta}^\epsilon(S)$ seems not unreasonable: $V_{n,\beta}^\epsilon(S)$ is independent of ϵ . However, Conjecture B is certainly false with $V_{n,\beta}^\epsilon(S)$ in place of $H_{n,\beta}(S)$ since the class

$$[S] \in K_{\text{var}}$$

often appears in $V_{n,\beta}^\epsilon(S)$. Examples of the latter phenomenon can be found already for elliptically fibered $K3$ surfaces in the fiber class.

The classes of $K3$ surfaces derived equivalent to S also appear in the elliptically fibered cases.

We define a new quotient ring of varieties:

$$\mathbf{K}_{\text{var}}^D = \mathbf{K}_{\text{var}}^{\widehat{\mu}} / I^D ,$$

where I^D is the ideal generated by all differences

$$[X] - [\widetilde{X}] \in \mathbf{K}_{\text{var}}^{\widehat{\mu}}$$

where X and \widetilde{X} are $K3$ surfaces which are derived equivalent.⁵ Then, we could hope the class

$$\overline{\mathbf{V}}_{n,m\alpha}^\epsilon(S) \in \mathbf{K}_{\text{var}}^D[\mathbf{L}^{-1}] ,$$

obtained from $\mathbf{V}_{n,m\alpha}^\epsilon(S)$, is a universal polynomial in the motivic powers of the class of the underlying $K3$ surface,

$$\overline{[\text{Sym}^r S]} \in \mathbf{K}_{\text{var}}^D[\mathbf{L}^{-1}] .$$

The coefficients of such a polynomial would lie in $\mathbb{Q}[\mathbf{L}, \mathbf{L}^{-1}]$.

An interesting related question immediately arises. Let S be nonsingular projective $K3$ surface with a positive *irreducible* class $\alpha \in \text{Pic}(S)$. The moduli space $P_n(S, \alpha)$ is known to be nonsingular [20, 29]. Is there a formula for

$$\overline{[P_n(S, \alpha)]} \in \mathbf{K}_{\text{var}}^D[\mathbf{L}^{-1}]$$

as a universal polynomial in the motivic powers $\overline{[\text{Sym}^r S]}$?

While we have (modest) computational evidence for Conjectures A and B constraining $\mathbf{H}_{m,\beta}^\epsilon(S)$, the above hopes for $\mathbf{V}_{m,\beta}^\epsilon(S)$ are simply speculations.

4. CALCULATION OF KAWAI-YOSHIOKA

In order to formulate Conjecture C which completely determines $\mathbf{H}_{n,m,h}$, we first review the calculation of Kawai-Yoshioka.

⁵Or perhaps I^D should be the ideal generated by the differences of *all* derived equivalent nonsingular projective Calabi-Yau varieties. Alternatively, Chow motives may be a more natural framework for the entire discussion.

Let $P_n(S, h)$ denote the moduli space of stable pairs on the $K3$ surface S for a positive irreducible class α satisfying

$$2h - 2 = \langle \alpha, \alpha \rangle.$$

The moduli space $P_n(S, h)$ is nonsingular of dimension $2h - 1 + n$. The Poincaré polynomial⁶ of $P_n(S, h)$,

$$\tilde{H}(P_n(S, h)) = \sum_{i=0}^{2h-1+n} \dim_{\mathbb{Q}} H^{2i}(P_n(S, h), \mathbb{Q}) u^{2i} \in \mathbb{Z}[u],$$

has been calculated by Kawai-Yoshioka. By Theorem 5.158 of [20],

$$\begin{aligned} & (u^2 \tilde{y} - 1) (1 - \tilde{y}^{-1}) \cdot \sum_{h=0}^{\infty} \sum_{n=1-h}^{\infty} \tilde{H}(P_n(S, h)) u^{-2h} \tilde{y}^n q^h = \\ & \prod_{n=1}^{\infty} \frac{1}{(1 - \tilde{y} q^n)(1 - u^2 \tilde{y} q^n)(1 - q^n)^{20}(1 - \tilde{y}^{-1} q^n)(1 - (u^2 \tilde{y})^{-1} q^n)}. \end{aligned}$$

In order to fit our motivic conventions in Section 6, we define

$$H(P_n(S, h)) = u^{-2h+1-n} \tilde{H}(P_n(S, h))$$

and rewrite the Kawai-Yoshioka formula as

$$\begin{aligned} & u^{-1} (u^2 \tilde{y} - 1) (1 - \tilde{y}^{-1}) \cdot \sum_{h=0}^{\infty} \sum_{n=1-h}^{\infty} H(P_n(S, h)) u^n \tilde{y}^n q^h = \\ & \prod_{n=1}^{\infty} \frac{1}{(1 - \tilde{y} q^n)(1 - u^2 \tilde{y} q^n)(1 - q^n)^{20}(1 - \tilde{y}^{-1} q^n)(1 - (u^2 \tilde{y})^{-1} q^n)}. \end{aligned}$$

After the substitution $y = u\tilde{y}$, we find

$$\begin{aligned} & (uy - 1) (u^{-1} - y^{-1}) \cdot \sum_{h=0}^{\infty} \sum_{n=1-h}^{\infty} H(P_n(S, h)) y^n q^h = \\ & \prod_{n=1}^{\infty} \frac{1}{(1 - u^{-1} y^{-1} q^n)(1 - u^{-1} y q^n)(1 - q^n)^{20}(1 - u y^{-1} q^n)(1 - u y q^n)}. \end{aligned}$$

The right side of the above formula is the generating series of Hodge polynomials of the Hilbert schemes of points of S .

⁶All the cohomology of $P_n(S, h)$ is even.

5. REFINED GOPAKUMAR-VAFA INVARIANTS

5.1. **Definition.** Refined Gopakumar-Vafa invariants were defined mathematically in [7] via refined stable pairs invariants.⁷ Following [7], we will define refined invariants

$$\mathbf{R}_{j_L, j_R}^h \in \mathbb{Z}$$

for $K3$ surfaces for all $h \in \mathbb{Z}$ and all half-integers

$$j_L, j_R \in \frac{1}{2}\mathbb{Z}_{\geq 0} = \left\{0, \frac{1}{2}, 1, \frac{3}{2}, \dots\right\} .$$

The definition uses formula (8.1) of [7] and the generating series of Hodge numbers of the Hilbert schemes of points of $K3$ surfaces of Section 4. If $h < 0$, the definition is simple:

$$\mathbf{R}_{j_L, j_R}^{h < 0} = 0 .$$

For the $h \geq 0$ case, we will use the following notation. For $j \in \frac{1}{2}\mathbb{Z}_{\geq 0}$, we define

$$[j]_x = x^{-2j} + x^{-2j+2} + \dots + x^{2j-2} + x^{2j} .$$

We define the refined invariants $\mathbf{R}_{j_L, j_R}^{h \geq 0}$ for $K3$ surfaces by

$$(5.1) \quad \sum_{h=0}^{\infty} \sum_{j_L} \sum_{j_R} \mathbf{R}_{j_L, j_R}^h [j_L]_y [j_R]_u q^h = \prod_{n=1}^{\infty} \frac{1}{(1 - u^{-1}y^{-1}q^n)(1 - u^{-1}yq^n)(1 - q^n)^{20}(1 - uy^{-1}q^n)(1 - uyq^n)} .$$

Here, the sums over j_L and j_R are both taken over $\frac{1}{2}\mathbb{Z}_{\geq 0}$.

The above definition of \mathbf{R}_{j_L, j_R}^h will be discussed further in Section 6. Formula (5.1) will be shown there to be consistent with the Kawai-Yoshioka calculation of Section 4 via the definition of the refined invariants in [7]. In fact, consistency with the Kawai-Yoshioka calculation *forces* formula (5.1) for \mathbf{R}_{j_L, j_R}^h .

⁷Other definition has been proposed in [14, 21]. Connecting the refined invariants for $K3$ surfaces \mathbf{R}_{j_L, j_R}^h defined here to the geometry of [14, 21] is the topic of the Appendix by R. Thomas.

The refined invariants R_{j_L, j_R}^h may be viewed as arising from the cohomology of $P_n(S, h)$, the moduli space of stable pairs on the $K3$ surface S with positive irreducible class α satisfying

$$\langle \alpha, \alpha \rangle = 2h - 2 .$$

Formula (5.1) also agrees with the refined invariant for $K3$ surfaces calculated in [15].⁸

Formula (5.1) uniquely determines R_{j_L, j_R}^h for $h \geq 0$. As a consequence, the following stabilization property holds: *for fixed i and j , the refined invariant $R_{\frac{h}{2}-i, \frac{h}{2}-j}^h$ is independent of h for sufficiently large h .*⁹

$R_{\frac{i}{2}, \frac{j}{2}}^0$	$i=0$
$j=0$	1

$R_{\frac{i}{2}, \frac{j}{2}}^1$	$j=0$	1
$i=0$	20	
1		1

$R_{\frac{i}{2}, \frac{j}{2}}^2$	$j=0$	1	2
$i=0$	231		
1		21	
2			1

$R_{\frac{i}{2}, \frac{j}{2}}^3$	$j=0$	1	2	3
$i=0$	1981		1	
1		252		
2	1		21	
3				1

$R_{\frac{i}{2}, \frac{j}{2}}^4$	$j=0$	1	2	3	4
$i=0$	13938		21		
1		2233		1	
2	21		253		
3		1		21	
4					1

TABLE 1. All nonvanishing R_{j_L, j_R}^h for $h \leq 4$ for $K3$ surfaces

5.2. Unrefined BPS invariants. Let X be a nonsingular projective Calabi-Yau 3-fold, and let $\beta \in H_2(X, \mathbb{Z})$ be a curve class.

The relation of the refined BPS invariants N_{j_L, j_R}^β of X to the unrefined BPS invariants n_g^β of X is obtained by from their definitions in terms

⁸See equations (8.3) and (8.4) of [15] as well as Table 8 in Appendix D.2.

⁹Refined invariants for local \mathbb{P}^2 were found to stabilize in [CKK]. We expect stabilization to hold more generally.

of traces on the BPS Hilbert space \mathcal{H}_{BPS} arising from wrapping $M5$ branes on curves in X . The Hilbert space \mathcal{H}_{BPS} carries an

$$SU(2) \times SU(2) = SU(2)_L \times SU(2)_R$$

action. We denote the irreducible representations of $SU(2) \times SU(2)$ by pairs $[j_L, j_R]$ where $j_L, j_R \in \frac{1}{2}\mathbb{Z}_{\geq 0}$. The refined invariants arise via the formula:

$$(5.2) \quad \text{Tr}_{\mathcal{H}_{BPS}} y^{\sigma_3^L} u^{\sigma_3^R} q^H = \sum_{\beta} \sum_{j_L, j_R \in \frac{1}{2}\mathbb{Z}_{\geq 0}} \mathbf{N}_{j_L, j_R}^{\beta} [j_L]_y [j_R]_u q^{\beta} .$$

Similarly, the unrefined invariants arise as:

$$(5.3) \quad \text{Tr}_{\mathcal{H}_{BPS}} (-1)^{F_R} y^{\sigma_3^L} q^H = \sum_{\beta} \sum_{g \in \mathbb{Z}_{\geq 0}} \mathfrak{n}_g^{\beta} \left(y^{\frac{1}{2}} + y^{-\frac{1}{2}} \right)^{2g} q^{\beta} .$$

Here, $(-1)^{F_R}$ acts as $(-1)^{2j_R}$ on $[j_R]$. The matrix

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

is twice the Cartan element of $SU(2)$ measuring the spins of the BPS state. The operator H measures the mass of the BPS state.

Let I_g denote the $SU(2)$ representation associated to the cohomology of an abelian variety of dimension g ,

$$I_g = \left(2[0] + \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right)^{\otimes g} .$$

By taking the trace on I_g , we obtain the following identity:

$$\text{Tr}_{I_g} y^{\sigma_3} = \left(y^{\frac{1}{2}} + y^{-\frac{1}{2}} \right)^{2g} .$$

Setting $u = -1$ in (5.2) and using (5.3), we obtain the basic relationship between the refined and unrefined invariants:

$$(5.4) \quad \sum_{j_L, j_R \in \frac{1}{2}\mathbb{Z}_{\geq 0}} (-1)^{2j_R} (2j_R + 1) \mathbf{N}_{j_L, j_R}^{\beta} [j_L] = \sum_{g \in \mathbb{Z}_{\geq 0}} \mathfrak{n}_g^{\beta} I_g .$$

If we specialize (5.4) to the refined invariants \mathbf{R}_{j_L, j_R}^h of $K3$ surfaces and change variables

$$y \mapsto -y$$

in (5.1), we obtain

$$\sum_{j_L, j_R \in \frac{1}{2}\mathbb{Z}_{\geq 0}} (2j_R + 1) \mathbf{R}_{j_L, j_R}^h [j_L]_y = \sum_{g \in \mathbb{Z}_{\geq 0}} (-1)^g r_g^h \left(y^{\frac{1}{2}} - y^{-\frac{1}{2}} \right)^{2g}$$

and recover the KKV formula for the BPS counts r_g^h of $K3$ surfaces from (5.1),

$$\sum_{h=0}^{\infty} \sum_{g \geq 0} (-1)^g r_g^h \left(y^{\frac{1}{2}} - y^{-\frac{1}{2}} \right)^{2g} q^h = \prod_{n=1}^{\infty} \frac{1}{(1 - y^{-1}q^n)(1 - yq^n)(1 - q^n)^{20}(1 - y^{-1}q^n)(1 - yq^n)},$$

in accordance with [25, 31].

r_g^h	$h = 0$	1	2	3	4
$g = 0$	1	24	324	3200	25650
1		-2	-54	-800	-8550
2			3	88	1401
3				-4	-126
4					5

TABLE 2. All nonvanishing r_g^h for $h \leq 4$ for $K3$ surfaces

5.3. Mathieu moonshine. The important conceptual difference between the invariants $\mathbf{N}_{j_L, j_R}^\beta$ and \mathbf{n}_g^β is that the former are actual BPS degeneracies. Hence, we expect $\mathbf{N}_{j_L, j_R}^\beta$ to always be nonnegative. Moreover, if there is a symmetry acting on the Hilbert space which commutes with H and the spin operators, the multiplicities must fall in representations of the symmetry group. For E_n del Pezzo surfaces, the invariants $\mathbf{N}_{j_L, j_R}^\beta$ were observed in [15] to fall naturally in representations of E_n . The Weyl group of E_n acts on the geometry by Lefschetz monodromy.

Greg Moore pointed out to us at *String Math 2014* during the presentation [19] that the number $\mathbf{R}_{0,0}^2 = 231$ is the dimension of an irreducible representation of the Mathieu group \mathbf{M}_{24} , a sporadic group of order

244823040, which is conjecturally [11] related to the elliptic genus of the $K3$ surface. The dimensions of the 26 irreducible representations of the group M_{24} are

$$1, 23, 45, 231, 252, 253, 483, 770, 990, 1035, 1265, \\ 1771, 2024, 2277, 3312, 3520, 5313, 5544, 5796, 10395,$$

where the representations of dimension 45, 231, 770, 990, 1035 come in complex conjugated pairs. There is an extra real representation of dimension 1035.

We have checked all the values taken by refined invariants R_{j_L, j_R}^h in Table 1, except for the values 20 (and $21 = 20 + 1$), are expressible in a simple way in terms of the dimensions of the irreducible representations of M_{24} . Beside the dimensions which occur directly, we have:

$$\begin{aligned} R_{0,0}^3 &= 1981 = 2 \cdot 990 + 1 & R_{0,0}^4 &= 13938 = 2 \cdot 5313 + 3312 \\ R_{\frac{1}{2}, \frac{1}{2}}^4 &= 2233 = 2 \cdot 990 + 253 & R_{1,1}^5 &= 2254 = 1771 + 483 \\ R_{\frac{3}{2}, \frac{3}{2}}^6 &= 2255 = 1265 + 990. \end{aligned}$$

Since the dimensions of the representations of M_{24} are small, the significance is somewhat limited. The decompositions with a minimal numbers of summands given above are not always unique. For example,

$$R_{\frac{3}{2}, \frac{3}{2}}^6 = 2024 + 231 \quad \text{and} \quad R_{0,0}^4 = 10395 + 3520 + 23.$$

Nevertheless, the action of the Mathieu group is expected¹⁰ in the string compactification on $K3 \times T^2$ and the refined spacetime BPS spectrum is a natural place to see its action.

6. CONJECTURE C

Our motivic convention is the following.¹¹ If the moduli space of stable pairs P is a nonsingular variety of complex dimension d , then

¹⁰See [5, 12] for related constructions and [6] for subsequent developments.

¹¹Our conventions here distribute the signs in a slightly different manner than the conventions of [7], but there is no essential difference.

the associated motivic invariant is defined¹² to be

$$\mathbf{L}^{-\frac{d}{2}}[P] \in \mathbf{K}_{\text{var}}^{\widehat{\mu}}[\mathbf{L}^{-1}] .$$

Here, $[P]$ is the usual element associated to P in the Grothendieck ring of varieties. When considering the virtual Poincaré polynomial,

$$\widetilde{\mathbf{H}} : \mathbf{K}_{\text{var}}^{\widehat{\mu}}(\mathbf{L}^{-1}) \rightarrow \mathbb{Z}[u, u^{-1}] ,$$

we map $\mathbf{L}^{\pm\frac{1}{2}} \mapsto u^{\pm 1}$. We define $\mathbf{H}(P)$ to be the virtual Poincaré polynomial of $\mathbf{L}^{-\frac{d}{2}}[P]$.

Let $\alpha \in \text{Pic}(S)$ be a positive irreducible class of norm square

$$\langle \alpha, \alpha \rangle = 2h - 2 .$$

Then $P_n(S, h)$ is nonsingular of dimension $2h - 1 + n$. Hence, the relation

$$\mathbf{H}(P_n(S, h)) = u^{-2h+1-n} \widetilde{\mathbf{H}}(P_n(S, h))$$

of Section 4 is consistent with our motivic conventions.

An elementary verification based upon the interpretation of the Kawai-Yoshioka calculation in Section 4 and the definition of the refined invariants in Section 5 yields the following identity: the v^α coefficient of the product

$$(6.1) \quad \prod_{j_L, j_R, m_L, m_R, m, j} (1 + u^{-m+1+2j-2m_R} y^{m-2m_L} v^\alpha)^{(-1)^{2(j_L+j_R)} \mathbf{R}_{j_L, j_R}^h}$$

exactly equals

$$(6.2) \quad \sum_{n=1-h}^{\infty} \mathbf{H}(P_n(S, h)) y^n .$$

The product $\prod_{j_L, j_R, m_L, m_R, m, j}$ in (6.1) signifies

$$(6.3) \quad \prod_{j_L \in \frac{1}{2}\mathbb{Z}_{\geq 0}} \prod_{j_R \in \frac{1}{2}\mathbb{Z}_{\geq 0}} \prod_{m_L = -j_L}^{j_L} \prod_{m_R = -j_R}^{j_R} \prod_{m=1}^{\infty} \prod_{j=0}^{m-1}$$

where m_L and m_R increase by steps of 1.

¹²The motivic invariant of P may depend on a choice of orientation. We have made an implicit choice here as explained in Section (7.1) – the only choice possible if $\text{Pic}(P)$ has no 2-torsion.

The product (6.1) occurs in the definition of the refined invariants [7, Equation (8.1)]. The equality of (6.1) and (6.2) is a geometric constraint verified by definition (5.1). In fact, definition (5.1) is uniquely determined by the above constraint.

To state our last conjecture, let $\alpha \in \text{Pic}(S)$ be a positive, primitive class of norm square

$$\langle \alpha, \alpha \rangle = 2h - 2.$$

We will consider the motivic partition function for classes which are multiples of α ,

$$Z_h = \exp \left(\sum_{k=1}^{\infty} \sum_{n \in \mathbb{Z}} H_{n, k\alpha} y^n v^{k\alpha} \right).$$

For fixed k , the motivic invariant $H_{n, k\alpha}$ vanishes for sufficiently negative n . Assuming Conjectures A and B, we rewrite the partition function as

$$Z_h = \exp \left(\sum_{k=1}^{\infty} \sum_{n \in \mathbb{Z}} H_{n, k, h[k]} y^n v^k \right)$$

where we define

$$2h[k] - 2 = \langle k\alpha, k\alpha \rangle = k^2(2h - 2), \quad h[k] = k^2(h - 1) + 1.$$

The variable v^α has now been replaced by just v .

Conjecture C. *For all h , the partition function*

$$Z_h = \exp \left(\sum_{k=1}^{\infty} \sum_{n \in \mathbb{Z}} H_{n, k, h[k]} y^n v^k \right)$$

equals the product

$$\prod_{k=1}^{\infty} \prod_{j_L, j_R, m_L, m_R, m, j} (1 + u^{-m+1+2j-2m_R} y^{m-2m_L} v^k)^{(-1)^{2(j_L+j_R)} R_{j_L, j_R}^{h[k]}}.$$

The product $\prod_{j_L, j_R, m_L, m_R, m, j}$ appearing in Conjecture C is just as before (6.3). Conjecture C determines every $H_{n, k, h[k]}$ in terms of the

refined Gopakumar-Vafa invariants obtained from primitive class geometry. Such a relation may be viewed as a divisibility invariance property.

If we substitute $u = -1$ in $\mathbf{H}_{n,k\alpha}$, we recover the stable pairs invariants $R_{n,k\alpha}$ defined in [31] for $K3$ surfaces. An unwinding of the definitions then shows Conjecture C implies the KKV conjecture (proven in [31]) for stable pairs invariants in all classes.

7. FIRST PREDICTIONS

7.1. Virtual motives. Before presenting examples, we quickly review the theory of virtual motives following [4, 16].

Joyce and collaborators introduce the notion of an *oriented d -critical locus* as a framework for defining motivic invariants within classical (non-derived) algebraic geometry. Moduli spaces of stable pairs carry such a structure. We review the aspects which are most relevant for us and refer the reader to [16] for the omitted details.

A *d -critical locus* is a variety M which can locally be realized as $\text{Crit}(f)$ for a holomorphic f on a smooth space U (a *critical chart*), with a weak notion of compatibility among the critical charts. The compatibility is strong enough however to define a *virtual canonical bundle*

$$K_M^{\text{vir}} \in \text{Pic}(M^{\text{red}}).$$

Given a critical chart (U, f) , there is a canonical isomorphism

$$(7.1) \quad K_M^{\text{vir}}|_{\text{Crit}(f)} \simeq K_U^{\otimes 2}|_{\text{Crit}(f)}.$$

An *orientation* is a choice of square root of the virtual canonical bundle

$$(K_M^{\text{vir}})^{1/2} \in \text{Pic}(M^{\text{red}}).$$

From the data of a oriented d -critical locus, a virtual motive $[M]^{\text{vir}}$ can be defined. There are two ingredients:

- The motivic vanishing cycle of Denef and Loeser [8],
- A principal \mathbb{Z}_2 bundle determined by the choice of orientation.

A local virtual motive can be associated to the motivic vanishing cycle. After a motivic twist by the principal \mathbb{Z}_2 bundle, Joyce and collaborators [16] show the local motives glue together.

We review the motivic vanishing cycle following [8]. For our examples, we will only require the case where f is of the form

$$f = \prod_{i=1}^n z_i^{n_i}.$$

Here (z_1, \dots, z_n) are coordinates in a neighborhood U of the origin in \mathbb{C}^n . Put $U_0 = f^{-1}(0)$.

For an index set $I \subset \{1, \dots, n\}$ define E_I by the equations $z_i = 0$ for all $i \in I$, and define

$$E_I^\circ = E_I - \cup_{j \notin I} E_j.$$

Let $m_I = \gcd\{n_i \mid i \in I\}$ and define $\tilde{E}_I \rightarrow E_I^\circ$ by

$$\tilde{E}_I = \left\{ (z, w) \in E_I^\circ \times \mathbb{C} \mid w^{m_I} = \prod_{j \notin I} z_j^{n_j} \right\}$$

with a natural projection to E_I° .

The group μ_{m_I} of roots of unity acts on \tilde{E}_I by its action on w . In fact, \tilde{E}_I is a Galois μ_{m_I} -cover of E_I° . Denoting the action by ρ_I , we obtain an element $[\tilde{E}_I, \rho_I]$ in the ring of equivariant motives over U . The motivic nearby cycle of f is

$$(7.2) \quad \text{MF}_{U,f}^{\text{mot}} = \sum_{I \neq \emptyset} (1 - \mathbf{L})^{|I|-1} [\tilde{E}_I, \rho_I],$$

where $\mathbf{L} = [\mathbb{A}^1]$ as before.

The *motivic vanishing cycle* of f is

$$\text{MF}_{U,f}^{\text{mot},\phi} = \mathbf{L}^{-\dim U/2} ([U_0] - \text{MF}_{U,f}^{\text{mot}})$$

We only need the motivic vanishing cycle in three special cases:

- $f = 0$
- $f = z_1^2 z_2^2$
- $f = z_1^2 z_2$

In case $f = 0$, we have $X = U_0 = U$ is nonsingular, $\mathrm{MF}_{U,f}^{\mathrm{mot}}$ is empty, and so the motivic vanishing cycle is $\mathbf{L}^{-\dim U/2}[U]$. Here, we match the conventions of Section 6.

In case $f = z_1^2 z_2^2$, for every nonempty $I \subset \{1, 2\}$, we have $m_I = 2$ and \tilde{E}_I is a disconnected double cover of E_I° . So each summand in the motivic nearby cycle (7.2) is the product of the respective E_I° with the absolute equivariant motive

$$[\mu_2, \rho]$$

where ρ denotes the action of μ_2 on itself. We therefore obtain

$$\mathrm{MF}_{U,f}^{\mathrm{mot}} = [\mu_2, \rho] ([E_1^\circ] + [E_2^\circ] + (1 - \mathbf{L}) [E_{12}^\circ]).$$

Using (2.2), we find the following expression:

$$\begin{aligned} \mathrm{MF}_{U,f}^{\mathrm{mot},\phi} = \\ \mathbf{L}^{-\dim U/2} (\mathbf{L}^{1/2} ([E_1^\circ] + [E_2^\circ]) + (1 - (1 - \mathbf{L}^{1/2})(1 - \mathbf{L})) E_{12}^\circ) \end{aligned}$$

which simplifies to

$$(7.3) \quad \mathbf{L}^{-\dim M/2} ([M] + [E_{12}^\circ] (\mathbf{L}^{1/2} - \mathbf{L})) .$$

In case $f = z_1^2 z_2$, we have $m_1 = 2$ and $m_I = 1$ otherwise. Thus,

$$\mathrm{MF}_{U,f}^{\mathrm{mot}} = [\tilde{E}_1, \rho_1] + [E_2^\circ] + (1 - \mathbf{L}) [E_{12}^\circ]$$

and

$$(7.4) \quad \mathrm{MF}_{U,f}^{\mathrm{mot},\phi} = \mathbf{L}^{-\dim U/2} \left([E_1^\circ] - [\tilde{E}_1, \rho_1] + \mathbf{L}[E_{12}^\circ] \right)$$

As expected, E_2° has cancelled out (E_2° is not part of $\mathrm{Crit}(f)$).

The principal \mathbb{Z}_2 bundle associated with a choice of orientation is given by the local isomorphisms

$$(K_M^{\mathrm{vir}})^{1/2} |_{\mathrm{Crit}(f)} \simeq K_U |_{\mathrm{Crit}(f)}$$

which are square roots of the canonical isomorphism (7.1).

In the case $f = 0$ or $f = z_1^2 z_2^2$, the principal bundle \mathbb{Z}_2 bundle is trivial in a punctured neighborhood of each E_I° , essentially since there is no ramification in the Galois μ_2 covers described above. As we

shall see, the principal \mathbb{Z}_2 bundle plays an important role in the case $f = z_1^2 z_2$.

Next we globalize the $f = 0$ geometry. Suppose M is nonsingular. As a d -critical locus, M can be described by a single critical chart $(M, 0)$, so

$$K_M^{\text{vir}} \cong K_M^{\otimes 2}.$$

For the natural choice of orientation K_M , we have

$$[M]^{\text{vir}} = \mathbb{L}^{-\dim M/2}[M],$$

If there is no 2-torsion in $\text{Pic}(M)$, then K_M is the only possible orientation, as will be the case in our example.

In a second geometry which will arise, M^{red} is a union of two nonsingular components E_1, E_2 meeting transversally along a nonsingular irreducible divisor E_{12} . In such case, M *must* be nonreduced. We will also have $E_1 - E_2$ and $E_2 - E_1$ nonsingular, so nilpotents can occur only along E_{12} . We *assume* the simplest possible scheme structure compatible with the situation: $(z_1^2 z_2, z_1 z_2^2)$ in the neighborhood of any point of any E_{ij} , where $z_1 = 0$ and $z_2 = 0$ are local equations for E_1 and E_2 respectively. In other words, we take

$$f = z_1^2 z_2^2$$

as the superpotential.

Consider the natural isomorphism

$$K_M^{\text{vir}}|_{E_1^\circ} \cong K_{E_1^\circ}^{\otimes 2}$$

These bundles extend to respective line bundles $K_M^{\text{vir}}|_{E_1}$ and $K_{E_1}^{\otimes 2}$ on E_1 . Direct computation shows the isomorphism vanishes to order 2 along E_{12} . So

$$K_M^{\text{vir}}|_{E_1} \cong K_{E_1}^{\otimes 2}(-2E_{12}),$$

with the analogous identification on E_2 .

So there is again a natural orientation $(K_M^{\text{vir}})^{1/2}$ determined by

$$(K_M^{\text{vir}})^{1/2}|_{E_i} \cong K_{E_i}(-E_{12})$$

for $i = 1, 2$. If in addition there is no 2-torsion in $\text{Pic}(E_1)$ or $\text{Pic}(E_2)$, then the orientation is unique. Such uniqueness will occur in our example.

With the unique orientation, we have a globalization of (7.3),

$$(7.5) \quad [M]^{\text{vir}} = \mathbb{L}^{-\dim M/2}([M] + [E_{12}] (\mathbb{L}^{1/2} - \mathbb{L})).$$

A third geometry will arise: M^{red} is irreducible, nonsingular, and contains a nonsingular divisor $D \subset M$ *precisely* along which M is nonreduced. We *assume* the simplest possible scheme structure compatible with the situation, $(z_1^2, z_1 z_2)$ in the neighborhood of any point of D , where $z_1 = 0$ is a local equation for M^{red} and $z_1 = z_2 = 0$ are local equations for D . In other words, we take

$$f = z_1^2 z_2$$

as the superpotential,¹³ locally identifying E_1 with M^{red} and E_{12} with D .

Consider the natural isomorphism

$$K_M^{\text{vir}}|_{M^{\text{red}}-D} \cong K_{M^{\text{red}}-D}^{\otimes 2}$$

These bundles extend to respective line bundles K_M^{vir} and $K_{M^{\text{red}}}^{\otimes 2}$ on M^{red} . Direct computation shows the isomorphism vanishes to order 1 along D . So

$$(7.6) \quad K_M^{\text{vir}} \cong K_{M^{\text{red}}}^{\otimes 2}(-D).$$

It is apparent that there is no natural orientation $(K_M^{\text{vir}})^{1/2}$ in this general situation.

For the moduli space of stable pairs, we know that M is an oriented d -critical locus by general theory. Hence, we conclude D must be even:

$$\mathcal{O}_{M^{\text{red}}}(D) \simeq L^{\otimes 2}$$

¹³The superpotential $z_1^2 z_2$ together with the associated \mathbb{Z}_2 -monodromy was first analyzed in [33, Example 4.5] to calculate nontrivial refined stable pairs invariants of local \mathbb{P}^1 .

for some line bundle L on M^{red} .¹⁴ Then, we have an orientation

$$(K_M^{\text{vir}})^{1/2} = K_{M^{\text{red}}} \otimes L^{-1},$$

which is the only possibility if $\text{Pic}(M)$ has no 2-torsion.

In the above oriented situation, the principal \mathbb{Z}_2 bundle of square roots of (7.1) in a critical chart naturally ramifies when extended to D .

Let

$$\pi : \widetilde{M} \rightarrow M$$

be the double cover of M branched along D , with the natural involution ι and ramification divisor \widetilde{D} . The principal \mathbb{Z}_2 bundle modifies the local virtual motive $\mathbf{L}^{-\dim M/2}[M - D]$ of $M - D$ to¹⁵

$$\mathbf{L}^{-(\dim M+1)/2} \left([M - D] - [\widetilde{M} - \widetilde{D}, \iota] \right).$$

Comparison with (7.4) shows how to extend the motive globally. The result is

$$(7.7) \quad \mathbf{L}^{-(\dim M+1)/2} \left([M - D] - [\widetilde{M} - \widetilde{D}, \iota] + \mathbf{L}[D] \right)$$

We now compute the virtual Poincaré polynomial of (7.7). To convert a \mathbb{Z}_2 -equivariant motive $[V, \iota]$ to a virtual Poincaré polynomial, decompose $H_c^*(V)$ into its even and odd parts under ι^* :

$$H_c^*(V) = H_c^+(V) \oplus H_c^-(V)$$

and then take the virtual Poincaré polynomial, which we write as

$$\widetilde{H}(V) = \widetilde{H}_c^+(V) + \widetilde{H}_c^-(V)$$

In our conventions the virtual Poincaré polynomial of $[V, \iota]$ is then

$$(7.8) \quad \widetilde{H}(V, \iota) = \widetilde{H}_c^+(V) - u\widetilde{H}_c^-(V),$$

¹⁴The same calculation was applied in [16, Example 2.39] to the situation

$$(M^{\text{red}}, D) = (\mathbb{P}^1, p)$$

to show that a certain d -critical locus was *not* orientable since the class of a point is not even in $\text{Pic}(\mathbb{P}^1)$.

¹⁵In [4], the \mathbb{Z}_2 twists are only defined in a quotient of the equivariant motivic ring. We presume the computation holds in the equivariant motivic ring itself if other approaches to virtual motives are followed [23].

see [24, Corollary 7.2]. The minus sign in (7.8) is consistent with the evaluation

$$\tilde{H}(\mathbb{L}^{1/2}) = \tilde{H}(1 - [\mu_2, \rho]) = u.$$

Finally, the virtual Poincaré polynomial of (7.7) is

$$(7.9) \quad u^{-(\dim M+1)} \left(u\tilde{H}_c^-(\tilde{M}) + u^2\tilde{H}_c(D) \right).$$

7.2. Poincaré polynomials. Some elucidation of (7.8) is in order here. Let $K_0^{\hat{\mu}}(\text{HS})$ denote the Grothendieck ring of the category of Hodge structures with a $\hat{\mu}$ -action. There is a Hodge characteristic map [24],

$$\chi_h : K_{\text{var}}^{\hat{\mu}} \rightarrow K_0^{\hat{\mu}}(\text{HS}),$$

which can be extended to

$$(7.10) \quad \chi_h : K_{\text{var}}^{\hat{\mu}}[\mathbb{L}^{-1}] \rightarrow K_0^{\hat{\mu}}(\text{HS})$$

since $\chi_h(\mathbb{L})$ is invertible. We will define a *virtual Poincaré polynomial* map

$$P : K_0^{\hat{\mu}}(\text{HS}) \rightarrow \mathbb{Z}[u, u^{-1}]$$

which then determines a virtual Poincaré polynomial¹⁶ map

$$\tilde{H} = P \circ \chi_h : K_{\text{var}}^{\hat{\mu}}[\mathbb{L}^{-1}] \rightarrow \mathbb{Z}[u, u^{-1}].$$

The definition of P is chosen so that \tilde{H} is a ring homomorphism.

For simplicity of exposition, we focus on the special case of μ_2 -equivariant Hodge structures (the only case which appears in the examples considered here) and say a few words about how to extend to the general case.

By [24, Cor. 7.2], the motivic convolution product in $K_{\text{var}}^{\hat{\mu}}$ descends to a product in $K_0^{\hat{\mu}}(\text{HS})$, denoted $*$. Restricting to elements of $K_0^{\mu_2}(\text{HS})$, the product is shown to satisfy

$$(7.11) \quad H * H' = H^+ \otimes (H')^+ + H^+ \otimes (H')^- + H^- \otimes (H')^+ + H^- \otimes (H')^-(-1),$$

¹⁶Equivalent definitions have appeared before, see [34, Appendix A.4] and the references there.

where H^\pm are the even and odd parts of the μ_2 -action on H (and similarly for H').

Forgetting the μ_2 action, we let $Q(H)$ be the Poincaré polynomial of a Hodge structure H . Equation (7.8) can be rephrased as defining P to be

$$P(H) = Q(H^+) - uQ(H^-).$$

Then (7.11) implies $P(H * H')$ is

$$Q(H^+ \otimes (H')^+) - uQ(H^+ \otimes (H')^-) - u(H^- \otimes (H')^+) + u^2Q(H^- \otimes (H'))$$

which equals $P(H)P(H')$ as desired.

The full result [24, Cor. 7.2] extends (7.11) to a formula for $*$ valid for any

$$H, H' \in K_0^{\widehat{\mu}}(\text{HS}),$$

expressed in terms of characters of $\widehat{\mu}$. Then $P(H)$ can be defined by $Q(H)$ for H a trivial representation of $\widehat{\mu}$ and $-uQ(H)$ if H transforms by a nontrivial character of $\widehat{\mu}$. The verification of

$$P(H * H') = P(H)P(H')$$

in the general case is a bit more involved, relying on the precise form of [24, Cor. 7.2] and of the computation of the characters of the cohomology of the Fermat curves used in the definition of motivic convolution [32]. The result from [32] also appears as [24, Lemma 7.1].

7.3. Elliptically fibered K3 surfaces. Let S be an elliptically fibered K3 surface,

$$\pi : S \rightarrow \mathbb{P}^1,$$

with section s and fiber class f . No singular point of any fiber lies on s . We will compute motivic stable pair invariants in classes

$$s, f, s + f \in \text{Pic}(S)$$

with small Euler characteristic following the method of [16]. For our definition in Section 2, we will consider various families

$$\epsilon : T \rightarrow (\Delta, 0)$$

depending upon the class.

We start with the fiber class $f \in \text{Pic}(S)$. If (F, τ) is a stable pair of class $[F] = f$ and $\chi(F) = 0$, then $F = \mathcal{O}_E$ for some fiber E of π . Since the fibers are parametrized by \mathbb{P}^1 ,

$$P_0(S, f) \simeq \mathbb{P}^1$$

and

$$[P_0(S, f)]^{\text{vir}} = \mathbf{L}^{-1/2}[\mathbb{P}^1].$$

Note $\text{Pic}(\mathbb{P}^1)$ has no torsion, so there is no choice in the motivic invariant.

If $\chi(F) = 1$, then the coker(τ) of the stable pair (F, τ) is a point, and $P_1(S, f) \cong S$, so

$$[P_1(S, f)]^{\text{vir}} = \mathbf{L}^{-1}[S].$$

Again, $\text{Pic}(S)$ has no torsion, so there is no choice in the motivic invariant.

Next consider the section $s \in \text{Pic}(S)$. We see $P_1(S, s)$ is a point and

$$P_2(S, s) \cong \mathbb{P}^1 .$$

Hence, the coefficient of v^s in the motivic partition function Z^{mot} is

$$q + \mathbf{L}^{-1/2}[\mathbb{P}^1]q^2 + \dots .$$

Since $s, f \in \text{Pic}(S)$ are irreducible classes, for any 1-rigid family ϵ , the above moduli identifications are valid on T .

The class $s + f$ is primitive but *not* irreducible. In the Euler characteristic 0 case,

$$P_0(S, s + f) \cong \mathbb{P}^1$$

is determined by the location of a fiber of π . Hence,

$$[P_0(S, s + f)]^{\text{vir}} = \mathbf{L}^{-1/2}[\mathbb{P}^1] .$$

Again, T plays no interesting role.

The more interesting geometry occurs in Euler characteristic 1. The moduli space $P_1(T, s + f)$ has two components, E_1 where the point is on the fiber and E_2 where the point is on s :

- $E_1 \cong S$
- $E_2 \cong \mathbb{P}^1 \times \mathbb{P}^1$

For E_2 , the first \mathbb{P}^1 parametrizes the point of the fiber and the second \mathbb{P}^1 parametrizes the location of the point $\text{coker}(\tau)$ on the section. The two components meet along

$$E_1 \cap E_2 \cong \mathbb{P}^1$$

embedded in S as the section s and in $\mathbb{P}^1 \times \mathbb{P}^1$ as the diagonal.

We assume the local superpotential is the second form discussed in Section 7.1. By (7.5), we have

$$[P_1(X, s + f)]^{\text{vir}} = \mathbf{L}^{-1} ([P_1(X, s + f)] + [\mathbb{P}^1](\mathbf{L}^{1/2} - \mathbf{L}))$$

For the classes $s, f, s + f \in \text{Pic}(S)$ to order q , the motivic partition function is

$$\begin{aligned} Z^{\text{mot}} &= 1 + v^s (q + \dots) \\ &\quad + v^f (\mathbf{L}^{-1/2}[\mathbb{P}^1] + q\mathbf{L}^{-1}[S] + \dots) \\ &\quad + v^{s+f} (\mathbf{L}^{-1/2}[\mathbb{P}^1] + q\mathbf{L}^{-1}([P_1(X, s + f)] + [\mathbb{P}^1](\mathbf{L}^{1/2} - \mathbf{L})) + \dots) \\ &\quad + \dots \end{aligned}$$

We now calculate the coefficient of v^{s+f} in $\log(Z^{\text{mot}})$. The q^0 coefficient is simply $\mathbf{L}^{-1/2}[\mathbb{P}^1]$. The q coefficient is

$$\begin{aligned} &\mathbf{L}^{-1} \left([P_1(X, s + f)] + [\mathbb{P}^1] (\mathbf{L}^{1/2} - \mathbf{L}) \right) - \mathbf{L}^{-1/2}[\mathbb{P}^1] \\ &= \mathbf{L}^{-1} \left([S] + [\mathbb{P}^1 \times \mathbb{P}^1] - [\mathbb{P}^1] + [\mathbb{P}^1] (\mathbf{L}^{1/2} - \mathbf{L}) \right) - \mathbf{L}^{-1/2}[\mathbb{P}^1] \\ &= \mathbf{L}^{-1}[S] + \mathbf{L}^{-1}[\mathbb{P}^1]([\mathbb{P}^1] - 1 + \mathbf{L}^{1/2} - \mathbf{L}) - \mathbf{L}^{-1/2}[\mathbb{P}^1] . \end{aligned}$$

The last expression is easily simplified to $\mathbf{L}^{-1}[S]$. We conclude the q^0 and q^1 coefficients of v^{s+f} in $\log(Z^{\text{mot}})$ agree with the q^0 and q^1 coefficients of v^f in $\log(Z^{\text{mot}})$.

The above calculation provides nontrivial evidence for Conjectures A and B. In fact, if more naive approaches to the motivic theory are taken (for example using the actual moduli spaces or even the Behrend function on the moduli spaces), the agreement we have found fails. We

have verified the prediction to order q^2 , but we do not include the more involved calculations here.

We next turn to the class $2f$. We have

$$P_0(S, 2f) \cong \text{Sym}^2(\mathbb{P}^1) \cong \mathbb{P}^2 .$$

However, in the algebraic twistor family, the scheme structure is *not* reduced¹⁷ precisely along the diagonal curve $D \subset \mathbb{P}^2$, a plane conic. The discussion of Section 7.1 therefore applies. The virtual canonical bundle is

$$K_{\mathbb{P}^2}^{\otimes 2}(-D) \cong \mathcal{O}_{\mathbb{P}^2}(-8) ,$$

and the algebraic twistor family is uniquely oriented by $\mathcal{O}_{\mathbb{P}^2}(-4)$.

The double cover of \mathbb{P}^2 branched along D is a nonsingular quadric surface Q containing an isomorphic copy \tilde{D} of D . The motive of $D \cong \tilde{D}$ is $L + 1$, so the motive of $Q - \tilde{D}$ is $L^2 + L$ and the motive of $\mathbb{P}^2 - D$ is L^2 . Therefore the odd part of the motive of $Q - \tilde{D}$ is L . Hence (7.9) yields

$$\mathbf{H}_{0,2,0[2]} = \tilde{\mathbf{H}}([P_0(X, 2f)]^{\text{vir}}) = 1 + (u + u^{-1}) ,$$

in complete agreement with Conjecture C.

8. DUALITY AND NOETHER-LEFSCHETZ THEORY

8.1. Heterotic-Type II duality. The Yau-Zaslow conjecture originates in heterotic-Type II duality in 6d, where the heterotic string is compactified on the four torus T^4 and the Type II string on $K3$. By the adiabatic argument [37], this can be extended to 4d $N = 2$ supersymmetric theories, which are obtained from *dual pairs* of heterotic string compactifications on $K3 \times T^2$ and Type II string compactifications on Calabi-Yau 3-folds X . The latter are $K3$ fibrations over \mathbb{P}^1 . This construction of dual pairs requires a match between the vector- and hypermultiplet moduli spaces of the heterotic and the type II compactifications. The heterotic moduli parametrize the metric of $K3 \times T^2$, the bundle data of the heterotic compactification, and the

¹⁷We thank R. Thomas for the verification.

heterotic dilaton S , which is in a vector multiplet. In Type IIA compactifications, the complexified Kähler moduli (t, S) of X parametrize the vector multiplet moduli space¹⁸, which is of complex dimension $h^{11}(X)$. In particular, the heterotic dilaton

$$S = \frac{4\pi}{g_{het}^2} + i\theta$$

is identified with the complexified volume of the base \mathbb{P}^1 .

The simplest example is the STU-model, see [22] for review. Here, the Calabi-Yau 3-fold X is an elliptic fibration over $\mathbb{P}^1 \times \mathbb{P}^1$ and a $K3$ fibration over \mathbb{P}^1 with

$$h^{11}(X) = 3 .$$

The three vector moduli are identified on the heterotic side with the heterotic dilaton S , the complex modulus T , and the complexified Kähler modulus U of the heterotic torus T^2 ¹⁹.

An impressive consequence of the proposed duality is that a perturbative heterotic one-loop amplitude predicts *all higher* genus amplitudes

$$F(\lambda, t) = \sum_g \lambda^{2g-2} F_g(t)$$

of X in the infinite base limit (the dependence on the $K3$ fiber classes). For the STU model,

$$(8.1) \quad \lim_{S \rightarrow \infty} F(\lambda, S, T, U) = F_{het}^{1-loop}(\lambda, T, U) .$$

We will use this relation in the holomorphic limit to extend Conjecture C to a proposal for the refined invariants of the STU model.

8.2. Refined Noether-Lefschetz theory.

¹⁸The complex moduli of X together with the Ramond fields and the type II dilaton parametrize the hypermultiplet moduli space, which is of quaternionic dimension $h^{21}(X) + 1$. This makes the duality much richer, but we focus on the vector moduli.

¹⁹As $h^{21}(X) = 243$, the heterotic hypermultiplet moduli space is of quaternionic dimension 244.

8.2.1. *Overview.* We pass now from the string point of view to the more precise mathematical perspective advanced in [25, 31] as Noether-Lefschetz correspondences. For our study of refined invariants, the Noether-Lefschetz numbers of [25, 31] also require refinement.

8.2.2. Λ -*polarization.* Following the notation of [31, Section 1.1], let Λ be a fixed rank r primitive²⁰ sublattice

$$\Lambda \subset U \oplus U \oplus U \oplus E_8(-1) \oplus E_8(-1)$$

with signature $(1, r - 1)$, and let $v_1, \dots, v_r \in \Lambda$ be an integral basis. The discriminant is

$$\Delta(\Lambda) = (-1)^{r-1} \det \begin{pmatrix} \langle v_1, v_1 \rangle & \cdots & \langle v_1, v_r \rangle \\ \vdots & \ddots & \vdots \\ \langle v_r, v_1 \rangle & \cdots & \langle v_r, v_r \rangle \end{pmatrix}.$$

The sign is chosen so $\Delta(\Lambda) > 0$.

A Λ -*polarization* of a $K3$ surface S is a primitive embedding

$$j : \Lambda \rightarrow \text{Pic}(S)$$

satisfying two properties:

- (i) the lattice pairs $\Lambda \subset U^3 \oplus E_8(-1)^2$ and $\Lambda \subset H^2(S, \mathbb{Z})$ are isomorphic via an isometry which restricts to the identity on Λ ,
- (ii) $\text{Im}(j)$ contains a quasi-polarization.

By (ii), every Λ -polarized $K3$ surface is algebraic.

The period domain M of Hodge structures of type $(1, 20, 1)$ on the lattice $U^3 \oplus E_8(-1)^2$ is an analytic open set of the 20-dimensional nonsingular isotropic quadric Q ,

$$M \subset Q \subset \mathbb{P}((U^3 \oplus E_8(-1)^2) \otimes_{\mathbb{Z}} \mathbb{C}).$$

Let $M_\Lambda \subset M$ be the locus of vectors orthogonal to the entire sublattice $\Lambda \subset U^3 \oplus E_8(-1)^2$.

Let Γ be the isometry group of the lattice $U^3 \oplus E_8(-1)^2$, and let

$$\Gamma_\Lambda \subset \Gamma$$

²⁰A sublattice is primitive if the quotient is torsion free.

be the subgroup restricting to the identity on Λ . By global Torelli, the moduli space \mathcal{M}_Λ of Λ -polarized $K3$ surfaces is the quotient

$$\mathcal{M}_\Lambda = M_\Lambda/\Gamma_\Lambda.$$

We refer the reader to [10] for a detailed discussion.

8.2.3. *Noether-Lefschetz divisors.* Let (\mathbb{L}, ι) be a rank $r + 1$ lattice \mathbb{L} with an even symmetric bilinear form $\langle \cdot, \cdot \rangle$ and a primitive embedding

$$\iota : \Lambda \rightarrow \mathbb{L}.$$

Two data sets (\mathbb{L}, ι) and (\mathbb{L}', ι') are isomorphic if and only if there exist an isometry relating \mathbb{L} and \mathbb{L}' which takes ι to ι' . The first invariant of the data (\mathbb{L}, ι) is the discriminant $\Delta \in \mathbb{Z}$ of \mathbb{L} .

An additional invariant of (\mathbb{L}, ι) can be obtained by considering any vector $v \in \mathbb{L}$ for which²¹

$$(8.2) \quad \mathbb{L} = \iota(\Lambda) \oplus \mathbb{Z}v.$$

The pairing

$$\langle v, \cdot \rangle : \Lambda \rightarrow \mathbb{Z}$$

determines an element of $\delta_v \in \Lambda^*$. Let $G = \Lambda^*/\Lambda$ be the quotient defined via the injection $\Lambda \rightarrow \Lambda^*$ obtained from the pairing $\langle \cdot, \cdot \rangle$ on Λ . The group G is abelian of order given by the discriminant $|\Delta(\Lambda)|$. The image

$$\delta \in G/\pm$$

of δ_v is easily seen to be independent of v satisfying (8.2). The invariant δ is the *coset* of (\mathbb{L}, ι)

By elementary arguments, two data sets (\mathbb{L}, ι) and (\mathbb{L}', ι') of rank $r + 1$ are isomorphic if and only if the discriminants and cosets are equal.

²¹Here, \oplus is used just for the additive structure (not orthogonal direct sum).

Let v_1, \dots, v_r be an integral basis of Λ as before. The pairing of \mathbb{L} with respect to an extended basis v_1, \dots, v_r, v is encoded in the matrix

$$\mathbb{L}_{h,d_1,\dots,d_r} = \begin{pmatrix} \langle v_1, v_1 \rangle & \cdots & \langle v_1, v_r \rangle & d_1 \\ \vdots & \ddots & \vdots & \vdots \\ \langle v_r, v_1 \rangle & \cdots & \langle v_r, v_r \rangle & d_r \\ d_1 & \cdots & d_r & 2h - 2 \end{pmatrix}.$$

The discriminant is

$$\Delta(h, d_1, \dots, d_r) = (-1)^r \det(\mathbb{L}_{h,d_1,\dots,d_r}).$$

The coset $\delta(h, d_1, \dots, d_r)$ is represented by the functional

$$v_i \mapsto d_i.$$

The Noether-Lefschetz divisor $P_{\Delta,\delta} \subset \mathcal{M}_\Lambda$ is the closure of the locus of Λ -polarized $K3$ surfaces S for which $(\text{Pic}(S), j)$ has rank $r + 1$, discriminant Δ , and coset δ . By the Hodge index theorem²², $P_{\Delta,\delta}$ is empty unless $\Delta > 0$. By definition, $P_{\Delta,\delta}$ is a reduced subscheme.

Let h, d_1, \dots, d_r determine a positive discriminant

$$\Delta(h, d_1, \dots, d_r) > 0.$$

The Noether-Lefschetz divisor $D_{h,(d_1,\dots,d_r)} \subset \mathcal{M}_\Lambda$ is defined by the weighted sum

$$(8.3) \quad D_{h,(d_1,\dots,d_r)} = \sum_{\Delta,\delta} m(h, d_1, \dots, d_r | \Delta, \delta) \cdot [P_{\Delta,\delta}]$$

where the multiplicity $m(h, d_1, \dots, d_r | \Delta, \delta)$ is the number of elements β of the lattice (\mathbb{L}, ι) of type (Δ, δ) satisfying

$$\langle \beta, \beta \rangle = 2h - 2, \quad \langle \beta, v_i \rangle = d_i.$$

If the multiplicity is nonzero, then $\Delta | \Delta(h, d_1, \dots, d_r)$ so only finitely many divisors appear in the above sum.

²²The intersection form on $\text{Pic}(S)$ is nondegenerate for an algebraic $K3$ surface. Hence, a rank $r + 1$ sublattice of $\text{Pic}(S)$ which contains a quasi-polarization must have signature $(1, r)$ by the Hodge index theorem.

8.2.4. *Refined Noether-Lefschetz numbers.* Let X be a nonsingular projective Calabi-Yau 3-fold fibered in $K3$ surfaces,

$$\pi : X \rightarrow \mathbb{P}^1 .$$

Let $L_1, \dots, L_r \in \text{Pic}(X)$ determine a 1-parameter family of Λ -polarized $K3$ surfaces,

$$(X, L_1, \dots, L_r, \pi) .$$

The 1-parameter family determines a morphism

$$\iota : \mathbb{P}^1 \rightarrow \mathcal{M}_\Lambda .$$

The Noether-Lefschetz number $\text{NL}_{h,d_1,\dots,d_r}^\pi$ is defined [25, 31] by the following conditions:

- if $\Delta(h, d_1, \dots, d_r) < 0$, then $\text{NL}_{h,d_1,\dots,d_r}^\pi = 0$,
- if $\Delta(h, d_1, \dots, d_r) = 0$, then $\text{NL}_{h,d_1,\dots,d_r}^\pi = -2$,
- if $\Delta(h, d_1, \dots, d_r) > 0$, the Noether-Lefschetz number is defined by the classical intersection product

$$\text{NL}_{h,(d_1,\dots,d_r)}^\pi = \int_{\mathbb{P}^1} \iota_\pi^* [D_{h,(d_1,\dots,d_r)}] .$$

Our refinements of $\text{NL}_{h,(d_1,\dots,d_r)}^\pi$ will not be numbers, but rather representations²³ of $SU(2) \times SU(2)$ lying in the space

$$\mathbb{Z}_{\geq 0} [0, 0] \oplus \mathbb{Z}_{\geq 0} [0, \frac{1}{2}] .$$

The first refinement is defined by

$$\text{RNL}_{h,d_1,\dots,d_r}^{\pi,\circ} = \text{NL}_{h,d_1,\dots,d_r}^\pi [0, 0]$$

and carries no more data than the Noether-Lefschetz number.

The definition of the second refinement $\text{RNL}_{h,d_1,\dots,d_r}^{\pi,\diamond}$ is more subtle. Again, we consider three cases based upon the discriminant:

- if $\Delta(h, d_1, \dots, d_r) < 0$, then $\text{RNL}_{h,d_1,\dots,d_r}^{\pi,\diamond} = 0$,
- if $\Delta(h, d_1, \dots, d_r) = 0$, then $\text{RNL}_{h,d_1,\dots,d_r}^{\pi,\diamond} = [0, \frac{1}{2}]$.

²³As before, We denote the irreducible representations of $SU(2) \times SU(2)$ by pairs $[j_L, j_R]$ where $j_L, j_R \in \frac{1}{2}\mathbb{Z}_{\geq 0}$.

If $\Delta(h, d_1, \dots, d_r) > 0$, we divide the effective sum (8.3) defining $D_{h,(d_1,\dots,d_r)}$ into two parts

$$D_{h,(d_1,\dots,d_r)} = S_l + T_l$$

where S_l is the sum of the divisors on the right side of (8.3) not containing $\iota(\mathbb{P}^1)$ and T_l is the sum of such divisors containing $\iota(\mathbb{P}^1)$. The final case of the definition is:

- if $\Delta(h, d_1, \dots, d_r) > 0$, then

$$\text{RNL}_{h,d_1,\dots,d_r}^{\pi,\diamond} = \int_{\mathbb{P}^1} \iota_{\pi}^* S_l \cdot [0, 0] - \frac{1}{2} \int_{\mathbb{P}^1} \iota_{\pi}^* T_l \cdot [0, \frac{1}{2}] .$$

The motivation of the second refinement is to record the geometric components of the Noether-Lefschetz locus over the base \mathbb{P}^1 . Such loci here are unions of points and lines — the points correspond to the representation $[0, 0]$ and the lines to the representation $[0, \frac{1}{2}]$.

8.3. Refined Pairs/Noether-Lefschetz correspondence. We predict a refined Pairs/Noether-Lefschetz correspondence which intertwines three theories associated to the 1-parameter family

$$\pi : X \rightarrow \mathbb{P}^1$$

of Λ -polarized $K3$ surfaces with Calabi-Yau total space:

- (i) the refined Noether-Lefschetz theory of π ,
- (ii) the refined Gopakumar-Vafa invariants of X in fiber classes,
- (iii) the refined invariants \mathbf{R}_{j_L, j_R}^h of the $K3$ fibers.

Let $\mathbf{R}_{j_L, j_R, (d_1, \dots, d_r)}^X$ denote the refined Gopakumar-Vafa invariants of [7] defined via the stable pairs moduli spaces of X for π -vertical curve classes of degrees d_1, \dots, d_r with respect to line bundles L_1, \dots, L_r corresponding to a basis of Λ . An r -tuple (d_1, \dots, d_r) is positive if the associated degree with respect to a quasi-polarization $\lambda^\pi \in \Lambda$ is positive.

For our proposed R/NL refined correspondence, the $K3$ invariant \mathbf{R}_{j_L, j_R}^h must be divided into two parts,

$$(8.4) \quad \mathbf{R}_{j_L, j_R}^h = \mathbf{R}_{j_L, j_R}^{h,\circ} + \mathbf{R}_{j_L, j_R}^{h,\diamond} .$$

Speculation [Refined P/NL correspondence]. *A 1-parameter family of Λ -polarized K3 surfaces*

$$\pi : X \rightarrow \mathbb{P}^1$$

with Calabi-Yau total space determines a division (8.4) satisfying the following property. For degrees (d_1, \dots, d_r) positive with respect to the quasi-polarization λ^π ,

$$\begin{aligned} \sum_{j_L, j_R} \mathbf{N}_{j_L, j_R}^{X, (d_1, \dots, d_r)} [j_L, j_R] &= \sum_{j_L, j_R} \sum_{h=0}^{\infty} \mathbf{R}_{j_L, j_R}^{h, \circ} [j_L, j_R] \otimes \mathbf{RNL}_{h, (d_1, \dots, d_r)}^{\pi, \circ} \\ &+ \sum_{j_L, j_R} \sum_{h=0}^{\infty} \mathbf{R}_{j_L, j_R}^{h, \diamond} [j_L, j_R] \otimes \mathbf{RNL}_{h, (d_1, \dots, d_r)}^{\pi, \diamond} . \end{aligned}$$

By vanishing properties of the Noether-Lefschetz numbers, the above summations over h are finite for given (d_1, \dots, d_r) . We expect the counts $\mathbf{R}_{j_L, j_R, (d_1, \dots, d_r)}^X$ to be invariant under deformations of X as a family of Λ -polarized K3 surfaces.

8.4. STU example. For the STU model, we have a precise conjecture for the division (8.4) of \mathbf{R}_{j_L, j_R}^h which is consistent with several basic calculations.

We follow the STU conventions of [22] with the lattice

$$\Lambda = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} .$$

The Noether-Lefschetz numbers for the STU model

$$\pi : X \rightarrow \mathbb{P}^1$$

are determined in [22] to be

$$\mathbf{NL}_{h, (d_1, d_2)}^{\pi} = \text{Coeff}_{q^{1+d_1 d_2 - h}} \left(-2E_4(q)E_6(q) \right)$$

where E_4 and E_6 are the Eisenstein series,

$$-2E_4(q)E_6(q) = -2 + 528q + 270864q^2 + 10393152q^3 + \dots .$$

The refinement is easily seen to be given by

$$\mathbf{RNL}_{h, (d_1, d_2)}^{\pi, \diamond} = \left[0, \frac{1}{2} \right]$$

if $1 + d_1 d_2 - h = 0$ and

$$\text{RNL}_{h,(d_1,d_2)}^{\pi,\diamond} = \text{NL}_{h,(d_1,d_2)}^{\pi} \cdot [0, 0]$$

otherwise. The Betti number of X are

$$u^{-3}[X] = u^{-3} + 3u^{-1} + 488 + 3u + u^3 .$$

We define $\text{R}_{j_L, j_R}^{h,\diamond}$ for the STU model by a formula parallel to (5.1) but using only part of the generating series of Hodge numbers of the Hilbert schemes of points of $K3$ surfaces:

$$(8.5) \quad \sum_{h=0}^{\infty} \sum_{j_L} \sum_{j_R} \text{R}_{j_L, j_R}^{h,\diamond} [j_L]_y [j_R]_u q^h = \prod_{n=1}^{\infty} \frac{1}{(1 - u^{-1}y^{-1}q^n)(1 - u^{-1}yq^n)(1 - uy^{-1}q^n)(1 - uyq^n)} .$$

$\text{R}_{\frac{i}{2}, \frac{j}{2}}^{0,\diamond}$	$i=0$
$j=0$	1

$\text{R}_{\frac{i}{2}, \frac{j}{2}}^{1,\diamond}$	$j=0$	1
$i=0$		
1		1

$\text{R}_{\frac{i}{2}, \frac{j}{2}}^{2,\diamond}$	$j=0$	1	2
$i=0$	1		
1		1	
2			1

$\text{R}_{\frac{i}{2}, \frac{j}{2}}^{3,\diamond}$	$j=0$	1	2	3
$i=0$	1		1	
1		2		
2	1		1	
3				1

$\text{R}_{\frac{i}{2}, \frac{j}{2}}^{4,\diamond}$	$j=0$	1	2	3	4
$i=0$	3		1		
1		3		1	
2	1		3		
3		1		1	
4					1

TABLE 3. All nonvanishing $\text{R}_{j_L, j_R}^{h,\diamond}$ for $h \leq 4$

We expect $\text{R}_{j_L, j_R}^{h,\diamond}$ to always be nonnegative and bounded by R_{j_L, j_R}^h . Then, $\text{R}_{j_L, j_R}^{h,\diamond}$ is uniquely defined by equations (5.1), (8.4), and (8.5).

Conjecture D. *A refined P/NL correspondence holds for fiber classes of the STU model:*

$$\begin{aligned} \sum_{j_L, j_R} \mathbf{N}_{j_L, j_R}^{STU, (d_1, d_2)} [j_L, j_R] &= \sum_{j_L, j_R} \sum_{h=0}^{\infty} \mathbf{R}_{j_L, j_R}^{h, \circ} [j_L, j_R] \otimes \mathbf{RNL}_{h, (d_1, d_2)}^{\pi, \circ} \\ &+ \sum_{j_L, j_R} \sum_{h=0}^{\infty} \mathbf{R}_{j_L, j_R}^{h, \diamond} [j_L, j_R] \otimes \mathbf{RNL}_{h, (d_1, d_2)}^{\pi, \diamond} , \end{aligned}$$

for degrees (d_1, d_2) positive with respect to the quasi-polarization.

Conjectures C and D together predict the refined invariants of the STU model in fiber classes. Let $(d_1, d_2) = (0, 1)$ be the fiber class of the elliptic fibration

$$\mu : X \rightarrow \mathbb{P}^1 \times \mathbb{P}^1 .$$

For refined invariants of the STU model in class $(0, 1)$, the conjectures predict:

$$(8.6) \quad 488[0, 0] + \left[\frac{1}{2}, 0\right] + \left[\frac{1}{2}, 1\right] .$$

After expanding formula (8.1) of [7] with the refined invariants (8.6) for the STU model X in class $(d_1, d_2) = (0, 1)$, we obtain predictions for the Betti realizations of the following stable pairs motives:

$$\begin{aligned} u^{-2}[P_0(X, (0, 1))] &= u^{-2} + 2 + u^2 , \\ u^{-3}[P_1(X, (0, 1))] &= u^{-3} + 3u^{-1} + 488 + 3u + u^3 . \end{aligned}$$

The above predictions exactly match the expected geometry

$$\begin{aligned} P_0(X, (0, 1)) &\cong \mathbb{P}^1 \times \mathbb{P}^1 , \\ P_1(X, (0, 1)) &\cong X . \end{aligned}$$

In fact, the predictions for the fiber class $(0, 1)$ case match for the moduli spaces $P_m(X, (0, 1))$ of stable pairs for all Euler characteristics m .

Conjecture D proposes an exact solution for the Betti realization of the stable pairs motivic invariants for the STU model X in fiber classes. Further values of the refined invariants for the STU model are given below.

Checking the above prediction for $P_0(X, (n, 1))$ is easy for all $n \geq 0$. Further checks in the case $(d_1, d_2) = (1, 1)$ have been completed (and match Conjecture D). Determining the moduli space and the superpotential becomes harder as the Euler characteristic and the curve class increase.

$N_{\frac{i}{2}, \frac{j}{2}}^{(0,1)}$	j=0	1	2
i=0	488		
1	1		1

$N_{\frac{i}{2}, \frac{j}{2}}^{(1,1)}$	j=0	1	2	3
i=0	280964	1		
1	1	488	1	
2		1		1

$N_{\frac{i}{2}, \frac{j}{2}}^{(2,1)}$	j=0	1	2	3	4
i=0	15928440	2		1	
1	2	281452	2		
2		2	488	1	
3			1		1

$N_{\frac{i}{2}, \frac{j}{2}}^{(3,1)}$	j=0	1	2	3	4	5
i=0	410133618	4	488	1		
1	3	16209892	4		1	
2	488	4	281452	3		
3	1		2	488	1	
4				1		1

TABLE 4. Refined invariants for the STU model in fiber classes

9. SUMMARY

Our goal in the paper is to present a conjectural framework for the evaluation of the motivic stable pairs invariants of $K3$ surfaces in all curve classes. The first step is a definition using suitably Noether-Lefschetz transverse algebraic families. Conjectures A and B predict a deformation invariance for the associated Betti realization. These new Betti properties go beyond the older numerical invariance. Conjecture C then reduces the entire theory to the primitive (and irreducible) case which is determined by the Kawai-Yoshioka calculation.

In Conjecture D, we propose a refined Pairs/Noether-Lefschetz correspondence for the STU model X . Together, Conjectures A-D provide a rich non-toric setting where motivic invariants are well-behaved.

For Calabi-Yau 3-folds, there are three basic approaches to curve counting (along with several variations):

- (i) the Gromov-Witten theory of stable maps,
- (ii) the Donaldson-Thomas theory of stable pairs,
- (iii) the Gopakumar-Vafa theory of stable sheaves with $\chi = 1$.

While (i) and (ii) are foundationally well settled and proven equivalent in many cases, the precise formulation of (iii) is not yet clear (and the equivalence is not understood). As we have already mentioned, for a motivic theory, only the sheaf theoretic approaches (ii) and (iii) are at present possible.

When the moduli space of stable sheaves with 1-dimensional support and Euler characteristic 1 is *nonsingular*, approach (iii) yields a clear proposal. Nonsingularity of the moduli spaces is certainly rare. However, in the Appendix by R. Thomas, nonsingularity is proven for the stable sheaves associated to $K3$ surfaces in algebraic families transverse to the relevant Noether-Lefschetz divisors. Nonsingularity is proven for all (even imprimitive) curve classes. The outcome provides a direct approach to the GV motivic theory of $K3$ surfaces which matches precisely with our stable pair predictions. In particular, the GV approach predicts the divisibility invariance of the original KKV formula.

A path proving Conjectures A, B, and C is to start with the GV moduli spaces as discussed in the Appendix and wall-cross to stable pairs.²⁴ Various difficulties arise: the most significant of which may be the inability to achieve Euler characteristic 1 by twisting by line bundles in the imprimitive case. Nevertheless, proving a Pairs/GV

²⁴See [36] where exactly the same strategy is applied to study the Euler characteristics of moduli spaces of stable pairs on $S \times \mathbb{C}$. The difficulties left open [36] arise here as well.

correspondence appears the most promising approach to Conjectures A, B, and C at the moment.

APPENDIX A. REFINED KKV FROM REFINED GOPAKUMAR-VAFA

by R. P. Thomas

Gopakumar and Vafa have suggested a conjectural approach to defining BPS numbers via moduli of sheaves. S. Katz [18] proposed using stable sheaves of Euler characteristic 1. The papers [14, 21] make BPS and motivic predictions of GV invariants via the geometry of such stable sheaves. We show here for fibre classes of sufficiently Noether-Lefschetz transverse $K3$ -fibred 3-folds, the proposals of [14, 21] work perfectly for both the BPS numbers and their refinements: we recover the generating series of Hodge numbers of Hilbert schemes of points on $K3$ surfaces (5.1) as considered in the current paper.

By a flat family of stable sheaves on the fibres of a projective family

$$\pi: \mathcal{Y} \rightarrow B,$$

we mean a coherent sheaf \mathcal{F} on \mathcal{Y} which is flat over B , such that \mathcal{F}_b is stable on \mathcal{Y}_b for any closed point $b \in B$. We will need the following standard result.

Lemma A.1. *Let \mathcal{F} be a flat family of stable sheaves on the fibres of $\pi: \mathcal{Y} \rightarrow B$. Then $\pi_* \mathcal{H}om(\mathcal{F}, \mathcal{F}) = \mathcal{O}_B$.*

Proof. Stable sheaves are simple, so

$$\mathrm{Hom}(\mathcal{F}_b, \mathcal{F}_b) / \mathbb{C} \cdot \mathrm{id} = 0$$

for all closed point $b \in B$. It follows by base change that

$$\pi_* \mathcal{H}om(\mathcal{F}, \mathcal{F}) / \mathcal{O}_B \cdot \mathrm{id} = 0,$$

which gives the result. \square

Now fix $T \xrightarrow{\pi} (\Delta, 0)$, a $K3$ -fibred 3-fold with central fibre $\pi^{-1}(0)$ given by

$$\iota: S \hookrightarrow T.$$

Suppose the curve Δ is transverse to NL_β for some fixed

$$\beta \in H^2(S, \mathbb{Z}) \cap H^{1,1}(S).$$

By shrinking Δ if necessary, we may assume that π is smooth and intersects NL_β only in 0.

Consider the moduli space $\mathcal{M}_{1,\iota_*\beta}(T)$ of stable²⁵ dimension 1 sheaves on T with fundamental class $\iota_*\beta$ and holomorphic Euler characteristic $\chi = 1$. The last condition ensures that no semistable sheaves exist, so $\mathcal{M}_{1,\iota_*\beta}(T)$ is projective. By simplicity, any stable sheaf must be scheme theoretically supported on a single fibre of π . By the Noether-Lefschetz condition, there is an open and closed component

$$\mathcal{M}^* \subset \mathcal{M}_{1,\iota_*\beta}(T)$$

of sheaves with scheme theoretic support on S . Hence, the obvious map

$$(A.2) \quad \iota_*: \mathcal{M}_{1,\beta}(S) \longleftrightarrow \mathcal{M}^*$$

is a set-theoretic bijection.

Proposition A.3. *The map (A.2) is an isomorphism of schemes.*

Proof. The statement is local, so we may work on a Zariski open subset $U \subset \mathcal{M}$. Shrinking U if necessary, we may assume there is a universal sheaf \mathcal{F} on $U \times T$. We must show that \mathcal{F} is the push forward of a sheaf on $U \times S$. The classifying map of the resulting flat family of stable sheaves on S will define the inverse map to (A.2).

Consider the composition

$$\Gamma(\mathcal{O}_{\Delta \times U}) \xrightarrow{p^*} \Gamma(\mathcal{O}_{T \times U}) \xrightarrow{\text{id}} \text{Hom}(\mathcal{F}, \mathcal{F}) \cong \Gamma(\mathcal{O}_U),$$

where the final isomorphism is given by Lemma A.1. The map is $\Gamma(\mathcal{O}_U)$ -linear so defines a U -point of $\Delta \times U$. Its ideal is the kernel of the above composition, and by its definition this ideal annihilates \mathcal{F} .

Thus we get a map

$$f: U \longrightarrow \Delta$$

such that \mathcal{F} is the pushforward of a sheaf F on $T \times_\Delta U \subset T \times U$. Since \mathcal{F} is flat over U , so is F .

²⁵Stability is taken with respect to a fixed polarization of T .

On closed points f is the constant map to $0 \in \Delta$. To prove f is a constant map of schemes, we may replace U by the formal neighbourhood of any closed point. The result will follow below from Hodge theory and the Noether-Lefschetz transversality condition.

We have shown that F is a flat family of stable sheaves on the smooth family

$$T \times_{\Delta} U \xrightarrow{\pi} U$$

of $K3$ surfaces over U . The fundamental class of F lies in

$$F^1 H_{dR}^2(T \times_{\Delta} U/U),$$

the part of the Hodge filtration defined by $\Omega_{T \times_{\Delta} U/U}^{\geq 1}$.

Via the isomorphism [3, Proposition 3.8]

$$H_{dR}^2(T \times_{\Delta} U/U) \cong H^2(S, \mathbb{C}) \otimes_{\Delta} \mathcal{O}_U,$$

the class of F is $\beta \otimes 1$. Since it lies in F^1 , its projection

$$[\beta \otimes 1]^{0,2} \in H_{dR}^2(T \times_{\Delta} U/U) / F^1 H_{dR}^2(T \times_{\Delta} U/U)$$

vanishes identically. This class is the pull back via f of the analogous class

$$[\beta \otimes 1]^{0,2} \in H_{dR}^2(\widehat{T}/\widehat{\Delta}) / F^1 H_{dR}^2(\widehat{T}/\widehat{\Delta}),$$

where $\widehat{\Delta}$ is the completion of Δ at 0, and \widehat{T} is the completion of T along S .

But the scheme theoretic Noether-Lefschetz locus is defined precisely by the vanishing of $[\beta \otimes 1]^{0,2}$, and we assumed this is $\{0\} \subset \Delta$. Therefore $f|_{\Delta}$ is the constant map to $0 \in \Delta$. \square

By Proposition A.3, \mathcal{M}^* is a moduli space of stable sheaves on S . By [40, Corollary 3.5], \mathcal{M}^* is nonsingular, nonempty, and deformation equivalent to $\text{Hilb}^g(S)$, where $2g - 2 = \beta^2$. In particular, \mathcal{M}^* has canonical orientation data.

We expect the Chow support map to the complete linear system in class β ,

$$(A.4) \quad \mathcal{M}^* \longrightarrow |\mathcal{O}(\beta)|,$$

to factor through a Lagrangian fibration onto a projective space

$$(A.5) \quad \mathbb{P}^g \subset |\mathcal{O}(\beta)|.$$

The proof in most cases is well-known:

- If $\beta^2 > 0$ and β is nef, then (A.4) is the classical *Beauville integrable system*, and the image \mathbb{P}^g is the whole linear system $|\mathcal{O}(\beta)|$. The generic element C of $|\mathcal{O}(\beta)|$ is a nonsingular irreducible curve of genus

$$g = 1 + \beta^2/2,$$

and the fibre of (A.4) over C is the nonsingular Lagrangian torus $\text{Pic}_g C \subset \mathcal{M}^*$.

- If $\beta^2 = 0$ and β is nef of divisibility $m > 0$, then β/m is the class of a fibre in an elliptic fibration $S \rightarrow \mathbb{P}^1$. The sheaves parameterized by \mathcal{M}^* are (the pushforward to S of) rank m sheaves on a single fibre, and the fibration (A.4) maps this sheaf to its support in $\mathbb{P}^1 = |\mathcal{O}(\beta/m)|$. In turn this \mathbb{P}^1 embeds in $\mathbb{P}^m = |\mathcal{O}(\beta)|$ as in (A.5) by the m th Veronese embedding.
- If $\beta^2 = -2$, \mathcal{M}^* is a single point, and the claim is trivial.
- If $\beta^2 < -2$, \mathcal{M}^* is empty.

The remaining case where $\beta^2 \geq 0$ with β not nef is likely to follow from the above cases after using wall-crossing and the methods of [1, Section 11]. Alternatively, it is not hard to show directly that the fibres of the map (A.4) are Lagrangian; that the image should then be \mathbb{P}^g is a standard conjecture (proved by Hwang when the image is nonsingular). We leave the matter open here.

Finally, for the cases treated above, we can follow the calculation of [21, Section 8.3] (which in turn follows [14]) to determine the refined Gopakumar-Vafa invariants using the perverse Leray filtration for the

fibration.²⁶ At the level of Poincaré polynomials, the generating function²⁷ obtained is:

$$\prod_{m \geq 1} \frac{1}{(1 - t_L t_R q^m)(1 - t_L^{-1} t_R q^m)(1 - t_L t_R^{-1} q^m)(1 - t_L^{-1} t_R^{-1} q^m)(1 - q^m)^{20}}.$$

Substituting (u, v) for (t_L, t_R) gives the refined KKV generating function (5.1) of KKP.

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²⁶The cited references [14, 21] use the fibration $\text{Hilb}^g(S) \rightarrow \mathbb{P}^g$ induced by an elliptic fibration $S \rightarrow \mathbb{P}^1$. This is a deformation of a Beauville-Mukai system (relative compactified Pic_g) of any complete linear system of genus g in a primitive curve class on a K3 surface. (For instance if the surface is an elliptic fibration with a section, we may take the curve class to be a section plus g fibres.) Thus, when our curve class β is *primitive*, our fibration $\mathcal{M}^* \rightarrow \mathbb{P}^g$ can be deformed to $\text{Hilb}^g(S) \rightarrow \mathbb{P}^g$ to deduce that the Hodge numbers associated to the perverse Leray filtration agree with those in [14, 21].

We are grateful to Davesh Maulik and Junliang Shen for pointing out that for multiple curve classes, there is no such deformation *through Lagrangian fibrations*, due to results of Markman [26, Theorem 1.5 and Proposition 1.7]. Fortunately however, Shen and Yin have recently proved the beautiful result that the perverse Leray Hodge numbers are equal to the ordinary Hodge numbers for any smooth holomorphic symplectic variety with a Lagrangian fibration [35, Theorem 0.2]. Since \mathcal{M}^* and $\text{Hilb}^g(S)$ are deformation equivalent (even if not as Lagrangian fibrations) their ordinary Hodge numbers agree. Thus the perverse Hodge numbers of $\mathcal{M}^* \rightarrow \mathbb{P}^g$ are the same as those of $\text{Hilb}^g(S) \rightarrow \mathbb{P}^g$ used in [14, 21].

²⁷See the equation before [21, Equation (8.3)], before the specialization $t_R = -1$.

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