Enumerative Geometry of Curves, Maps, and Sheaves

Part II: Stable maps

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(i) Moduli of Stable maps

Let X be a nonsingular projective variety $/\mathbb{C}$

We will consider maps

$$f : C \to X$$

target

algebraic morphism

Complete connected nodal curve of genus $g = 1 - \chi(\mathcal{O}_C)$

$$f_* [C] = \beta \in H_2(X, \mathbb{Z})$$

Curve class

$\overline{M}_g(X, \beta)$ is the moduli space of stable maps of genus g curves to X representing β.

- $[f : C \to X] \in \bar{M}_g(X, \beta)$ is stable

  if and only if $|\mathrm{Aut}(f)| < \infty$.

- An automorphism of f is an automorphism of C which commutes with f:

$$\begin{tikzcd} C \arrow[dr, "f"] \arrow[dd, "\phi"'] \arrow[ddr, phantom, "\circlearrowright"] & \\ & X \\ C \arrow[ur, "f"'] & \end{tikzcd}$$

$$\mathrm{Aut}(f) \subset \mathrm{Aut}(C)$$

if $|\mathrm{Aut}(C)| < \infty$
then $|\mathrm{Aut}(f)| < \infty$
and f is stable

When are two stable maps

$$[f : C \to X], \quad [\hat{f} : \hat{C} \to X]$$

isomorphic? If and only if

$\exists\ \phi : C \xrightarrow[\sim]{\text{isom}} \hat{C}$ which commutes with $f, \hat{f}$:

$$\begin{tikzcd}
C \arrow[rd, "f"] \arrow[d, "\phi"'] \arrow[r, phantom, "\circlearrowright"] & \\
\hat{C} \arrow[r, "\hat{f}"'] & X
\end{tikzcd}$$

parallel definitions, Auts and isoms must respect the markings

$\overline{\mathcal{M}}_g(X, \beta)$ and $\overline{\mathcal{M}}_{g,n}(X, \beta)$

are Deligne-Mumford stack, but may be reducible, non-reduced, and very singular.

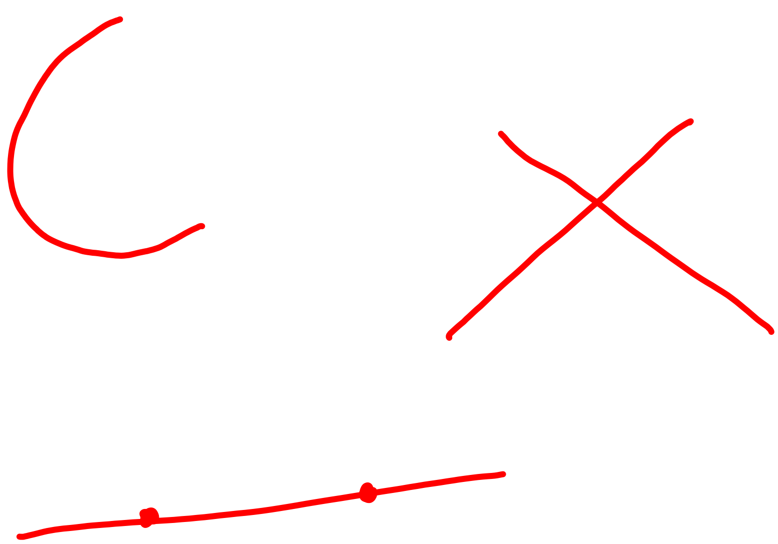
First examples:

- $\overline{\mathcal{M}}_{g,n}(X,0) = \overline{\mathcal{M}}_{g,n} \times X$ for $2g-2+n>0$

- $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^n, 1) = Gr(\mathbb{P}^1, \mathbb{P}^n)$

  class of the line $L \in H_2(\mathbb{P}^n, \mathbb{Z})$

- $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^2, 2)$ = classical space of complete conics



(ii) <u>Obstruction theory</u>

$\bar{M}_{g,n}(X,\beta)$ carries a Def-Obs theory with

$\chi(f^* T_X)$

$$\text{vir dim } \bar{M}_{g,n}(X,\beta) = \int_\beta c_1(X) + \dim_{\mathbb{C}} X(1-g) + 3g-3+n$$

dim of $\bar{M}_{g,n}$

The Def-Obs theory for a fixed domain curve $f: C \to X$ is

Def $\quad H^0(C, f^* T_X)$

Obs $\quad H^1(C, f^* T_X)$

higher obstructions vanish

$$\begin{array}{c} \mathcal{M}_C \times C \\ {\scriptstyle \pi}\swarrow \quad \searrow{\scriptstyle f} \\ \mathcal{M}_C \qquad X \end{array}$$

$\mathcal{M}_C(X,\beta)$ has Def-Obs theory

$\uparrow$ fixed domain

of vir dim $\chi(C, f^*T_X)$

$$\begin{array}{c} (R\pi_* f^*T_X)^\vee \\ \downarrow \\ L^\bullet_{\mathcal{M}_C} \end{array}$$

$\swarrow$ Artin stack

Then, since $\mathcal{M}_{g,n}$ is nonsingular,

we obtain a Def-obs theory

for $\overline{M}_{g,n}(X,\beta)$.

Behrend-Fantechi

Li-Tian

Gromov witten theory

$$[\overline{M}_{g,n}(X,\beta)]^{vir} \in A_{\text{virdim}}\left(\overline{M}_{g,n}(X,\beta)\right)$$

Exercise: The virtual class of $\overline{M}_{g,n}(X,0)$

is given by $c_{top}\left(\mathbb{E}_g^* \boxtimes T_X\right)$

on $\overline{M}_{g,n}(X,0) = \overline{M}_{g,n} \times X$

$\mathbb{E}_g \to \overline{M}_{g,n}$

Hodge bundle

with fiber

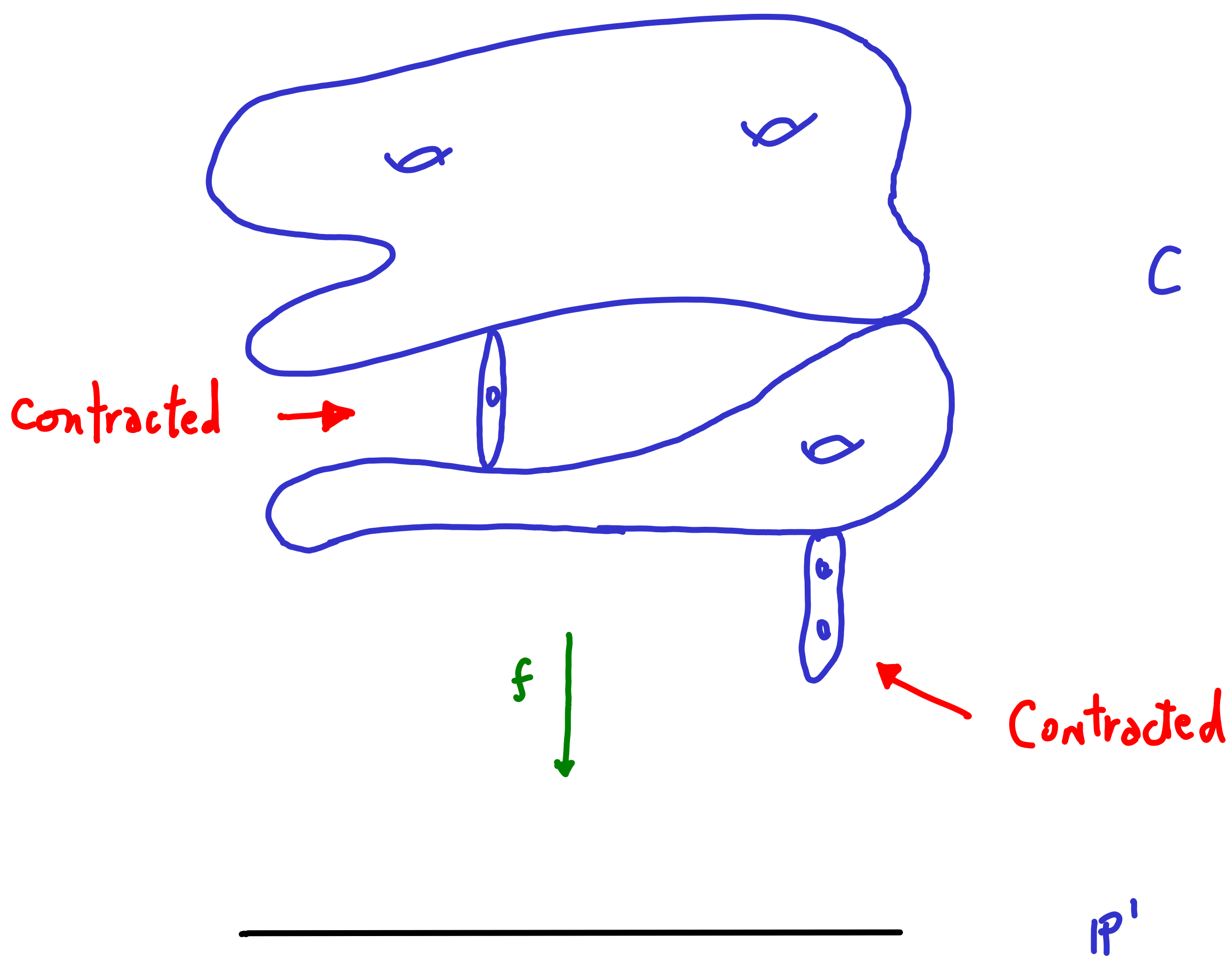
$H^0(C, \omega_C)$

(iii) Maps to $\mathbb{P}^1$

$\overline{M}_{g,n}(\mathbb{P}^1, d)$ has virtual dim $2g + 2d - 2 + n$

degree d maps

A general map looks like



We can connect Gromov-Witten theory to Hurwitz's older enumerative geometry of maps to $\mathbb{P}^1$.

How does a ramification condition appear?

$$\overline{\mathcal{M}}_{g,1}(\mathbb{P}^1, d) \xrightarrow{ev_1} \mathbb{P}^1$$

evaluation map

$$\begin{array}{c} \mathbb{L}_1 \\ \downarrow \\ \overline{\mathcal{M}}_{g,1}(\mathbb{P}^1, d) \end{array}$$

cotangent line on the domain at the marking

$\psi_1 \cdot ev_1^*(p) \sim$ imposition of a ramification condition over $p \in \mathbb{P}^1$

$$df : T_{C,x} \to T_{\mathbb{P}^1, f(x)}$$

differential of f at the marked point $x \in C$

We can rewrite as a section

$$df \in H^0(T^*_{C,1} \otimes f^* T_{\mathbb{P}^1})$$
$$\parallel$$
$$H^0(\mathbb{L}_1 \otimes f^* T_{\mathbb{P}^1})$$

After we also impose $ev_1^{-1}(p)$,

$f^* T_{\mathbb{P}^1} \Big|_{ev_1^{-1}(p)}$ is trivial.

So the vanishing of df restricted to $ev_1^{-1}(p)$ represents ψ_1 and occurs at critical points of f.

Theorem: Basic GW/Hurwitz Correspondence P 1999

$$\int_{\left[\overline{\mathcal{M}}_{g,\,2g+2d-2}(\mathbb{P}^1,d)\right]^{vir}} \prod_{i=1}^{2g+2d-2} \psi_i \, ev_i^*(p) = \mathcal{H}ur^{\circ}_{g,d}$$

Hurwitz Count of Connected Covers with Simple ramifications

Proof: Show the intersections avoid all pathologies of the moduli of maps to $\mathbb{P}^1$

Branch morphism Fantechi-P is useful here

$$br: \overline{\mathcal{M}}_{g,n}(\mathbb{P}^1,d) \to Sym^{2g+2d-2}(\mathbb{P}^1)$$

(iv) <u>Descendents</u>

For $\gamma \in H^*(X, \mathbb{Q})$, notation

$$\tau_k(\gamma) \longleftrightarrow \psi^k \, ev^*(\gamma)$$

and for integrals

$$\left\langle \tau_{k_1}(\gamma_1) \cdots \tau_{k_n}(\gamma_n) \right\rangle^X_{g,n,\beta}$$

redundant

$$\| $$

$$\int_{[\overline{M}_{g,n}(X,\beta)]^{vir}} \prod_{i=1}^{n} \psi_i^{k_i} \, ev_i^*(\gamma_i)$$

Restatement: $\left\langle \tau_1(p)^{2g+2d-2} \right\rangle^{\mathbb{P}^1}_{g,d} = Hur^{\circ}_{g,d}$

Two immediate questions:

(A) is there a such a statement for higher descendents?

(B) is there a generalization of Witten's Conjecture which controls descendents for the target X?

The answers are Yes in both cases with some qualifications

(A) for $\mathbb{P}^1 \Rightarrow$ full GW/Hurwitz Correspondence Okounkov-P

for $X \Rightarrow$ Relative GW theory, Descendent/Relative Maulik-P

Theorem: GW/Hurwitz correspondence Okounkov - P

$$\left\langle \prod_{i=1}^{n} \tau_{k_i}(p) \right\rangle^{\bullet}_{g,d} = \prod_{i=1}^{n} \frac{1}{k_i!} \mathrm{Hur}^{\bullet}_{g,d}\left(\prod_{i=1}^{n} \overline{(k_i+1)} \right)$$

disconnected domains

$\overline{(k_i+1)}$ is the completed cycle,

an object in the theory of symmetric functions

Kerov, Olshanski, Okounkov (and others)

$$\overline{(k+1)} = (k+1) + \text{corrections}$$

usual cycle

Examples (see "GW theory, Hurwitz theory and completed cycles" Okounkov – P 2006)

degenerate constant contributions

$$\overline{(1)} = (1) - \frac{1}{24}(\ )$$

First case

$$\overline{(2)} = (2)$$

$$\overline{(3)} = (3) + (1,1) + \frac{1}{12}\cdot(1) + \frac{7}{2880}(\ )$$

There is a simple formula for these correction coefficients

$$S(z) = \frac{\sinh(x/2)}{x/2} = \frac{e^{x/2} - e^{-x/2}}{x}$$

Then we have

$$(\overline{\kappa}) = \sum_{\mu} \rho_{\kappa,\mu} \, (\mu)$$

Partition $\mu = \{\mu_i\}$ of size $|\mu|$ and length $\ell(\mu)$

$$\rho_{\kappa,\mu} = (\kappa-1)! \, \frac{\prod \mu_i}{|\mu|!} \left[z^{k+1-|\mu|-\ell(\mu)} \right] S(z)^{|\mu|-1} \cdot \prod S(\mu_i z)$$

(B) Virasoro Constraints for arbitrary targets X

Eguchi
Hori
Xiong
S. Katz

For simplicity, we assume X has only even cohomology of type (p,p).

The general case is important even for $\dim_{\mathbb{C}} X = 1$!

Let X be a nonsingular projective variety of $\dim_{\mathbb{C}} X = r$.

$\gamma_0 = \mathrm{Id}$ class

Let $\{\gamma_a\}$ be a basis of X with $\gamma_a \in H^{2p_\alpha}(X, \mathbb{Q})$

By our assumptions, $\gamma_a \in H^{p_a, p_a}(X)$

As before, we define

$$\left\langle \prod \tau_{k_i}(\gamma_{a_i}) \right\rangle_{g,\beta}^X = \int_{[\bar{M}_{g,n}(X,\beta)]^{vir}} \prod \psi_i^{k_i} ev_i^*(\gamma_{a_i})$$

Let $$\phi = \sum_{a,\kappa} t_\kappa^a \, \tau_k(\gamma_a)$$

and $$F^X = \sum_{g \geq 0} \lambda^{2g-2} \sum_\beta q^\beta \sum_{n \geq 0} \frac{1}{n!} \langle \phi^n \rangle_{g,\beta}^X$$

Finally, let $\mathcal{Z}^X = \exp(\mathcal{F}^X)$

Virasoro Conjecture: $L_k \mathcal{Z}^X = 0$

for all $k \geq -1$

where

$$L_k = \sum_{m=0}^{\infty} \sum_{i=0}^{k+1} \Big([b_a + m]_i^k \, (C^i)_a^b \, \tilde{t}_m^a \, \partial_{b,m+k-i}$$

$$+ \frac{\lambda^2}{2} (-1)^{m+1} [-b_a - m]_i^k \, (C^i)^{ab} \, \partial_{a,m} \, \partial_{b,k-m-i-1} \Big)$$

Repeated indices summed

$$+ \frac{\lambda^{-2}}{2} \left(C^{k+1} \right)_{ab} t_0^a t_0^b$$

$$+ \frac{\delta_{k0}}{48} \int_X \big((3-r) c_r(x) - 2 c_1(x) c_{r-1}(x) \big)$$

Various terms require definition

- $g_{ab} = \int_X \gamma_a \gamma_b$ intersection pairing

  used to raise and lower indicies

- matrix $C_a^{\ b}$ is defined by

$$C_a^{\ b} \gamma_b = c_1(x) \cup \gamma_a$$

- Combinatorial coefficients

$$b_a = p_a + (1-r)/2$$

$$[x]_i^k = e_{k+1-i}(x, x+1, \ldots, x+k)$$

elementary symmetric function

. Variables with dilaton shift

$$\tilde{t}^a_m = t^a_m - \delta_{a0}\,\delta_{m1}$$

$$\partial_{a,m} = \frac{\partial}{\partial t^a_m}$$

Cases known

- X is a point Witten's Conjecture
- $\dim_{\mathbb{C}} X = 1$ Okounkov-P
- $QH^*(X)$ is semisimple Teleman, Givental-Teleman classification

  $X = \mathbb{P}^n$ or G/P
- for $X = CY3$, trivial
- for all X in genus 0 X. Liu-Tian, Getzler

Unknown for most varieties: hypersurfaces, surfaces of general type, Fanos, etc.

(v) Projective plane $\mathbb{P}^2$

The basic Gromov-Witten invariants count Severi degrees.

$$N_{g,d} = \left\langle \tau_0(p)^{3d-1+g} \right\rangle_{g,d}^{\mathbb{P}^2}$$

$$= \int_{[\overline{M}_{g,3d-1+g}(\mathbb{P}^2,d)]^{vir}} \prod_{i=1}^{3d-1+g} ev_i^*(p)$$

Classical dimension bounds imply that $N_{g,d}$ is the actual count of genus g, degree d curves through $3d-1+g$ general points in $\mathbb{P}^2$

<u>genus 0</u>: The fundamental equation comes from quantum cohomology (associativity of the quantum product $\star$)

$$N_{0,d} = \sum_{\substack{d_1+d_2=d \\ d_i>0}} N_{d_1} N_{d_2} \left[d_1^2 d_2^2 \binom{3d-4}{3d_1-2} - d_1^3 d_2 \binom{3d-4}{3d_1-1} \right]$$

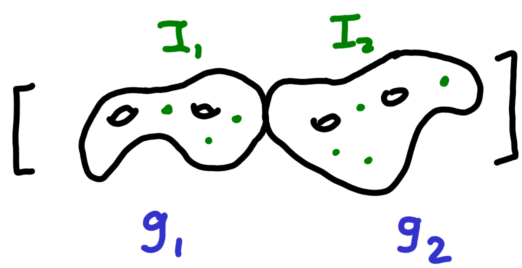
Kontsevich
WDVV

$N_{0,1}=1, \ N_{0,2}=1, \ N_{0,3}=12, \ N_{0,4}=620, \ldots$

initial condition

There are many good expositions.

$$\begin{tikzcd} \overline{\mathcal{M}}_{g,n}(X,\beta) \arrow[d, "\varepsilon"'] & \overline{\mathcal{M}}_{g,n}(X,\beta) \times_{\overline{\mathcal{M}}_{g,n}} D \arrow[l] \arrow[d] \\ \overline{\mathcal{M}}_{g,n} & D = \overline{\mathcal{M}}_{g_1, I_1+\bullet} \times \overline{\mathcal{M}}_{g_2, I_2+\star} \arrow[l, "\delta"'] \end{tikzcd}$$



The rule for computing the pull-back:

$$\delta^\star \left[\overline{\mathcal{M}}_{g,n}(X,\beta) \right]^{vir} =$$

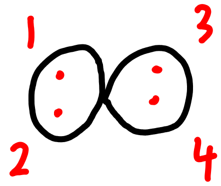
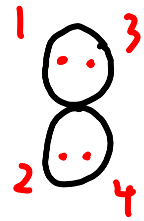
$$\sum_{\beta_1+\beta_2=\beta} \left[\overline{\mathcal{M}}_{g_1, I_1+\bullet}(X, \beta_1) \right]^{vir} \times \left[\overline{\mathcal{M}}_{g_2, I_2+\star}(X, \beta_2) \right]^{vir} \cdot ev_{\bullet,\star}^{\;\star}(\Delta)$$

class of diagonal in $X \times X$

Splitting Axioms in Gromov-Witten theory

Behrend, Behrend-Fantechi

WDVV equation are obtained from the splitting axioms and the relation

$[$  $] = [$  $]$ in $H^2(\bar{M}_{0,4})$

Exercise: Derive the recursion for $N_{0,d}$

Requires some geometric ideas.

We see that the cohomology of $\bar{M}_{g,n}$ constrains the Gromov-Witten invariants.

Also the opposite is true. The geometry of the moduli space of stable maps constrains the cohomology of $\bar{M}_{g,n}$.

Example: Proof of Pixton's relations

genus 1: Much more subtle, and less well-known.

Eguchi
Hori
Xiong

$$N_{1,d} = \frac{1}{12}\binom{d}{3} N_{0,d} + \sum_{\substack{d_1+d_2=d \\ d_i>0}} \frac{3d_1^2 d_2 - 2d_1 d_2}{9} \binom{3d-1}{3d_1-1} N_{0,d_1} N_{1,d_2}$$

$N_{1,1} = 0$, $N_{1,2} = 0$, $N_{1,3} = 1$, $N_{1,4} = 225$

Consequence of the Virasoro constraints for $\mathbb{P}^2$ (also can be derived from Getzler's relation in $\overline{M}_{1,4}$).

How to prove the $g=1$ recursion?

Step 1. Write L_1 explicitly for $\mathbb{P}^2$

Step 2. Extract the coefficient of $\lambda^0 (t_0^2)^{3d-1}$ in $\frac{L_1 Z^{\mathbb{P}^2}}{Z^{\mathbb{P}^2}}$.

You will find

$$-\frac{9}{(3d-1)!} \langle \tau_0(p)^{3d} \rangle_{1,d} + \cdots$$

which vanishes by the Virasoro constraints.

Step 3. Terms with the insertions

$T_1(H)$ and $T_2(1)$

require applications of

TRR in genus 1:

[and the String, divisor equation]

$$\left\langle T_r(\gamma) \cdot T_0(p)^m \right\rangle_{1,d}^{\mathbb{P}^2}$$

$$\|$$

$$\frac{d^2}{24} \left\langle T_0(\gamma) \cdot T_0(p)^m \right\rangle_{0,d}^{\mathbb{P}^2}$$

$$+ \sum_{\substack{d_1+d_2=d \\ m_1+m_2=m}} \binom{m}{m_1} \left\langle T_{r-1}(\gamma)\, T_0(p)^{m_1}\, T_0(\gamma^a) \right\rangle_{0,d_1}^{\mathbb{P}^2} g^{ab} \left\langle T_0(\gamma^b)\, T_0(p)^{m_2} \right\rangle_{1,d_2}^{\mathbb{P}^2}$$

genus $g \geq 2$

Are there higher genus recursions?

Answer is Yes, but much more complicated forms involving additional recursions for descendents

- Using Virasoro + TRR $\Rightarrow$ Gathmann

- There are also recursions using degenerations   Z. Ran Caporaso-Harris

For other surface, many open questions:

- Virasoro constraints for the Enriques surface Not known even in genus 1

- Can not be formulated in symplectic geometry in general (requires Hodge decomposition)

Very strange state of affairs which led to some doubts about the constraints.

Derivation of the Eguchi-Hori-Xiong $g=1$ recursion from L_1 for $\mathbb{P}^2$.

Notes written by Longting Wu:

Let us set $p^0=1$, $p^1=H$, $p^2=H^2$. Then variables

$$t_m^i \rightsquigarrow \tau_m(H^i)$$

By taking the coefficient $(t_0^2)^{3d-1}$ of

$$L_1 \exp(F^{\mathbb{P}^2}) = 0$$

we get

$$\frac{15}{4}(3d-1)\langle \tau_1(H^2)\underbrace{\tau_0(H^2)\cdots\tau_0(H^2)}_{3d-2}\rangle_{1,d} - 9N_{1,d}$$

$$-\frac{3}{4}\langle \tau_2(1)\underbrace{\tau_0(H^2)\cdots\tau_0(H^2)}_{3d-1}\rangle_{1,d} - 6\langle \tau_1(H)\underbrace{\tau_0(H^2)\cdots\tau_0(H^2)}_{3d-1}\rangle_{1,d}$$

$$+\frac{d^2}{8}N_{0,d} + \frac{1}{4}\sum_{\substack{d_1+d_2=d\\ d_i>0}}(d_1d_2)\binom{3d-1}{3d_1-1}N_{0,d_1}N_{1,d_2} - \frac{d}{32}N_{0,d}$$

$$=0 \quad *$$

We set

$$N_{1,d} = \langle \underbrace{\tau_0(H^2)\cdots\tau_0(H^2)}_{3d} \rangle_{1,d}$$

pull back relation

$$\psi_1 = \frac{1}{12}[\ ]$$

in $\overline{M}_{1,1}$

$$N_{0,d} = \langle \tau_0(H^2)\cdots\tau_0(H^2) \rangle_{0,d}$$

Using the topology recursion for genus 1, we get

$$\langle \tau_1(H^2)\underbrace{\tau_0(H^2)\cdots\tau_0(H^2)}_{3d-2} \rangle_{1,d} = \frac{1}{24}d^2 N_{0,d} - \frac{d}{8}N_{0,d} + \sum_{\substack{d_1+d_2=d \\ d_i>0}} \binom{3d-2}{3d_1-2}(d_1 d_2) N_{0,d_1} N_{1,d_2}$$

$$\langle \tau_1(H)\underbrace{\tau_0(H^2)\cdots\tau_0(H^2)}_{3d-1} \rangle_{1,d} = \frac{1}{24}d^3 N_{0,d} - \frac{d^2}{8}N_{0,d} + \sum_{\substack{d_1+d_2=d \\ d_i>0}} \binom{3d-1}{3d_1-1} d_1^2 d_2 N_{0,d_1} N_{1,d_2}$$

$$\langle \tau_2\underbrace{\tau_0(H^2)\cdots\tau_0(H^2)}_{3d-1} \rangle_{1,d} = \frac{d^2(3d-1)}{24}N_{0,d} - \frac{(3d-2)d}{8}N_{0,d} + \sum_{\substack{d_1+d_2=d \\ d_i>0}} \binom{3d-1}{3d_1-1}(3d_1-2)d_1 d_2 N_{0,d_1} N_{1,d_2}$$

Plug these into previus relation $*$

We get recursion

$$N_{1,d} = \frac{1}{12}\binom{d}{3} N_{0,d} + \sum_{\substack{d_1+d_2=d \\ d_i>0}} \frac{3d_1^2 d_2 - 2d_1 d_2}{9} \binom{3d-1}{3d_1-1} N_{0,d_1} N_{1,d_2}$$