Descendents for stable pairs on 3-folds

Rahul Pandharipande

ETHZ

Hangzhou   28 April 2020

(ZOOM)

Idea of descendents

GW theory / X nonsingular projective variety

$$\overline{M}_{g,1}(X,\beta)$$

π ↙   ↘ ev

$$\overline{M}_g(X,\beta) \qquad X$$

$$\widetilde{T}_k(\gamma) \in H^*\left(\overline{M}_g(X,\beta)\right)$$

Nonstandard definition!

$$\gamma \in H^*(X)$$

$$\widetilde{T}_k(\gamma) = \pi_*\left(\psi_1^k \cup ev^*(\gamma)\right)$$

ψ_1 is the cotangent line class

The study of descendent classes
in GW theory is a long story

- Tautological ring of $\overline{M}_{g,n}$

  Pixton's relations

- Witten's Conjecture

  Kontsevich, OP, Mirzakhani

- Virasoro Constraints

  Eguchi, Hori, Xiong
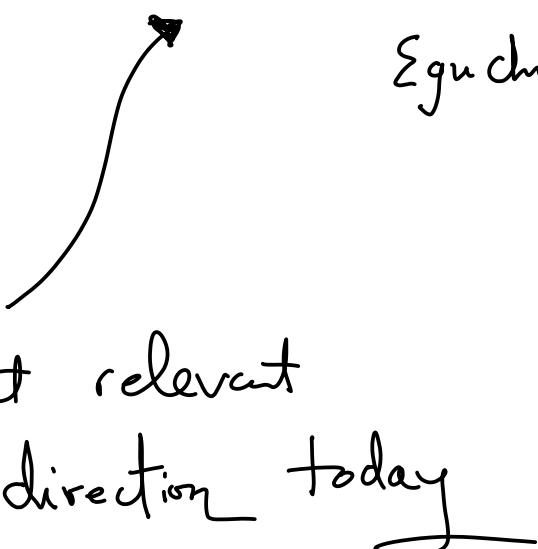
most relevant
direction today

The topic today is descendent classes on moduli spaces of sheaves

- ### dim 1

$$\mathcal{U}_{C,2,\mathcal{L}} = \text{moduli space of rank 2 stable bundles on a fixed curve } C \text{ of genus } g \text{ with a fixed } \det = \mathcal{L}.$$

$$\boxed{\deg \mathcal{L} = 1}$$

Universal bundle

$$\mathcal{E}$$
$$\downarrow$$
$$\mathcal{U} \times C$$
$$\pi_1 \swarrow \qquad \searrow \pi_2$$
$$\mathcal{U} \qquad\qquad C$$
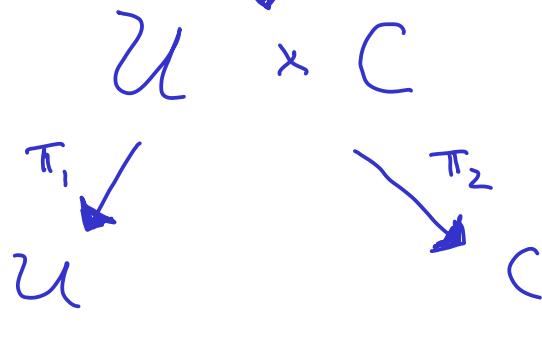
$$T_k(\gamma) = \pi_{1*}\left(\mathrm{ch}_k(\mathcal{E}) \cup \pi_2^*(\gamma) \right)$$

$$\gamma \in H^*(C)$$

Theorem: $H^*(\mathcal{U})$ is generated by descendents.

Mumford, Kirwan, Zagier,... (also find relations).

- dim 2

  Moduli of sheaves on Surfaces

  Descendent theory started with Donaldson.

  A more recent example: Quot schemes
  on Surfaces

  Let S be a nonsingular projective Surface

  $\text{Quot}_S (\not{c}^r, \beta, n)$ of quotients

  $$\not{c}^r \otimes \mathcal{O}_S \longrightarrow Q \to 0$$

  where

  $\text{rk } Q = 0$

  $c_1(Q) = \beta$

  $\chi(Q) = n$

deformation theory

$$0 \to \mathcal{S} \to \mathbb{C}^r \to Q \to 0$$

def $\qquad \mathrm{Ext}^0(\mathcal{S}, Q)$

Obs $\qquad \mathrm{Ext}^1(\mathcal{S}, Q)$

higher $\qquad \mathrm{Ext}^2(\mathcal{S}, Q) = \mathrm{Ext}^0(Q, \mathcal{S} \otimes k) = 0$

Since Q is torsion

so we have a 2-term obstruction theory and a virtual fundamental class

$$\left[\mathrm{Quot}_S(\mathbb{C}^r, \beta, n) \right]^{vir} \quad \text{of} \quad \dim$$

$$r \cdot \eta + \beta^2$$

Marian – Oprea – P   2018

Oprea – P   2019

$$F \quad \text{universal quotient}$$

$$\downarrow$$

$$\text{Quot}\left(\mathbb{C}^r, \beta, n\right) \times S$$

$$\pi_1 \swarrow \qquad \qquad \searrow \pi_2$$

$$\text{Quot} \qquad \qquad \qquad S$$

$$T_k(\gamma) = \pi_{1*}\left(\text{ch}_k F \cup \pi_2^*(\gamma)\right)$$

$$\uparrow$$
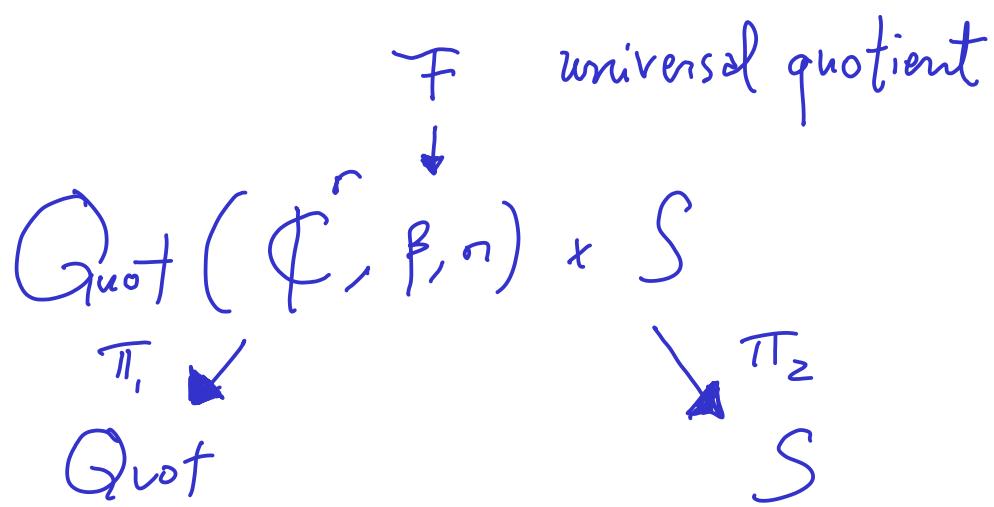
$$\gamma \in H^*(S)$$

descendent series

$$\left\langle T_{k_1}(\gamma_1) \cdots T_{k_\ell}(\gamma_\ell) \right\rangle_{r,\beta}^{S} = \sum_{n \in \mathbb{Z}} q^n \int_{[\text{Quot}(r,\beta,n)]^{vir}} T_{k_1}(\gamma_1) \cdots T_{k_\ell}(\gamma_\ell) \, c\left(T^{vir}\right)$$

$$\mathord{\curvearrowright} \; \mathbb{Q}((q))$$

Conj [D. Johnson, Oprea, P 2020]

The series is Laurent expansion of

a **rational** function in q.

The virtual geometry of

$$Quot_S (\mathbb{C}^r, \beta, n)$$ [MOP]

Lehn's
Conj about
Segre integral
Hilb (S, n)

Nekrasov
Partition function

[Arbesfeld, ...]

Donaldson/SW
invariants

$S = K3$

K theory

χ_y-genus

Voisin

[OP, Woonam Lin]
see also Duerr-Kabanov-Okonek

Ex. S is a minimal surface of general type
with a nonsing canonical curve of genus 3

$$\langle 1 \rangle^S_{2, K_S} = (-1)^{\chi(\mathcal{O}_S)} \cdot \frac{(128q^4 - 64q^3 + 8q^2 - 16q + 8)}{q (1-4q)^4}$$

SW invariant

- dim 3

The main topic of the lecture concerns the moduli of stable pairs on 3-folds

X is nonsingular proj 3-fold

$\beta \in H_2(X, \mathbb{Z})$ curve class

$n \in \mathbb{Z}$

$P_n(X, \beta)$ is the moduli of stable pairs

$[\mathcal{F}, s] \in P_n(X, \beta)$

- $\mathcal{F}$ is pure sheaf of dim 1.
- $\mathcal{O}_X \xrightarrow{s} \mathcal{F}$ is a section with coker of dim 0

Stable pair

χ
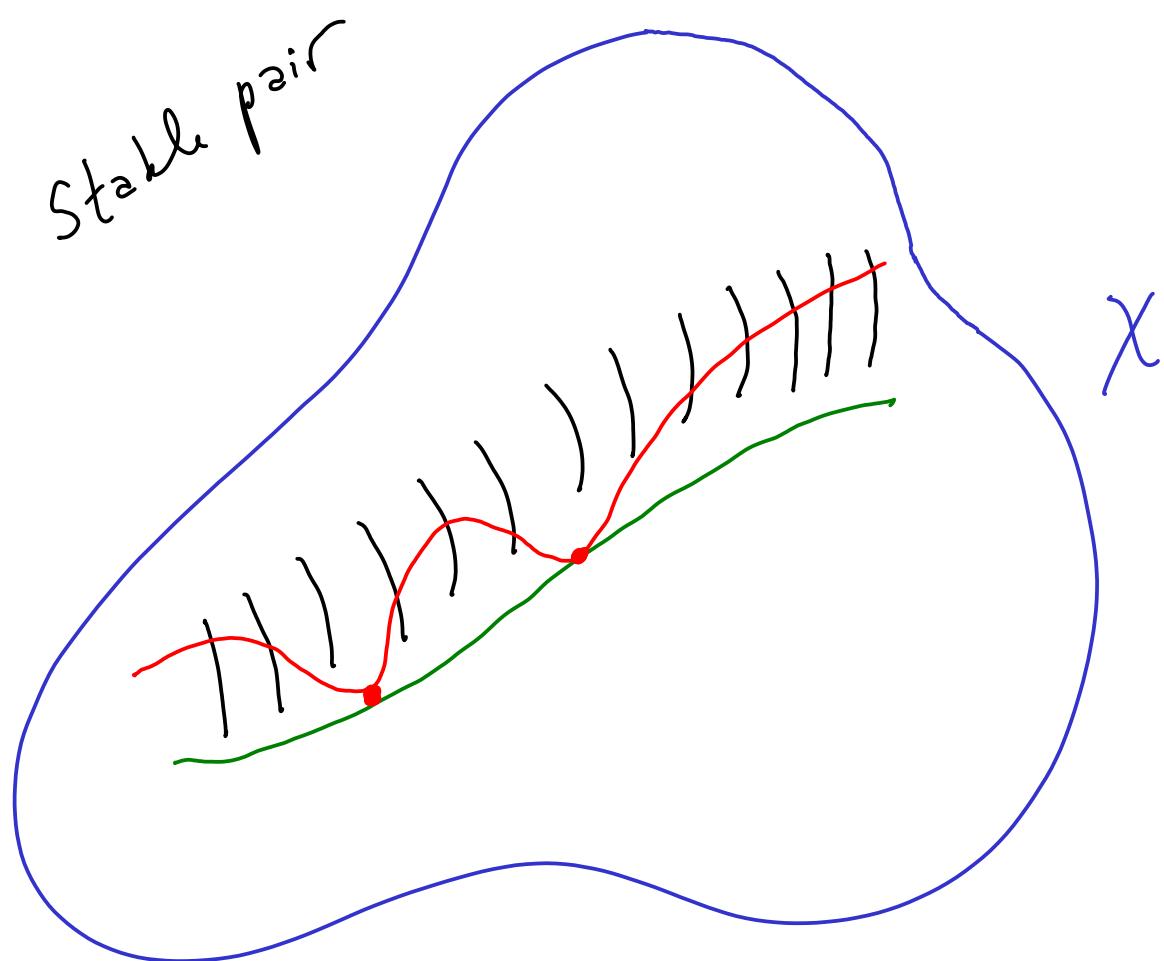
s section

F))))) sheaf $n = \chi(F)$

Supp (F) ～ $\beta = [Supp(F)]$

Le Potier, a series of papers P-Thomas

Ex. $X = \mathbb{P}^3$

Then $P_n(X, d) \supset$ Classical locus

Parameterizing :

$C \subset \mathbb{P}^3$ non sing irreducible curve
                           of degree d

$\mathcal{F} \to C$ a line bundle of degree l

$s \in H^0(C, \mathcal{F})$ a nonzero section

here $n = l - \text{genus}(c) + 1$

Note: $P_n(x, d)$ contains more degenerate objects

if we consider $\mathbb{I} = [\mathcal{O}_X \xrightarrow{\wedge} f]$ as
an object in $D^b_{Coh}(X)$, then

def $\quad Ext^1_0 (\mathbb{I}, \mathbb{I})$

Obs $\quad Ext^2_0 (\mathbb{I}, \mathbb{I})$

higher vanish

So we have a virtual fundamental class

$$[P_n(X, \beta)]^{Vir} \text{ of } \dim \int_\beta c_1(X)$$

See my paper with R. Thomas
"Counting curves via Stable pairs..."

Descendents

$$F$$ universal sheaf

$$\downarrow$$

$$P_n(x,\beta) \times X$$

$$\pi_1 \swarrow \qquad \searrow \pi_2$$

$$P_n(X,\beta) \qquad\qquad X$$
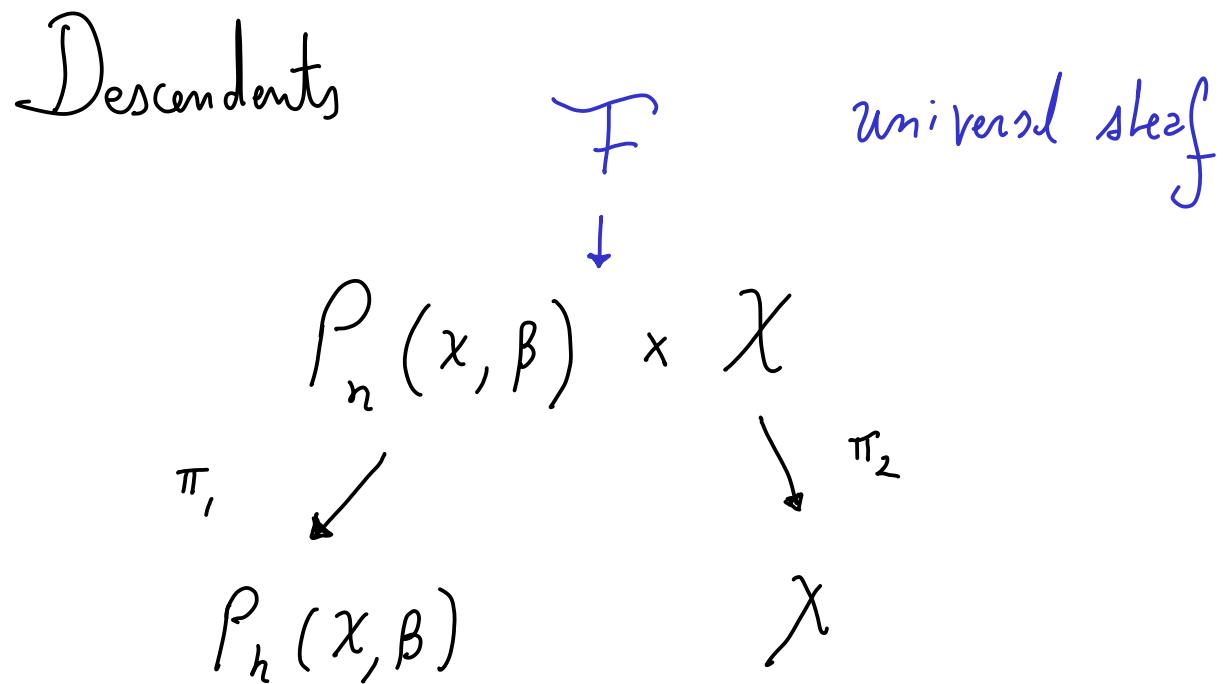
Definition:

$$T_k(\gamma) = \pi_{1*}\left(Ch_{k+2}(F) \cup \pi_2^*(\gamma) \right)$$

$$\gamma \in H^*(X)$$

Better convention

$$Ch_k(\gamma) = \pi_{1*}\left(Ch_k(F-\mathcal{O}) \cup \pi_2^*(\gamma) \right)$$

$-\mathbb{I}$

Conjectures

MNOP ~2005
Pixton-P ~2012
OOP 2019

GW/Pairs
Correspondence

Rationality

MNOP ~2005
Pixton-P ~2012

Virasoro

OOP 2019
Moreira OOP
Moreira 2020 (hopefully)

M = Maulik

N = Nekrasov

O = Okounkov

OO = Oblomkov Okounkov

Rationality

$$\left\langle \, ch_{k_1}(\gamma_1) \cdots ch_{k_r}(\gamma_r) \, \right\rangle_{\beta}^{\chi}$$

$$=$$

$$\sum_{n \in \mathbb{Z}} q^n \int_{[P_n(X,\beta)]^{vir}} ch_{k_1}(\gamma_1) \cdots ch_{k_r}(\gamma_r)$$
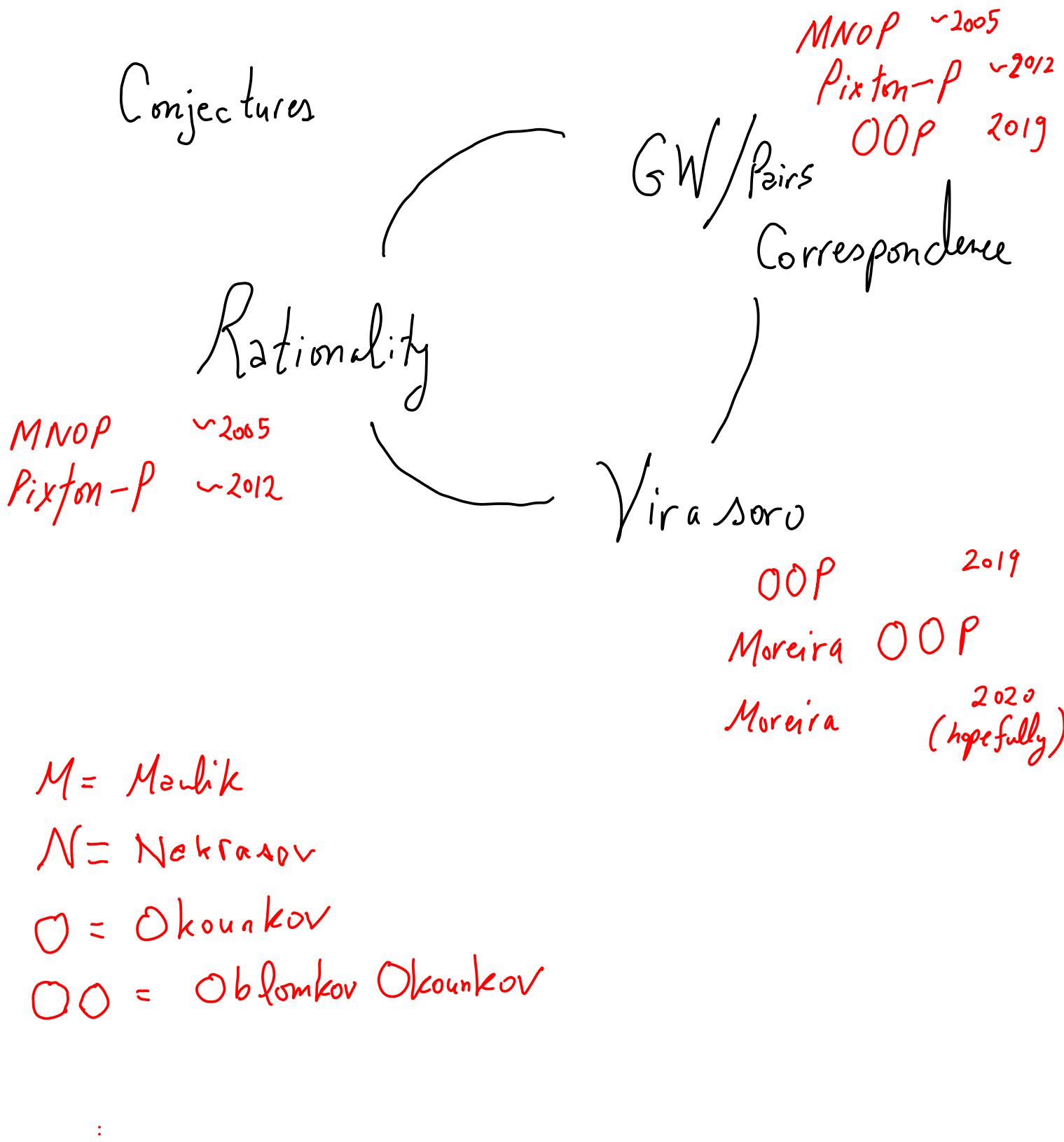
$$\in \mathbb{Q}((q))$$

Conj (Rationality)

$\left\langle ch_{k_1}(\gamma_1) \cdots ch_{k_r}(\gamma_r) \right\rangle_{\beta}^{\chi}$ is the Laurent expansion of a rational function in q.

Virasoro

Universal relations among descendent series

X is a nonsingular proj 3-fold
with only (p,p) cohomology

Main example: X is a toric 3-fold

Moreira treats the general case

Definition: Let $\mathbb{D}^X$ be
the commutative $\mathbb{Q}$-algebra
with generators $\{ch_i(\gamma) \mid i \geq 0, \gamma \in H^*(X, \mathbb{Q})\}$

subject to the basic relations

$$Ch_i(\lambda \cdot \gamma) = \lambda \, ch_i(\gamma) \qquad \lambda \in \mathbb{Q}$$

$$Ch_i(\gamma + \hat{\gamma}) = Ch_i(\gamma) + Ch_i(\hat{\gamma}) \qquad \gamma, \hat{\gamma} \in H^*(x).$$

In order to define the Virasoro Constraints,

We require 3 constructions in $\mathbb{D}^x$

(i) Define the derivation   for $k \geq -1$

$$R_k : \mathbb{D}^x \to \mathbb{D}^x \qquad by$$

$$R_k(Ch_i(\gamma)) = \left(\prod_{n=0}^{k} (i+d-3+n) \right) Ch_{i+k}(\gamma)$$

for R_{-1} we have $R_{-1}(Ch_i(\gamma)) = ch_{i-1}(\gamma).$

(ii) Define $ch_a ch_b(\gamma) \in \mathbb{D}^x$ by

$$Ch_a ch_b(\gamma) = \sum_i ch_a(\gamma_i^L) \, ch_b(\gamma_i^R)$$

where $\sum_i \gamma_i^L \otimes \gamma_i^R$ is th Künneth

decomposition of $\gamma \cdot \Delta \in H^*(X \times X)$.

The notation

$$(-1)^{d^L d^R} (a + d^L - 3)! \, (b + d^R - 3)! \; Ch_a ch_b(\gamma)$$

will be used for

$$\sum_i (-1)^{d(\gamma_i^L) d(\gamma_i^R)} (a + d(\gamma_i^L) - 3)! \, (b + d(\gamma_i^R) - 3)! \cdot$$
$$Ch_a(\gamma_i^L) \, Ch_b(\gamma_i^R)$$

$d(\;)$ denotes the complex degree!

(iii) define operator $T_k : \mathbb{D}^x \to \mathbb{D}^x$ by

$$T_k = -\frac{1}{2} \sum_{a+b=k+2} (-1)^{d^L d^R} (a+d^L-3)! \, (b+d^R-3)! \, ch_a \, ch_b \, (c_1)$$

$$+ \frac{1}{24} \sum_{a+b=k} a! \, b! \, ch_a \, ch_b \, (c_1 c_2)$$

- in sums $a, b \geq 0$
- factorials with negative arguments vanish
- $c_1, c_2 \in H^\bullet(X)$

  are the Chern classes of $Tan(X)$.

Define the Constraint operator

$$\mathcal{L}_k = T_k + R_k + (k+1)! \, R_{-1} \, ch_{k+1}(p)$$

for $k \geq -1$

Virasoro Conj [Moreira OOP]

X has only (p,p) cohomology

$\beta \in H_2(X, \mathbb{Z})$ curve class

$D \in \mathbb{D}^X$ any element.

Then
$$\left\langle \mathcal{L}_k (D) \right\rangle^X_\beta = 0$$

for $k \geq -1$. $\bullet$

Should view the point class here as
$$p = \frac{c_1 c_2}{24}$$

Ex. $X = \mathbb{P}^3$

$$\mathcal{L}_1(D) = \left(-4\, ch_3(H) + R_1 + 2\, ch_2(p) R_{-1}\right) D$$

Try $D = ch_3(p)$ and $\beta = L$

$$\Downarrow$$

$$-4 \left\langle ch_3(H)\, ch_3(p)\right\rangle_L^{\mathbb{P}^3} + 12 \left\langle ch_4(p)\right\rangle_L^{\mathbb{P}^3} +$$

$$+ 2 \left\langle ch_2(p)\, ch_2(p)\right\rangle_L^{\mathbb{P}^3} = 0$$

Check

$$\frac{3}{4} q - \frac{3}{2} q^2 + \frac{3}{4} q^3$$

$$\frac{1}{12} q - \frac{5}{6} q^2 + \frac{1}{12} q^3$$

$$q + 2q^2 + q^3$$

Thrm (Moreira OOP 2020)   Let X be toric 3-fold.

for all $D \in \mathbb{D}_t^X$,

the Virasoro Constraints hold

$$\left\langle \mathcal{L}_k (D) \right\rangle_\beta^X = 0.$$

Def: $\mathbb{D}_t^X \subset \mathbb{D}^X$

generated by $\{ \mathrm{ch}_i(\gamma) \mid i \geq 0, \ \gamma \in h^{>0}(x) \}$.

Stationary descendents

Path of Proof

X is a nonsingular proj toric 3 fold

GW Virasoro Constraints hold
Semi simple / Givental-Teleman theory
~ 2010

Lose control of
descendents of
1 here

GW/Pairs descendent
Correspondence

MNoP ~ 2005
Pixton-P ~ 2012
OOP 2019
(using older papers
on local curves).

Formula in
Stationary Case

Transfer Virasoro Constraints
from GW to Pairs in Stationary Case

Actually, I would like
run the whole argument
in the other direction.

Challenge: Prove Virasoro
constraints for stable pairs
on the moduli of Sheaves!

Challenge: Control the
descendents of $1 \in H^*(X)$.

Comments on the *Conjectural* functional equation

X nonsingular proj 3-fold

Rationality :

$$\left\langle\, T_{k_1}(\gamma_1)\cdots T_{k_r}(\gamma_r)\,\right\rangle^X_\beta \quad \text{is} \quad \text{a} \quad \text{rational fraction of } q$$

Functional equation :

q

dependence

$$d_\beta = \int_\beta c_1(x)$$

$$\left\langle\, T_{k_1}(\gamma_1)\cdots T_{k_r}(\gamma_r)\,\middle|\, q\,\right\rangle^X_\beta =$$

$$(-1)^{\sum_i k_i}\; q^{-d_\beta} \left\langle\, T_{k_1}(\gamma_1)\cdots T_{k_r}(\gamma_r)\,\middle|\, \tfrac{1}{q}\,\right\rangle^X_\beta$$

8   M. MOREIRA, A. OBLOMKOV, A. OKOUNKOV, AND R. PANDHARIPANDE

subject to the natural relations

$$
\begin{aligned}
\tau_i(\lambda \cdot \gamma) &= \lambda \, \tau_i(\gamma), \\
\tau_i(\gamma + \widehat{\gamma}) &= \tau_i(\gamma) + \tau_i(\widehat{\gamma})
\end{aligned}
$$

for $\lambda \in \mathbb{Q}$ and $\gamma, \widehat{\gamma} \in H^*(X)$. The subalgebra $\mathbb{D}_{\mathrm{GW}}^{X+} \subset \mathbb{D}_{\mathrm{GW}}^X$ of stationary descendents is generated by

$$
\left\{ \tau_i(\gamma) \,\middle|\, i \geqslant -2, \gamma \in H^{>0}(X, \mathbb{Q}) \right\}.
$$

We will use Getzler's renormalization $\mathfrak{a}_k$ of the Gromov-Witten descendents[6]:

$$
(7) \qquad \sum_{n=-\infty}^{\infty} z^n \tau_n = \mathsf{Z}^0 + \sum_{n>0} \frac{(\imath u z)^{n-1}}{(1 + z c_1)_n} \mathfrak{a}_n + \frac{1}{c_1} \sum_{n<0} \frac{(\imath u z)^{n-1}}{(1 + z c_1)_n} \mathfrak{a}_n,
$$

$$
\mathsf{Z}^0 = \frac{z^{-2} u^{-2}}{\mathcal{S}\left(\frac{zu}{\theta}\right)} - z^{-2} u^{-2},
$$

where we use standard notation for a Pochhammer symbol

$$
(a)_n = \frac{\Gamma(a + n)}{\Gamma(a)}.
$$

For example[7],

$$
(8) \qquad \tau_0(\gamma) = \mathfrak{a}_1(\gamma) + \frac{1}{24} \int_X \gamma c_2,
$$

$$
(9) \qquad \tau_1(\gamma) = \frac{\imath u}{2} \mathfrak{a}_2(\gamma) - \mathfrak{a}_1(\gamma \cdot c_1).
$$

For $k \geqslant 2$ and $\gamma \in H^{>0}(X)$, we have the general formula

$$
(10) \quad \tau_k(\gamma) = \frac{(\imath u)^k}{(k+1)!} \mathfrak{a}_{k+1}(\gamma) - \frac{(\imath u)^{k-1}}{k!} \left(\sum_{i=1}^{k} \frac{1}{i} \right) \mathfrak{a}_k(\gamma \cdot c_1)
$$

$$
+ \frac{(\imath u)^{k-2}}{(k-1)!} \left(\sum_{i=1}^{k-1} \frac{1}{i^2} + \sum_{1 \leqslant i < j \leqslant k-1} \frac{1}{ij} \right) \mathfrak{a}_{k-1}(\gamma \cdot c_1^2).
$$

For the special cases $k = -1$ and $k = -2$, we have

$$
\tau_{-1}(\gamma) = (\imath u)^{-2} \mathfrak{a}_{-1}(\gamma) + (\imath u)^{-3} \mathfrak{a}_{-2}(c_1 \cdot \gamma), \quad \tau_{-2}(\gamma) = -(\imath u)^{-3} \mathfrak{a}_{-2}(\gamma).
$$

---

[6]We use $\imath$ for the square root of -1. The genus variable u will usually occur together with $\imath$.

[7]The constant term $\frac{1}{24} \int_X \gamma c_2$ in the formula does not contribute unless $\gamma \in H^2(X)$.

0.6. Stationary GW/PT correspondence.

We define the subalgebras

$$\mathbb{D}^X_{\mathrm{GW}_\circ} \subset \mathbb{D}^X_{\mathrm{GW}} \quad \text{and} \quad \mathbb{D}^{X+}_{\mathrm{GW}_\circ} \subset \mathbb{D}^{X+}_{\mathrm{GW}}$$

by requiring the descendent indices for the generators to be non-negative: $\mathbb{D}^X_{\mathrm{GW}_\circ}$ is generated by

$$\left\{ \tau_i(\gamma) \,\middle|\, i \geq 0\,, \gamma \in H^*(X) \right\}$$

and $\mathbb{D}^{X+}_{\mathrm{GW}_\circ}$ is generated by

$$\left\{ \tau_i(\gamma) \,\middle|\, i \geq 0\,, \gamma \in H^{>0}(X) \right\}.$$

The stationary GW/PT transformation is the linear map

$$\mathfrak{C}^\bullet : \mathbb{D}^{X+}_{\mathrm{PT}} \to \mathbb{D}^{X+}_{\mathrm{GW}}$$

satisfying

$$\mathfrak{C}^\bullet(1) = 1$$

and defined on monomials by

$$\mathfrak{C}^\bullet\left(\widetilde{\mathrm{ch}}_{k_1}(\gamma_1) \ldots \widetilde{\mathrm{ch}}_{k_m}(\gamma_m) \right) = \sum_{P \text{ set partition of } \{1,\ldots,m\}} \prod_{S \in P} \mathfrak{C}^\circ\left(\prod_{i \in S} \widetilde{\mathrm{ch}}_{k_i}(\gamma_i) \right),$$

where the operations $\mathfrak{C}^\circ$ are

$$(11) \quad \mathfrak{C}^\circ\left(\widetilde{\mathrm{ch}}_{k+2}(\gamma) \right) = \frac{1}{(k+1)!}\mathfrak{a}_{k+1}(\gamma) + \frac{(\imath u)^{-1}}{k!}\sum_{|\mu|=k-1}\frac{\mathfrak{a}_{\mu_1}\mathfrak{a}_{\mu_2}(\gamma \cdot c_1)}{\mathrm{Aut}(\mu)}$$

$$+ \frac{(\imath u)^{-2}}{k!}\sum_{|\mu|=k-2}\frac{\mathfrak{a}_{\mu_1}\mathfrak{a}_{\mu_2}(\gamma \cdot c_1^2)}{\mathrm{Aut}(\mu)} + \frac{(\imath u)^{-2}}{(k-1)!}\sum_{|\mu|=k-3}\frac{\mathfrak{a}_{\mu_1}\mathfrak{a}_{\mu_2}\mathfrak{a}_{\mu_3}(\gamma \cdot c_1^2)}{\mathrm{Aut}(\mu)},$$

$$(12) \quad \mathfrak{C}^\circ\left(\widetilde{\mathrm{ch}}_{k_1+2}(\gamma)\widetilde{\mathrm{ch}}_{k_2+2}(\gamma') \right) = -\frac{(\imath u)^{-1}}{k_1!k_2!}\mathfrak{a}_{k_1+k_2}(\gamma\gamma') - \frac{(\imath u)^{-2}}{k_1!k_2!}\mathfrak{a}_{k_1+k_2-1}(\gamma\gamma' \cdot c_1)$$

$$- \frac{(\imath u)^{-2}}{k_1!k_2!}\sum_{|\mu|=k_1+k_2-2}\max(\max(k_1,k_2),\max(\mu_1+1,\mu_2+1))\frac{\mathfrak{a}_{\mu_1}\mathfrak{a}_{\mu_2}}{\mathrm{Aut}(\mu)}(\gamma\gamma' \cdot c_1),$$

$$(13) \quad \mathfrak{C}^\circ\left(\widetilde{\mathrm{ch}}_{k_1+2}(\gamma)\widetilde{\mathrm{ch}}_{k_2+2}(\gamma')\widetilde{\mathrm{ch}}_{k_3+2}(\gamma'') \right) = \frac{(\imath u)^{-2}|k|}{k_1!k_2!k_3!}\mathfrak{a}_{|k|-1}(\gamma\gamma'\gamma''), \quad |k| = k_1+k_2+k_3.$$

In equations (11)-(13) above, the parts of μ are *positive* integers,

$$k\,, \; k_i \geq 0\,,$$

and all ocurrences of $\mathfrak{a}_0$ and $\mathfrak{a}_{-1}$ are set to 0 except in certain cases for lower values of k and k_i which are described below.[8] Because of the stationary assumption, all higher $\mathfrak{C}^\circ$ operations vanish. We will reintroduce these formulas in Sections 3 and 5 at the first instance of their application.

---

[8]The corrections for the lower vaules are consistent with the rule $\mathfrak{a}_0 = 0$, $\mathfrak{a}_{-1}/(-1)! = -u^2\tau_{-2}$.

The End