Enumerative Geometry of Curves, Maps, and Sheaves

Part III : Sheaf Counting

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A fundamental property of Gromov-Witten theory is the uniform definition for all targets X.

[nonsingular, projective of any dimension]

Sheaf Counting is more delicate: the standard theories are for sheaves on X with $\dim_{\mathbb{C}} X \leq 3$

Recent work by R. Thomas and J. Oh on CY4-folds

What is the reason for the difference?

- Def-Obs theory for a stable map $f: C \to X$

$$\text{inf Aut} = 0 \quad [\text{map stability}]$$

$$\text{Def} = H^0(C, f^* T_X)$$

$$\text{Obs} = H^1(C, f^* T_X)$$

is always 2-term $\Rightarrow$ virtual fundamental class.

- For a sheaf $\mathcal{F} \to X$

$$\text{inf Aut} = \text{Ext}^0(\mathcal{F}, \mathcal{F})$$

Mostly killed by sheaf stability

$$\text{Def} = \text{Ext}^1(\mathcal{F}, \mathcal{F})$$

$$\text{Obs} = \text{Ext}^2(\mathcal{F}, \mathcal{F})$$

+ higher obstructions $\text{Ext}^k(\mathcal{F}, \mathcal{F})$

dim constraints on X are needed to kill

Dimension 1

Let X be a nonsingular projective curve of genus g.

- $\mathcal{U}_X(r,d)$ moduli of stable bundles
  $(r,d)=1$, already nonsingular of the expected dimension since
  $\mathrm{Ext}^2(\mathcal{F},\mathcal{F})=0$

many variations: Higgs bundles

- Quot scheme $\mathrm{Quot}_X(\mathbb{C}^n, r, d)$

$$0 \to G \to \mathbb{C}^n \otimes \mathcal{O}_X \to \mathcal{F} \to 0$$

rank r, degree d

Marian Oprea

$$\mathrm{Def} = \mathrm{Ext}^0(G,\mathcal{F})$$
$$\mathrm{Obs} = \mathrm{Ext}^1(G,\mathcal{F})$$
$$\mathrm{Ext}^{\geq 2}(G,\mathcal{F}) = 0$$

since $\dim_{\mathbb{C}} X = 1$

$\mathrm{Quot}_X(\mathbb{C}^n, r, d)$ is generally singular of mixed dimension, but carries a virtual fundamental class.

Exercise: Compute the virtual dimension,

$$\text{vir dim } \mathrm{Quot}_X(\mathbb{C}^n, r, d) = r(n-r)(1-g) + nd.$$

On an open set, $\mathrm{Quot}_X(\mathbb{C}^n, r, d)$ is a moduli space of bundles with sections.

Marian-Oprea transfer integrals on $\mathcal{U}_X(n-r, d)$ to $\mathrm{Quot}_X(\mathbb{C}^n, r, d)$ against the virtual class

$\Rightarrow$ leads to a proof of Verlinde formulas.

- $\mathrm{Quot}_X(\mathbb{C}^1, 0, d) = \mathrm{Sym}^d X$

$\mathrm{Quot}_X(\mathbb{C}^n, 0, d)$ = punctual Quot schemes of the curve X

[Exercise: $\mathrm{Quot}_X(\mathbb{C}^n, 0, d)$ is nonsingular of dimension nd and virtual class is the usual fundamental class.]

Tautological bundles on $\mathrm{Quot}_X(\mathbb{C}^n, 0, d)$ can be constructed as follows.

$E \to X$ vector bundle of rank e

$\Downarrow$

$E^{[d]} \to \mathrm{Quot}_X(\mathbb{C}^n, 0, d)$ vector bundle of rank de with fiber $H^0(X, F \otimes E)$

Interesting property: for $L \to X$ line bundle,

$$\int_{\mathrm{Quot}_X(\mathbb{C}^n, 0, d)} s(L^{[d]})^1 = (-1)^{(n-1)d} \int_{\mathrm{Quot}_X(\mathbb{C}^1, 0, d) = \mathrm{Sym}^d X} s(L^{[d]})^n$$

Segre class

$$s(B) = \frac{1}{c(B)}$$

Oprea-P 2019

Challenge: find a conceptual proof.

Dimension 2

Let X be a nonsingular projective surface

- The simplest theory is again for the Quot scheme

$\mathrm{Quot}_X(\mathbb{C}^n, \beta, d)$ of quotients

$$0 \to G \to \mathbb{C}^n \otimes \mathcal{O}_X \to \mathcal{F} \to 0$$

rank 0 [supported on curves]

$c_1(\mathcal{F}) = \beta$, $\chi(\mathcal{F}) = d$

Marian
Oprea
P 20

$$\mathrm{Def} = \mathrm{Ext}^0(G, \mathcal{F})$$

$$\mathrm{Obs} = \mathrm{Ext}^1(G, \mathcal{F})$$

Serre duality

$$\mathrm{Ext}^2(G, \mathcal{F}) = \mathrm{Ext}^0(\mathcal{F}, G \otimes K_X)^*$$

$$= 0$$ Since $\mathcal{F}$ is torsion

[We can remove the $\mathcal{F}$ is torsion assumption if X is Fano - a mostly unexplored direction]

$$\mathrm{vir\ dim}\ \mathrm{Quot}_X(\mathbb{C}^n, \beta, d) = nd + \int_X \beta^2$$

grows with d

What are the integrals?

For $\alpha \in K^0(X)$, define

$$\alpha^{[d]} = R\pi_{1*}(\mathcal{F} \otimes \pi_2^*\alpha) \in K^0(\mathrm{Quot}_X)$$

$\mathcal{F}$ universal quotient

$$\begin{tikzcd}
& \mathcal{F} \arrow[d] & \\
& \mathrm{Quot}_X(\mathbb{C}^n, \beta, d) \times X \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
\mathrm{Quot}_X(\mathbb{C}^n, \beta, d) & & X
\end{tikzcd}$$

Chern char here viewed as descendent insertion

$$Z^X_{n,\beta}(\alpha_1, \dots, \alpha_\ell \mid k_1, \dots, k_\ell)$$

$$\parallel$$

$$\sum_{d \in \mathbb{Z}} q^d \int_{[\mathrm{Quot}_X(\mathbb{C}^n, \beta, d)]^{vir}} \prod_{i=1}^{\ell} \mathrm{ch}_{k_i}(\alpha_i^{[d]})\, c(T^{vir}(\mathrm{Quot}_X(\mathbb{C}^n, \beta, d)))$$

total Chern class

Two basic ideas in the theory

(A) Rationality

Conjecture: $Z^{X}_{n,\beta}(\alpha_1, \dots, \alpha_\ell \mid k_1, \dots, k_\ell)$

is the Laurent expansion of

a rational function in q.

Oprea-P

Johnson-Oprea-P

W. Lim

Arbesfeld-J-L-O-P

$\Rightarrow$ proof in many cases,

but not yet all

(B) Exact solutions for

X = simply connected minimal surface

of general type with nonsingular

canonical curve.

Theorem [Oprea-P] :

genus of canonical curve

$$Z^{X}_{n,\,\ell K_X}(q) = (-1)^{\ell\cdot\chi(\mathcal{O}_X)}\, q^{\ell(1-g)} \cdot \sum_{1 \leq i_1 < \dots < i_{n-\ell} \leq n} A(r_{i_1}, \dots, r_{i_{n-\ell}})^{1-g}$$

where the sum is taken over all

$\binom{n}{n-\ell}$ choices of $n-\ell$ distinct roots

$$w = r_i(q)$$

of the equation $w^n - q\,(w-1)^n = 0$,

$$A(x_1, \dots, x_{n-\ell}) = \frac{(-1)^{\binom{n-\ell}{2}}}{n^{n-\ell}} \cdot \prod_{i=1}^{n-\ell} \frac{(1+x_i)^n (1-x_i)}{x_i^{n-1}}$$

$$\cdot \prod_{i<j} \frac{(x_i - x_j)^2}{1-(x_i-x_j)^2}$$

The result suggests a connection to Gromov-Witten curve counting via the appearance of $(-1)^{\chi(\theta_X)}$ and g

$$\langle 1 \rangle^X_{g, K_X} \qquad \text{Gromov} = \text{SW} \qquad \text{Taubes}$$

- A more sophisticated sheaf counting theory of surfaces was proposed by Vafa-Witten and defined mathematically by Tanaka-Thomas.

Sheaf counting on X approached via counting sheaves on the 3-fold

total space $\quad K_X \to X$

Much harder to calculate, rational functions replaced by modular forms, many results/conjectures by Göttsche-Kool

Dimension 3

Three is the most interesting dimension for counting, and there are many directions of study: Mirror symmetry, DT wallcrossing / Stability conditions, refined invariants, ...

The simplest place to start is with the Hilbert scheme of Curves.

Let X be a nonsingular projective 3-fold.

$I_n(X, \beta) =$ Hilbert scheme of curves $C \subset X$

$$0 \to \mathcal{I}_C \to \mathcal{O}_X \to \mathcal{O}_C \to 0$$

$$n = \chi(\mathcal{O}_C)$$

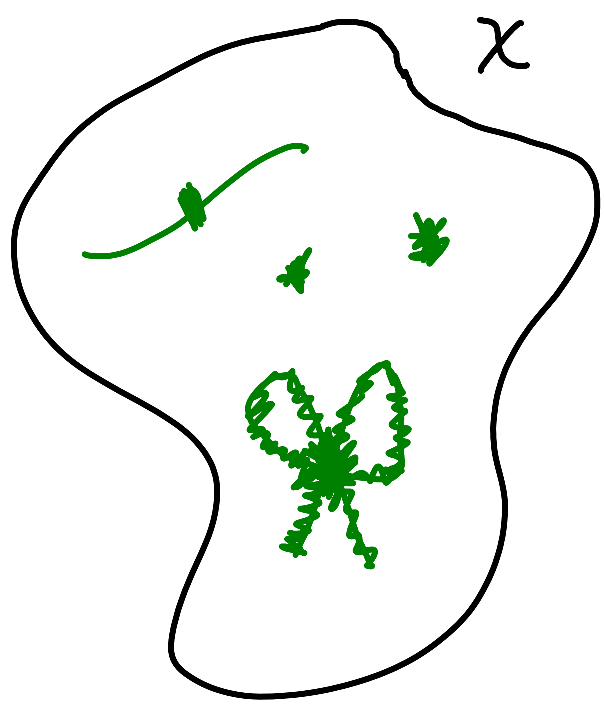
$$\beta = [C] \in H_2(X, \mathbb{Z})$$

We can consider the Hilbert scheme as a moduli space of ideal sheaves.

$$0 \to \mathcal{I}_C \to \mathcal{O}_X \to \mathcal{O}_C \to 0$$

We view Hilb as a moduli of ideal sheaves (with tracefree defs)

usually Hilb is viewed as a moduli of quotients



We really consider the entire Hilbert Scheme

R. Thomas
PhD Thesis

Consider the Def-obs theory

$$\mathrm{Ext}^0(\mathcal{I},\mathcal{I}) = \mathbb{C} \quad \text{scalars}$$

$$\mathrm{Ext}^1(\mathcal{I},\mathcal{I}) = \mathrm{Def}$$

$$\mathrm{Ext}^2(\mathcal{I},\mathcal{I}) = \mathrm{Obs}$$

$$\mathrm{Ext}^3(\mathcal{I},\mathcal{I}) \cong \mathrm{Ext}^0(\mathcal{I},\mathcal{I}\otimes K_X)^*$$

killed by traceless def theory

Conclusion:

$$\mathrm{Ext}^1_0(\mathcal{I},\mathcal{I}) = \mathrm{Def}$$

traceless Ext

$$\mathrm{Ext}^2_0(\mathcal{I},\mathcal{I}) = \mathrm{Obs}$$

$I_n(X,\beta)$ has a virtual fundamental class

Exercise: Calculate the virtual dimension

independent of n!

$$\text{vir dim } I_n(X,\beta) = \int_\beta c_1(X)$$

Integration against $[I_n(X,\beta)]^{vir}$

is Donaldson-Thomas theory.

Gromov-Witten theory also has an independence property for vir dim in dimension 3:

$$\text{vir dim } \overline{M}_g(X,\beta) = \int_\beta c_1(X)$$

Moreover the vir dim formula is the same.

independent of g!

Calabi-Yau 3-folds

CY3s are the perfect location for enumerative geometry: all problems have virtual dimension 0

Let X be a CY3

Let $\beta \in H_2(X, \mathbb{Z})$

$$\text{vir dim } \overline{M}_g(X, \beta) = \text{vir dim } I_n(X, \beta) = 0$$

Question: Is there a relationship

$$N_{g,\beta} = \int_{[\overline{M}_g(X,\beta)]^{vir}} 1 \quad \overset{?}{\sim} \quad I_{n,\beta} = \int_{[I_n(X,\beta)]^{vir}} 1$$

Both sides virtually count curves,
but some differences.

- Simplest hope:

Assume β is an indecomposable class

Can hope $N_{g,\beta} \overset{?}{=} I_{1-g,\beta}$

$\chi(C_g) = 1-g$

Let analyse the simplest case of such a geometry

$C \subset X$, $C \cong \mathbb{P}^1$ with normal bundle $N_{X/C} \cong \mathcal{O}(-1) \oplus \mathcal{O}(-1)$

GW calculation Faber-P 2000

$$\sum_{g\geq 0} u^{2g-2} N_{g,[C]} = \left(\frac{u/2}{\sin(u/2)}\right)^2 \frac{1}{u^2}$$

How are these integrals computed?

Everything can be moved to the moduli of maps to $C \cong \mathbb{P}^1$

Then the techniques are

- Localization (of the virtual class)
- Hodge integrals $\int_{\overline{M}_{g,1}} c(\mathbb{E}) \cdot \psi_1^k$ Hodge bundle
- Tricks

DT calculation MNOP I, II

Euler char

$$\sum_{n \in \mathbb{Z}} q^n I_{n,[C]} = \frac{q}{(1+q)^2} \cdot M(-q)^{e(X)}$$

$$M(q) = \prod_{n=1}^{\infty} (1-q^n)^{-n}$$

- Localization (of the virtual class)
- Box counting in 3-dimensions
- Tricks

Conclusion: simple hope taken literally fails.

- More sophisticated hope

$$M(-q)^{-e(X)} \sum_{n \in \mathbb{Z}} q^{n} I_{n,[C]} = \frac{q}{(1+q)^{2}}$$

$$=$$

Substitute $q = -e^{iu}$

$$\frac{-e^{iu}}{(1-e^{iu})^{2}} = -\frac{1}{(e^{iu/2} - e^{-iu/2})}$$

$$= \left(\frac{2i}{e^{iu/2} - e^{-iu/2}} \right)^{2} \frac{1}{2^{2}}$$

$$\sin(x) = \frac{e^{ix} - e^{-ix}}{2i}$$

$$= \left(\frac{u/2}{\sin(u/2)} \right)^2 \frac{1}{u^2}$$

$$= \sum_{g \geq 0} u^{2g-2} N_{g,[C]}$$

- GW/DT Correspondence of MNOP

Maulik
Nekrasov
Okounkov
P

Conjecture : The relationship found for $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$
$$\downarrow$$
$$\mathbb{P}^1$$
holds in general

Topological vertex and Box Counting allow for further examples

Aganagic Klemm Mariño Vafa

Let us write the conjecture precisely.

X is a CY3 fold

$$F'_{GW} = \sum_{g \geq 0} \sum_{\beta \neq 0} N_{g,\beta} \, u^{2g-2} v^{\beta}$$

GW theory of connected, nonconstant maps

$$Z'_{GW} = \exp(F')$$

disconnected theory, but nonconstant on every component

$$Z'_{GW} = 1 + \sum_{\beta \neq 0} Z'_{GW}(X,u)_{\beta} \, v^{\beta}$$

$$Z_{DT} = \sum_{\text{all } \beta} \sum_{n \in \mathbb{Z}} I_{n,\beta}\, q^n v^\beta$$

$$Z_{DT} = \sum_{\text{all } \beta} Z_{DT}(X,q)_\beta\, v^\beta$$

MNOP Conjecture 1:

$$Z_{DT}(X,q)_0 = M(-q)^{e(X)}$$

$\beta = 0$, hence about Hilbert schemes of points on X

STATUS:

Proven

Jun Li

Behrend-Fantechi

Levine-P

$$Z'_{DT} = Z_{DT} \big/ Z_{DT}(x,q)_0$$

idea: remove the constant contributions

$$Z'_{DT} = 1 + \sum_{\beta \neq 0} Z'_{DT}(x,q)_\beta \, v^\beta$$

MNOP Conjecture 2:

$Z'_{DT}(x,q)_\beta$ is the Laurent expansion of a rational function in q.

Also: $Z'_{DT}(x,q) = Z'_{DT}(x, 1/q)$

Status : Proven   Bridgeland, Toda
wallcrossing

MNOP Conjecture 3 :

$$Z'_{GW}(x, u) = Z'_{DT}(x, q)$$

after $\quad -e^{iu} = q$

Status : Open, but proven
in many cases
Cy3 toric geometries   MNOP
MOOP
Complete intersection Cy3s Pixton-P