Enumerative Geometry of Curves, Maps, and Sheaves

Part I: cotangent lines

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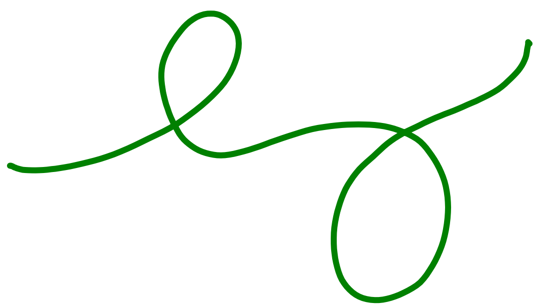
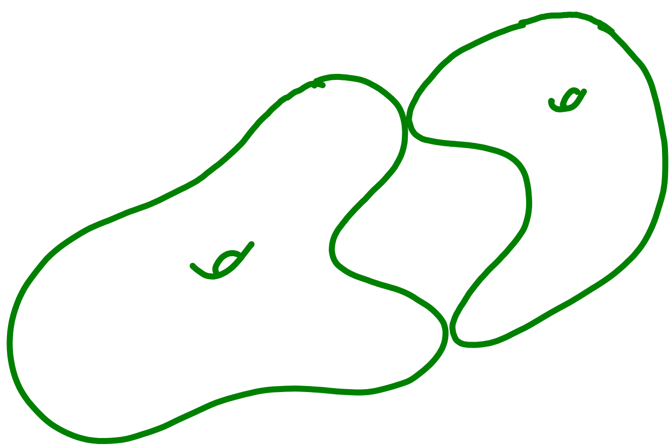
12 July 2021

(i) Deformations

Consider a nodal curve

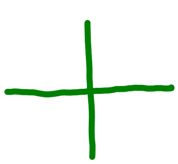


or perhaps



For us: nodal curves will be complex algebraic varieties of dim 1 with at worst nodal singularities

étale locally



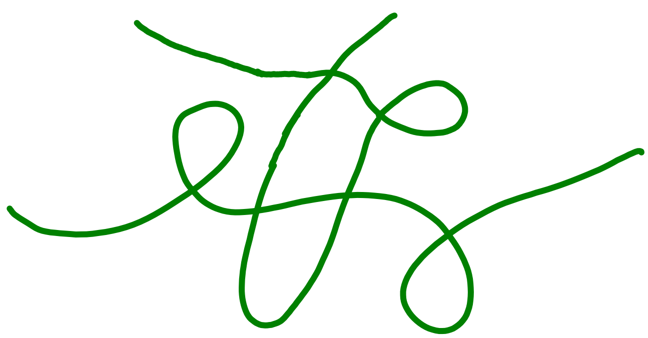
$(xy=0) \subset \mathbb{C}^2$

A nodal curve may be nonsingular

Usually a nodal curve will be assumed to be

- Connected
- Complete (= projective)

unless otherwise stated, but <u>not</u> irreducible.



We will start with some basics about the deformation theory of nodal curves.

Let C be a nodal curve.

If C is nonsingular of genus g, then

$$\mathrm{Def}(C) = H^1(\mathrm{Tan}_C)$$

dim $= 3g-3$
given by Riemann-Roch
for $g \geq 2$

$$H^2(\mathrm{Tan}_C) = 0$$

so the deformations are unobstructed

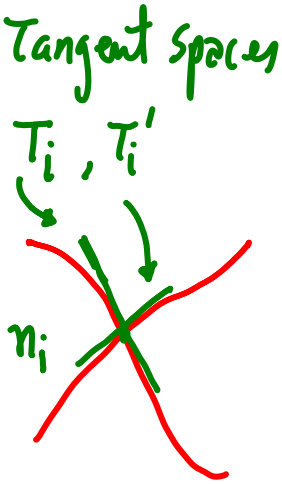
If C has nodes, the deformations are

$$\mathrm{Def}(C) = \mathrm{Ext}^1(\Omega_C, \mathcal{O}_C)$$

Sheaf of Kähler differentials

again $\mathrm{Ext}^2(\Omega_C, \mathcal{O}_C) = 0$

so the deformations are again unobstructed



But nodal curves are more interesting

Let $n_1, n_2, \ldots, n_\delta \in C$ be the nodes

By the local-to-global Ext sequence:

$$0 \to H^1(\mathcal{E}xt^0(\Omega_C, \mathcal{O}_C)) \to \mathrm{Ext}^1(\Omega_C, \mathcal{O}_C) \to H^0(\mathcal{E}xt^1(\Omega_C, \mathcal{O}_C)) \to 0$$

Moreover $\mathcal{E}xt^1(\Omega_C, \mathcal{O}_C)$ is a

sky scraper sheaf supported at the

nodes n_i. By a crucial local

calculation

smoothing of the nodes

$$\mathcal{E}xt^1(\Omega_C, \mathcal{O}_C) = \bigoplus_{i=1}^{\delta} T_i \otimes T_i'$$

Tangent Spaces at n_i

So we have

$$\mathrm{Def}(C) \xrightarrow{\mu} \bigoplus_{i=1}^{\delta} T_i \otimes T_i' \to 0$$

$\mathrm{Ker}(\mu)$ = deformations which preserve the nodes.

(ii) <u>Moduli</u>

noncompact → M_g moduli space of genus g $(g \geq 2)$ nonsingular curves

compact → $\overline{M}_g$ moduli space of genus g $(g \geq 2)$ Deligne-Mumford stable curves

C is nodal and connected, ω_C is ample

$\mathcal{M}_{g,n}$ moduli space of genus g n-pointed nonsingular curves

compact → $\overline{\mathcal{M}}_{g,n}$ Deligne-Mumford stable curves $(2g-2+n>0)$

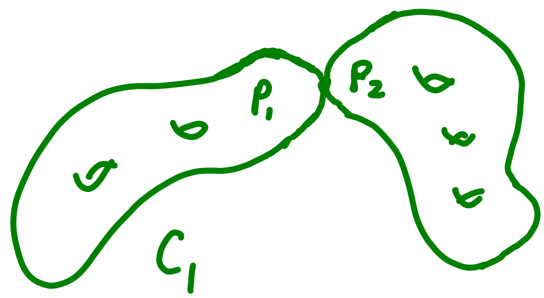
C is nodal and connected,
$p_1, p_2, \dots, p_n \in C$ distinct points in the nonsingular locus,
$\omega_C(\sum p_i)$ is ample

$\overline{\mathcal{M}}_{g,n}$ is a nonsingular, irreducible DM stack (or orbifold) of $\dim_{\mathbb{C}}$ $3g-3+n$

Proven by Deligne-Mumford 1960's
but goes back to Riemann 1860's

Tangent lines : $\begin{array}{c} T_i \\ \downarrow \\ \overline{\mathcal{M}}_{g,n} \end{array}$ $\mathbb{C}$- line bundle determined by the tangent space at the i^{th} point

Example :



C_1 genus g_1

C_2 genus g_2

moduli of stable curves

$$\overline{\mathcal{M}}_{g_1,1} \times \overline{\mathcal{M}}_{g_2,1} \xrightarrow{\delta} \overline{\mathcal{M}}_{g_1+g_2}$$

$\uparrow$ p_1 $\uparrow$ p_2

Normal bundle to δ is $T_{p_1} \otimes T_{p_2}$

$$T_{p_1} \otimes T_{p_2} \to \overline{\mathcal{M}}_{g_1,1} \times \overline{\mathcal{M}}_{g_2,2}$$

(iii) Cotangent lines

$$\begin{array}{c} T_i^* \\ \downarrow \\ \overline{\mathcal{M}}_{g,n} \end{array}$$

$\mathbb{C}$- line bundle determined by the cotangent space at the i^{th} point

We will use the notation

$$\begin{array}{c} \mathbb{L}_i = T_i^* \\ \downarrow \\ \overline{\mathcal{M}}_{g,n} \end{array}$$

First geometric result:

Proposition: $\mathbb{L}_i$ is nef on $\overline{\mathcal{M}}_{g,n}$

Proof: Let

$$\begin{array}{c} \mathcal{C} \\ \pi \big\downarrow \big\uparrow s_1, \dots, s_n \\ B \end{array}$$

be a family of stable curves of genus g and n marked points over a 1-dimensional base B

for k large,

$R^0\pi_*(\omega_\pi(\Sigma s_j)^k)$ is semipositive

bundle of degree ≥ 0

See "Projectivity of complete moduli" by J. Kollár §4.7

By GRR $\Rightarrow$ $\displaystyle\int_{\mathcal{C}} c_1(\omega_\pi(\Sigma s_j))^2 \geq 0$

algebraic surface $\to \mathcal{C}$

Now $\displaystyle\int_{\mathcal{C}} c_1(\omega_\pi(\Sigma s_j)) \cdot [s_i]$

$$\|$$

$$\text{degree } \omega_\pi\big|_{s_i} + \text{degree } \mathcal{O}_{\mathcal{C}}(s_i)\big|_{s_i}$$

But on s_i, $\omega_\pi \cong \mathcal{O}_{\mathcal{C}}(s_i)^*$

so $\displaystyle\int_{\mathcal{C}} c_1(\omega_\pi(\Sigma s_j)) \cdot [s_i] = 0$

Hence, by the Hodge index Theorem for surfaces $\Rightarrow$ $\displaystyle\int_{\mathcal{C}} [s_i]^2 \leq 0$

We use here $\mathbb{L}_i\big|_B \cong s_i^*(\mathcal{O}_{\mathcal{C}}(s_i)^*)$

Why is 0 possible? Hint: reducible curves.

$$\int_{\overline{\mathcal{M}}_{g,n}} [B] \cdot c_1(\mathbb{L}_i) \geq 0 \quad \square$$

standard notation:

$$\psi_i = c_1(\mathbb{L}_i) \in H^2(\overline{M}_{g,n})$$

We have the forgetful map

$$\pi : \overline{M}_{g,n+1} \to \overline{M}_{g,n}$$

forget $(n+1)^{st}$ marking

We define

$$\kappa_r = \pi_* \psi_{n+1}^{r+1} \in H^{2r}(\overline{M}_{g,n})$$

Kappa notation goes back to Mumford.

Convention of Arbarello-Cornalba

Another basic geometric fact:

$H^2(M_{g,n}, \mathbb{Q}) \cong Pic(M_{g,n}, \mathbb{Q})$ is generated by $\kappa_1, \psi_1, \dots, \psi_n$

By Nefness $\Rightarrow \displaystyle\int_{\overline{M}_{g,n}} \psi_1^{k_1} \cdots \psi_n^{k_n} \geq 0$

and the integral vanishes unless $\displaystyle\sum_{i=1}^{n} k_i = 3g-3+n$

Question: Can $\displaystyle\int_{\overline{M}_{g,n}} \psi_1^{k_1} \cdots \psi_n^{k_n}$ vanish if $\displaystyle\sum_{i=1}^{n} k_i = 3g-3+n$?

(iv) <u>Witten's Conjecture</u>

$$\langle \tau_{k_1} \cdots \tau_{k_n} \rangle_{g,n} = \int_{\overline{M}_{g,n}} \psi_1^{k_1} \cdots \psi_n^{k_n}$$

n is redundant

dim constraint: $\sum k_i = 3g-3+n$

$\langle \tau_{k_1} \cdots \tau_{k_n} \rangle_{g,n} = 0$ if constraint fails

Also 0 if $2g-2+n \leq 0$ by stability

To state Witten's Conjecture,

we form a generating series

$$\mathcal{F}_g(t_0, t_1, t_2, \dots) \overset{\text{definition}}{=} \sum_{\{n_i\}} \prod_{i=1}^{\infty} \frac{t_i^{n_i}}{n_i!} \langle \tau_0^{n_0} \tau_1^{n_1} \tau_2^{n_2} \cdots \rangle_g$$

$g \geq 0$ genus

Sum over all sequences of non-neg integers with only finitely many nonzero terms

$$\int_{\overline{\mathcal{M}}_{g, \Sigma n_i}} \psi_1^0 \cdots \psi_{n_0}^0 \psi_{n_0+1}^1 \cdots \psi_{n_0+n_1}^1 \cdots$$

A more compact way:

$$\mathcal{F}_g = \sum_{n=0}^{\infty} \frac{\langle \phi^n \rangle_g}{n!}$$

where $\phi = \sum_{i=0}^{\infty} t_i \tau_i$

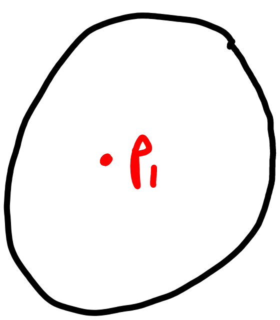
$\mathcal{F}_g$ contains the data of all integrals of ψ-classes over all moduli spaces $\overline{\mathcal{M}}_{g,n}$.

Called descendant integrals

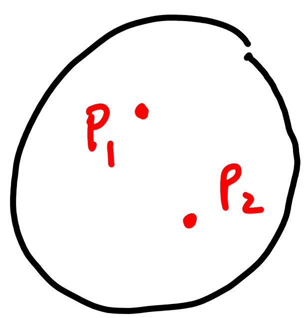
We can put all of the data for all genera together

$$\mathcal{F}(\lambda, t) = \sum_{g=0}^{\infty} \lambda^{2g-2} \mathcal{F}_g$$

[For $\mathcal{F}_0$, 1- and 2-point integrals are 0, since



and



are unstable]

Notation for partial derivatives:

$$\langle\langle \tau_{k_1} \cdots \tau_{k_n} \rangle\rangle = \frac{\partial}{\partial t_{k_1}} \frac{\partial}{\partial t_{k_2}} \cdots \frac{\partial}{\partial t_{k_n}} F$$

Witten's Conjecture / Kontsevich's Theorem (KdV):

For $n \geq 1$, we have:

$$(2n+1)\lambda^{-2} \langle\langle \tau_n \tau_0^2 \rangle\rangle =$$

$$\langle\langle \tau_{n-1} \tau_0 \rangle\rangle \langle\langle \tau_0^3 \rangle\rangle + 2 \langle\langle \tau_{n-1} \tau_0^2 \rangle\rangle \langle\langle \tau_0^2 \rangle\rangle$$

$$+ \frac{1}{4} \langle\langle \tau_{n-1} \tau_0^4 \rangle\rangle$$

Let $u = \dfrac{\partial^2 F}{\partial t_0^2}$

Then the $n=1$ equation $\Rightarrow$

$$\frac{\partial u}{\partial t_1} = u \frac{\partial u}{\partial t_0} + \frac{1}{12} \frac{\partial^3 u}{\partial t_0^3}$$ (set $\lambda = 1$)

Korteweg-de Vries equation

$t_0 \rightsquigarrow x$ space

$t_1 \rightsquigarrow t$ time

u height of wave

First written in the late 19^{th} century to model shallow water waves

What do water waves have to do with $\overline{\mathcal{M}}_{g,n}$?

Long and interesting story

Witten, 2d quantum gravity and intersection theory on moduli space

$$\text{KdV equations} \xleftrightarrow{\text{integrable systems}} \text{Matrix integrals } \int_{\text{matrices}} \xleftrightarrow{\text{2d Gravity}} \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{k_1} \cdots \psi_n^{k_n}$$

Proven by Kontsevich 1992

Okounkov-P, GW theory, Hurwitz numbers and Matrix models 2001

2nd approach via Hurwitz by Kazarian-Lando 2006

Can we calculate $\langle \tau_1 \rangle_1 = \int_{\overline{M}_{1,1}} \psi_1$?

Take equation for $n=3$:

$$7 \langle \tau_3 \tau_0^2 \rangle_1$$

$$\| $$

$$\langle \tau_2 \tau_0 \rangle_1 \langle \tau_0^3 \rangle_0 + \frac{1}{4} \langle \tau_2 \tau_0^4 \rangle_0$$

Set $\lambda = 1$ and all $t_i = 0$

After applying the String equation:

$$\langle \tau_3 \tau_0^2 \rangle_1 = \langle \tau_2 \tau_0 \rangle_1 = \langle \tau_1 \rangle_1$$

$$\langle \tau_2 \tau_0^4 \rangle_0 = \langle \tau_0^3 \rangle = 1$$

$\int_{\overline{M}_{0,3}} 1 = 1$

We find

$$7 \langle \tau_1 \rangle_1 = \langle \tau_1 \rangle_1 \cdot 1 + \frac{1}{4} \cdot 1$$

So $$6 \langle \tau_1 \rangle_1 = \frac{1}{4}$$

Finally $$\langle \tau_1 \rangle_1 = \frac{1}{24}$$

We have used the String equation

$$\left\langle \tau_0 \prod_{i=1}^{n} \tau_{k_i} \right\rangle_{g,n+1} = \sum_{j=1}^{n} \left\langle \tau_{k_j - 1} \prod_{i \neq j} \tau_{k_i} \right\rangle_{g,n}$$

Convention: $\tau_k = 0$ for $k < 0$.

Proof (String equation):

Consider the map

$$\pi : \overline{M}_{g,n+1} \to \overline{M}_{g,n}$$

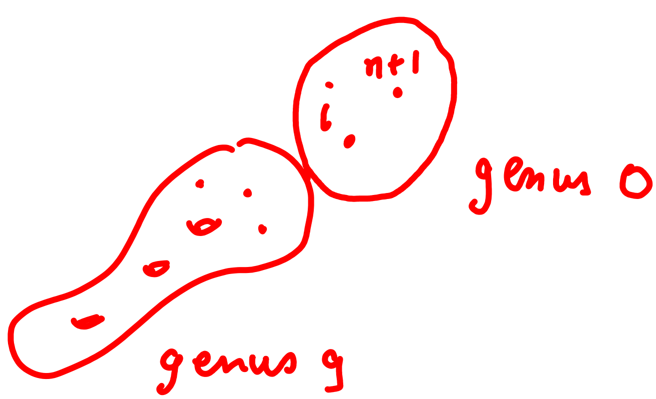
forgetting the last marking.

$$\left\langle \tau_0 \prod_{i=1}^{n} \tau_{k_i} \right\rangle_{g,n+1} = \int_{\overline{M}_{g,n+1}} \psi_1^{k_1} \cdots \psi_n^{k_n} \, \psi_{n+1}^{0}$$

We have the comparison equations

$$\psi_i = \pi^* \psi_i + \Delta_{i,n+1} \qquad \text{for } 1 \le i \le n$$

Divisor of curves



$$\int_{\overline{M}_{g,n+1}} \psi_1^{k_1} \cdots \psi_n^{k_n} \psi_{n+1}^{0} = \int_{\overline{M}_{g,n+1}} \prod_{i=1}^{n} \psi_i \left(\pi^* \psi_i + \Delta_{i,n+1}\right)^{k_i - 1}$$

$$= \int_{\overline{M}_{g,n+1}} \prod_{i=1}^{n} \psi_i \left(\pi^* \psi_i\right)^{k_i - 1}$$

Since

$\psi_i \Delta_{i,n+1} = 0$

$$= \int_{\overline{M}_{g,n+1}} \prod_{i=1}^{n} \left(\pi^* \psi_i^{k_i} + \pi^* \psi_i^{k_i - 1} \Delta_{i,n+1}\right)$$

$$\int_{\overline{M}_{g,n+1}} \prod_{i=1}^{n} \pi^* \psi_i^{k_i} = 0$$

$$= \sum_{j=1}^{n} \int_{\overline{M}_{g,n}} \psi_j^{k_j - 1} \prod_{i \neq j} \psi_i^{k_i}$$

The last expression equals $\rightarrow$ $$\sum_{j=1}^{n} \left\langle \tau_{k_{j-1}} \prod_{i\neq j} \tau_{k_i} \right\rangle_{g,n} \quad \square$$

Exercise: Use the string equation to prove $\langle \tau_{k_1} \cdots \tau_{k_n} \rangle_{0,n} = \binom{n-3}{k_1, \ldots, k_n}$.

The initial condition $\langle \tau_0^3 \rangle_{0,3} = 1$ is required.

Exercise: Prove the dilaton equation

$$\left\langle \tau_1 \prod_{i=1}^{n} \tau_{k_i} \right\rangle_{g,n+1} = (2g-2+n) \left\langle \prod_{i=1}^{n} \tau_{k_i} \right\rangle_{g,n}.$$

The proof of the dilaton equation uses the same geometry as the proof of the string equation.

A Harder Exercise: Prove the genus 1 formula of Eftekhary - Setayesh:

elementary symmetric function

$$\langle \tau_{k_1} \cdots \tau_{k_n} \rangle_{1,n} = \frac{1}{24} \binom{n}{k_1, \ldots, k_n} \cdot \left[1 - \sum_{i=2}^{n} \frac{\sigma_i(k_1, \ldots, k_n)}{i(i-1) \cdot \binom{n}{i}} \right]$$

A proof can be found in their paper "On the Structure of the Kappa ring"

What about removing τ_k for $k>1$?

It is possible using KdV + String

But easier with the Virasoro constraints.

(v) Virasoro Constraints

Recall $\mathcal{F}(\lambda, t) = \sum_{g=0}^{\infty} \lambda^{2g-2} \mathcal{F}_g$

Define $Z(\lambda, t) = \exp(\mathcal{F})$

The string and dilaton equations can be written using

$$L_{-1} = -\frac{\partial}{\partial t_0} + \frac{\lambda^{-2}}{2} t_0^2 + \sum_{i=0}^{\infty} t_{i+1} \frac{\partial}{\partial t_i}$$

$$L_0 = -\frac{3}{2} \frac{\partial}{\partial t_1} + \sum_{i=0}^{\infty} \frac{2i+1}{2} t_i \frac{\partial}{\partial t_i} + \frac{1}{16}$$

as $L_{-1} Z = 0$, $L_0 Z = 0$

String

$+ \langle \tau_0^3 \rangle_{0,3} = 1$

Dilaton

$+ \langle \tau_1 \rangle_{1,1} = \frac{1}{24}$

The bracket is $[L_{-1}, L_0] = -L_{-1}$

Let $\mathcal{L}_n = -u^{n+1} \frac{\partial}{\partial u}$ $n \geq -1$

holomorphic differential operator in the variable u.

Then, we have the Virasoro bracket

$$[\mathcal{L}_n, \mathcal{L}_m] = (n-m)\mathcal{L}_{n+m}$$

For $n > 0$, define

negative coefficient

Positive

$$L_n = -\frac{(2n+3)!!}{2^{n+1}} \frac{\partial}{\partial t_{n+1}}$$

$$+ \sum_{i=0}^{\infty} \frac{(2i+2n+1)!!}{(2i-1)!!\, 2^{n+1}}\, t_i \frac{\partial}{\partial t_{i+n}}$$

$$+ \frac{\lambda^2}{2} \sum_{i=0}^{n-1} \frac{(2i+1)!!\,(-2i+2n-1)!!}{2^{n+1}} \frac{\partial^2}{\partial t_i\, \partial t_{n-1-i}}$$

Then $[L_n, L_m] = (n-m)\, L_{n+m}$

Check algebraically

Virasoro Constraints: $L_n Z = 0$ for $n \geq -1$

Corollary: $\langle \tau_{k_1} \cdots \tau_{k_n} \rangle_{g,n} > 0$

whenever $\sum_{i=1}^{n} k_i = 3g-3+n$

Exercise: compute $\langle \tau_4 \rangle_{2,1} = \frac{1}{1152}$.

Proof (Virasoro Constraints): Two paths

Path I: KdV + String $\Rightarrow$ Virasoro

Dijkgraaf-Verlinde-Verlinde 1991

Path II: Hyperbolic geometry

Mirzakhani's study of

volumes and geodesics 2007

How to use $L_n Z = 0$?

$$Z = \exp(\mathcal{F})$$

Can be interpreted as descendent series for moduli of disconnected curves

generating series of descendent integrals over $\overline{\mathcal{M}}_{g,n}$, moduli of connected curves

However, we expand

$$L_n Z / Z = 0$$

$$\frac{\partial}{\partial t_i} Z / Z = \frac{\partial F}{\partial t_i}$$

So we obtain equations for F

and

$$\frac{\partial}{\partial t_i}\frac{\partial}{\partial t_j} Z / Z = \frac{\partial F}{\partial t_i}\frac{\partial F}{\partial t_j} + \frac{\partial^2 F}{\partial t_i \partial t_j}$$