Enumerative Geometry of Curves, Maps, and Sheaves

Part IV : Stable Pairs Descendents

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I. Descendents for curves and sheaves

We have discussed descendents for moduli spaces of stable maps to X.

Let us revisit the construction to define

$$\tau_k(\gamma) \in H^{2k+\delta-2}(\overline{M}_g(X,\beta))$$

power of ψ $\qquad$ $\gamma \in H^{\delta}(X)$

Idea is to use the correspondence

$$\begin{tikzcd} & \overline{M}_{g,1}(X,\beta) \arrow[dl, "\pi"'] \arrow[dr, "ev"] & \\ \overline{M}_g(X,\beta) & & X \end{tikzcd}$$

$$\tau_k(\gamma) = \pi_{\star}\left(\psi_1^k \cdot ev^{\star}(\gamma)\right)$$

For moduli spaces of sheaves on X, there is a parallel construction

$\dim_{\mathbb{C}} = r$

universal sheaf

$$
\begin{tikzcd}
 & \mathbb{I} \arrow[d] & \\
 & I \times X \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
I & & X
\end{tikzcd}
$$

$\gamma \in H^{\delta}(X)$

moduli space of sheaves

$$T_k(\gamma) = \pi_{1*}\left(ch_{k+r-1}(\mathbb{I}) \cdot \pi_2^*(\gamma) \right) \in H^{2k+\delta-2}(I)$$

descendent in sheaf theory

Examples of descendents in sheaf theory

- X is a nonsingular projective curve

  $\mathcal{L} \to X$ is a line bundle

  $\mathcal{U}_{X,2,\mathcal{L}}$ = moduli space of rank 2 stable bundles on X with fixed det $= \mathcal{L}$

  deg $\mathcal{L} = 1$

  [no semistables]

  Descendents defined via the universal bundle

  $$\mathcal{E} \longrightarrow \mathcal{U}_{X,2,\mathcal{L}}$$

  Theorem: $H^*(\mathcal{U}_{X,2,\mathcal{L}})$ generated by descendents. [Mumford, Kirwan, Zagier also found relations]

- X is a surface

Exactly parallel construction for moduli of sheaves on a surface

$\Rightarrow$ used in the theory of Donaldson invariants

We already saw related descendents in our discussion of

$$\int_{[\mathrm{Quot}_X(\mathbb{C}^n, \beta, d)]^{vir}} \prod \mathrm{ch}_{k_i}(\alpha_i^{[d]})$$

Chern classes after $R\pi_*$, so need GRR to relate to the descendent defined here

- X is a 3-fold

GW theory : $\displaystyle\int_{[\overline{M}_g(X,\beta)]^{vir}} \prod \tau_{k_i}(\gamma_i)$

differs slightly from the previous $\langle \tau_{k_1}(\gamma_1)\cdots\tau_{k_r}(\gamma_r)\rangle$ defined using $\overline{M}_{g,r}(X,\beta)$

DT theory : $\displaystyle\int_{[I_n(X,\beta)]^{vir}} \prod \tau_{k_i}(\gamma_i)$

(ideal sheaves)

$$\begin{tikzcd} & \mathbb{I} \arrow[d] & \\ & I \times X \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ I & & X \end{tikzcd}$$

$$\tau_k(\gamma) = \pi_{1*}\big(ch_{k+2}(\mathbb{I}) \cdot \pi_2^*(\gamma)\big)$$

$$\in H^{2k+\delta-2}\big(I_n(X,\beta)\big)$$

Question : Can we extend the GW/DT correspondence of MNOP to descendents ?

II. Stable pairs

The Hilbert scheme $I_n(X, \beta)$ has some shortcomings for the study of descendents. The moduli of stable pairs is better behaved.

Let X be a nonsingular projective 3 fold,

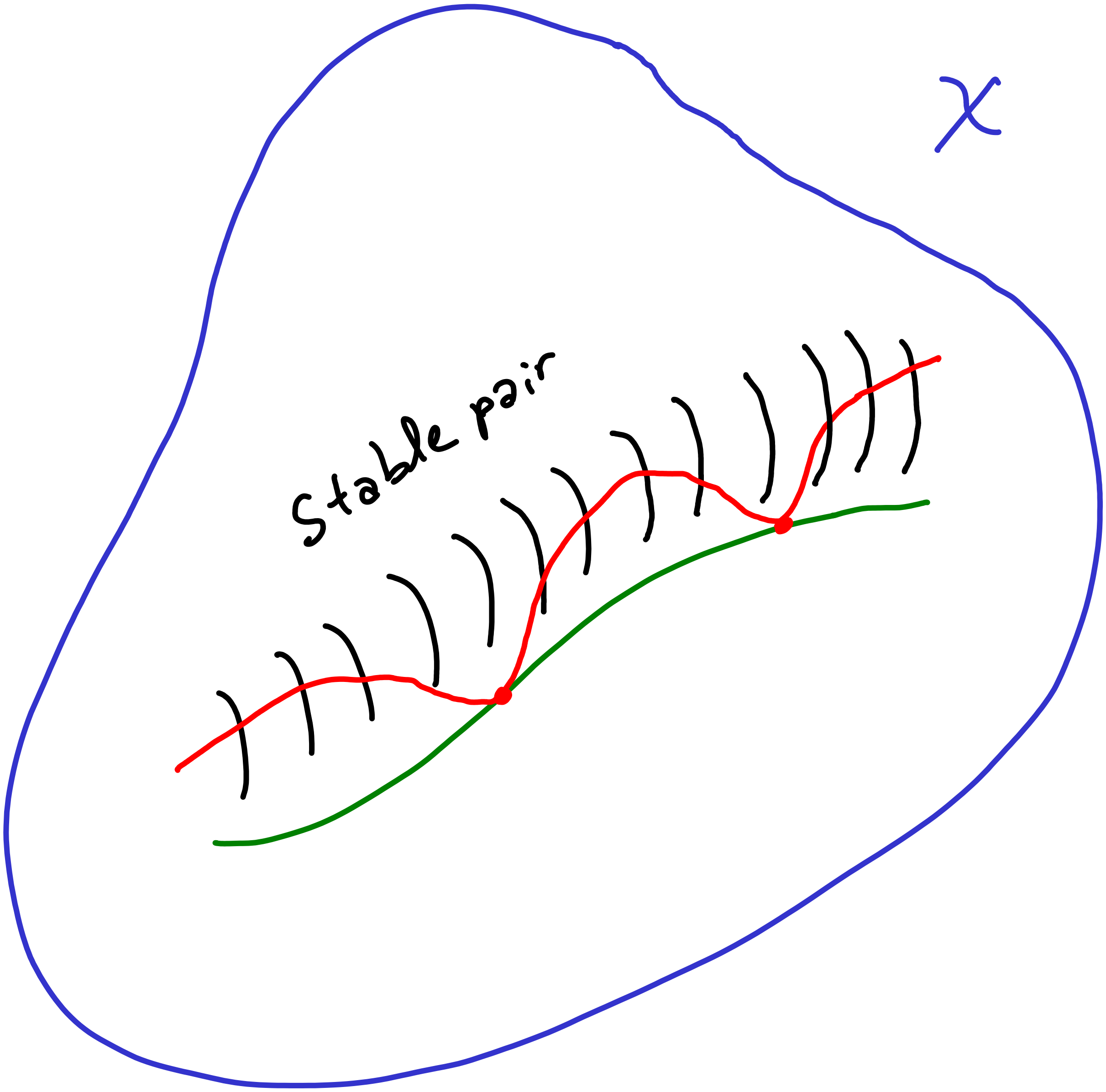
$\beta \in H_2(X, \mathbb{Z})$,

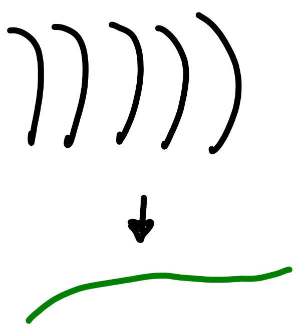
$n \in \mathbb{Z}$.

$P_n(X, \beta)$ is the moduli of stable pairs:

$$[\mathcal{F}, s] \in P_n(X, \beta)$$

- $\mathcal{F}$ is pure sheaf of dimension 1
- $\mathcal{O}_X \xrightarrow{s} \mathcal{F}$ is a section with coker of dimension 0



$\mathcal{F}$  sheaf $\quad n = \chi(\mathcal{F})$

$\downarrow$

$\mathrm{Supp}(\mathcal{F})$ $\quad \beta = [\mathrm{Supp}(\mathcal{F})]$

Construction of $P_n(X, \beta)$: use Le Potier,

See papers by P– R. Thomas

Example: $X = \mathbb{P}^3$

Then $P_n(X, d) \supset$ classical locus

which parameterizes ideal objects

$C \subset \mathbb{P}^3$ nonsingular irreducible curve of degree d

$F \to C$ line bundle of degree ℓ

$s \in H^0(C, F)$ a nonzero section

$$n = \ell - \text{genus}(C) + 1$$

Of course $P_n(X, d)$ also parameterizes more degenerate objects

We view $\mathbb{I} = [\mathcal{O}_X \xrightarrow{s} \mathcal{F}]$ as an object in $D^b_{Coh}(X)$. Then

$$\mathrm{Def} = \mathrm{Ext}^1_0(\mathbb{I}, \mathbb{I})$$

$$\mathrm{Obs} = \mathrm{Ext}^2_0(\mathbb{I}, \mathbb{I})$$

higher Ext^i_0's vanish

We have a virtual fundamental class $[P_n(X, \beta)]^{vir}$ of dimension $\int_\beta c_1(X)$

See "Counting curves via stable pairs" with R. Thomas

Descendents

$$\begin{tikzcd}[column sep=small] & \mathcal{F} \arrow[d] & \\ & P_n(X,\beta)\times X \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ P_n(X,\beta) & & X \end{tikzcd}$$

universal sheaf

$$\tau_k(\gamma) = \pi_{1*}\left(\mathrm{ch}_{k+2}(\mathcal{F}) \cdot \pi_2^*(\gamma)\right)$$

$\gamma \in H^*(X)$

We will use a better convention

$$\mathrm{ch}_k(\gamma) = \pi_{1*}\left(\mathrm{ch}_k(\mathcal{F}-\mathcal{O}) \cdot \pi_2^*(\gamma)\right)$$

no shift now

Conjectures for the descendent theory of stable pairs

MNOP 2005
Pixton-P 2012
OOP 2019

GW/Pairs Correspondence

MNOP 2005
Pixton-P 2012

Rationality

Virasoro

OOP 2019
Moreira OOP 2020
Moreira 2020

M = Maulik
N = Nekrasov
O = Okounkov
OO = Oblomkov, Okounkov

III. Rationality

Let X be a nonsingular projective 3fold

Define descendent generating series:

$$\left\langle ch_{k_1}(\gamma_1)\cdots ch_{k_r}(\gamma_r)\right\rangle_\beta^X \qquad \gamma_i \in H^*(X)$$

$$\| $$

$$\sum_{n\in\mathbb{Z}} q^n \int_{[P_n(X,\beta)]^{vir}} ch_{k_1}(\gamma_1)\cdots ch_{k_r}(\gamma_r)$$

moduli space are empty for $n<<0$

Rationality Conjecture I (P- R. Thomas):

$$\left\langle \mathrm{ch}_{k_1}(\gamma_1)\cdots\mathrm{ch}_{k_r}(\gamma_r)\right\rangle_\beta^X \in \mathbb{Q}((q))$$

is the Laurent expansion of a rational function in q.

Example (from my paper "Descendents for stable pairs on 3 folds")

with help from Oblomkov

$$\mathsf{Z}_{\mathsf{P}}\big(\mathbb{P}^3; q \mid \tau_9(1)\big)_{2\mathsf{L}} =$$

$$-\frac{(73q^{12}-825q^{11}-124q^{10}+5945q^9+779q^8-36020q^7+60224q^6-36020q^5+779q^4+5945q^3-124q^2-825q+73)q}{60480(1+q)^3(-1+q)^3}.$$

Rationality Conjecture II (formulated with Pixton)

$$\langle \mathrm{ch}_{k_1}(\gamma_1) \cdots \mathrm{ch}_{k_r}(\gamma_r) \rangle_\beta^X = Z(q) \in \mathbb{Q}(q)$$

has poles only at roots of unity and 0 and satisfies a

functional equation

$$Z\left(\frac{1}{q}\right) = (-1)^{\sum_{i=1}^{r} k_i} q^{-d_\beta} Z(q)$$

where $d_\beta = \int_\beta c_1(X)$.

Failure of Rationality for the Hilbert Scheme:

$$\begin{array}{c} \mathbb{I} \quad \text{universal sheaf} \\ \downarrow \end{array}$$

$$\begin{tikzcd} & I_n(X,\beta)\times X \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ I_n(X,\beta) & & X \end{tikzcd} \qquad \gamma \in H^*(X)$$

- $\mathrm{ch}_k(\gamma) = \pi_{1*}\left(\mathrm{ch}_k(\mathbb{I}) \cdot \pi_2^*(\gamma)\right)$

- $$\left\langle \mathrm{ch}_{k_1}(\gamma_1) \cdots \mathrm{ch}_{k_r}(\gamma_r) \right\rangle_\beta^{X,I}$$

$$\| $$

$$\sum_{n\in\mathbb{Z}} q^n \int_{[I_n(X,\beta)]^{vir}} \mathrm{ch}_{k_1}(\gamma_1) \cdots \mathrm{ch}_{k_r}(\gamma_r)$$

Since $\langle 1 \rangle_0^{X,I} = M(-q)^{\int_X c_3 - c_1 c_2}$,

$\langle 1 \rangle_0^{X,I}$ not rational in q

- We are interested in (see MNOP)

$$\frac{\langle ch_{k_1}(\gamma_1) \cdots ch_{k_r}(\gamma_r) \rangle_\beta^{X,I}}{\langle 1 \rangle_0^{X,I}}$$

But still not rational in q

Conjecture (Oblomkov-Okounkov-P):

Normalized series is a polynomial in $\left\{ \left(q \frac{d}{dq}\right)^i F_3(-q) \right\}_i$ with coefficients in $\mathbb{Q}(q)$.

$$F_3(q) = \sum_{n=1}^{\infty} n^2 \frac{q^n}{1-q^n}$$