The double ramification cycle in $\log CH^g(\overline{\mathcal{M}}_{g,n})$

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joint work with

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I. What is the double ramification cycle?

cycle class

$$DR_{g,A}$$

genus

$A=(a_1,\dots,a_n)$

$a_i \in \mathbb{Z}, \quad \sum_{i=1}^{n} a_i = 0$

- Informal definition :

$DR_{g,A}$ is the class of the locus of pointed curves $(C, p_1, \dots, p_n) \in \overline{\mathcal{M}}_{g,n}$ satisfying the condition

$$\text{"} \; \mathcal{O}_C\Big(\sum_{i=1}^{n} a_i p_i\Big) \cong \mathcal{O}_C \; \text{"}$$

not completely precise

Abel-Jacobi condition

- What is the issue with the AJ condition?

If C is nonsingular irreducible, then $\mathcal{O}_C\left(\sum_{i=1}^{n} a_i p_i\right) \cong \mathcal{O}_C$ is a well defined closed subscheme of virtual codim g.

another aspect

Otherwise, no issue in the nonsingular case

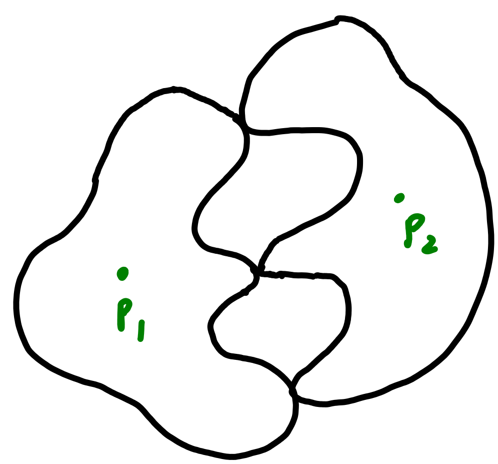
But what is the meaning of the condition

$$\text{"}\ \mathcal{O}_C\left(\sum_{i=1}^{n} a_i p_i\right) \cong \mathcal{O}_C\ \text{"}$$

when C is reducible?

Consider the case:

$$(C, p_1, p_2) = \left[\quad \right]$$



$A = (3, -3)$

- There are three main approaches to the definition of the double ramification cycle

Relative GW theory

$\Downarrow$

$$\boxed{DR_{g,A} \in CH^g(\bar{\mathcal{M}}_{g,n})}$$

$\nearrow\nearrow$ $\nwarrow\nwarrow$

Classical intersection theory on the moduli space of curves

log geometry
log intersection theory

There is a very long list of names related to the definitions of these theories.

Li-Ruan, J. Li, Abramovich-Chen-Gross-Siebert, Graber-Vakil, Eliashberg-Givental-Hofer, Holmes, Marcus-Wise, ...

The most elementary complete definition that I know :

of course, also the most recent definition

$g \geq 0$ genus

$A = (a_1, \ldots, a_n)$ multiplicity data $\sum_{i=1}^{n} a_i = 0$

We have

$$\bar{\mathcal{M}}_{g,A} \overset{i}{\subset} \mathcal{P}ic^{\circ}_{g,n}$$

inclusion defined by $\mathcal{L} \cong \mathcal{O}_C$

Artin stack of triples

$$(C, p_1, \ldots, p_n, \mathcal{L})$$

Connected nodal curve

p_i distinct points

$\mathcal{L} \to C$ line bundle, $\deg(\mathcal{L}) = 0$

nonsingular locally closed substack of pure codim g

nonsingular Artin stack

$$AJ_{g,A} \subset \mathfrak{Pic}^{\circ}_{g,n}$$

locus where $\mathcal{L} = \mathcal{O}_C\left(\sum_{i=1}^{n} a_i p_i\right)$

Then we define :

$$\boxed{DR_{g,A} = \iota^*\left[\overline{AJ}_{g,A}\right] \in CH^g(\overline{\mathcal{M}}_{g,n})}$$

Agreement with GW theory and log geometry is not trivial (closure is complicated)

Bae - Holmes - P - Schmitt - Schwarz 2020

See also upcoming survey Herr - Molcho - P - Wise 2021

If you fully embrace the above definition of $\mathsf{DR}_{g,A}$, the natural class here is

$$[\overline{\mathsf{AJ}}_{g,A}] \in CH^g(\mathfrak{Pic}_{g,n})$$

- Double ramification cycles

All calculated by versions of Pixton's formula

$$\mathsf{DR}_{g,A} \in CH^g(\overline{\mathcal{M}}_{g,n})$$

DR cycle

Janda-Pixton-P-Zvonkine 2016

$$[\overline{\mathsf{AJ}}_{g,A}] \in CH^g(\mathfrak{Pic}_{g,n})$$

universal DR cycle

BHPSS 2020

Intermediate for GW targets

$$\mathsf{DR}_{g,A,\beta}(X,L) \in CH^g(\overline{\mathcal{M}}_{g,n}(X,\beta))$$

$$\Sigma a_i = \beta \cdot c(L)$$

Janda-Pixton-P-Zvonkine 2018

II. Logarithmic intersection theory

What is log intersection theory?

Given any nonsingular variety X
with a normal crossings divisor $D \subset X$
we obtain a log scheme (X, D).

There are two related Chow constructions
lying over $CH^*(X)$

$$CH^*(X) \subset \log CH^*(X) \subset b\,CH^*(X)$$

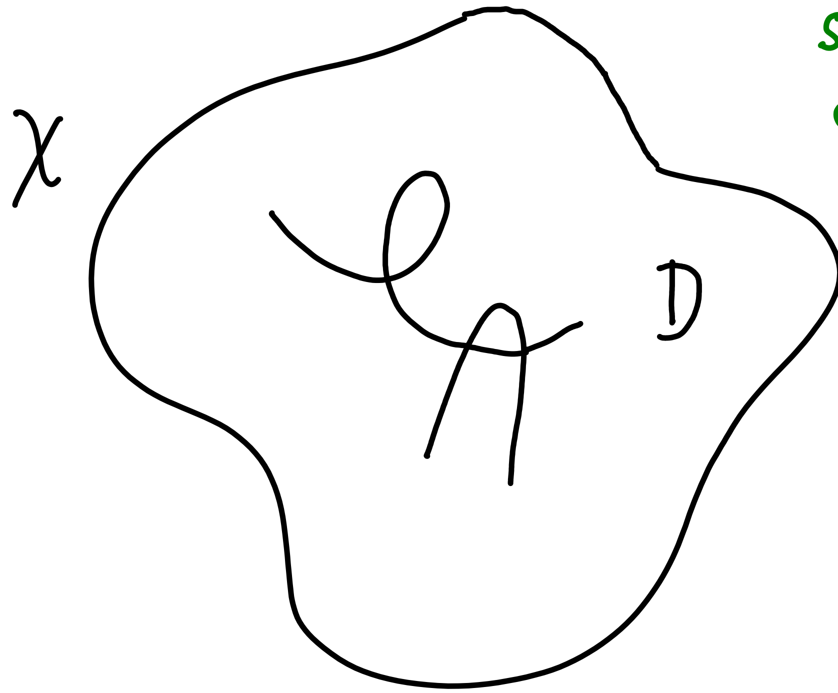
used by D. Holmes — Shokurov

Our main example is the log scheme

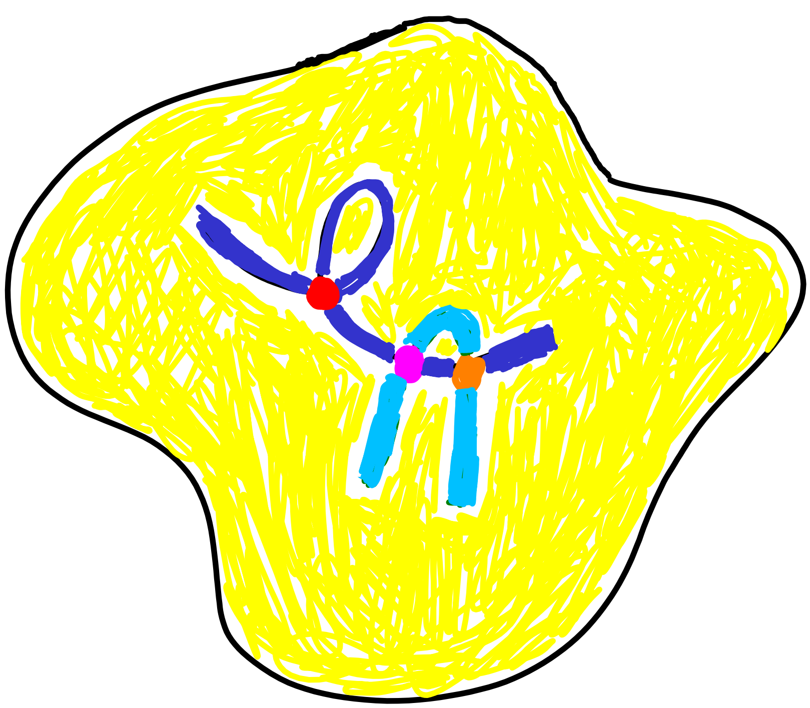
$$(\overline{\mathcal{M}}_{g,n}, \Delta)$$

$\leftarrow$ normal crossings divisor of reducible curves

Not assumed strict normal crossings



Basic Notion of stratification



Strata indicated by colors

A stratum $S \subset X$ is nonsingular and quasiprojective

$\bar{S} \subset X$ may be singular (mildly)

A simple blow-up of (X, D) is a blow up along a nonsingular stratum closure $\bar{S} \subsetneq X$.

$$Bl : (\hat{X}, \hat{D}) \to (X, D)$$

blowup

strict transform of D union the exceptional divisor E

Define a category $\mathcal{B}(X, D)$

- Objects are $(\tilde{X}, \tilde{D}) \xrightarrow{\tilde{\phi}} (X, D)$ where $\tilde{\phi}$ is a composition of simple blowups

- Morphisms are commutative diagrams

$$\begin{tikzcd}
(\tilde{\tilde{X}}, \tilde{\tilde{D}}) \arrow[rr, "\gamma"] \arrow[dr, "\tilde{\tilde{\phi}}"'] & & (\tilde{X}, \tilde{D}) \arrow[dl, "\tilde{\phi}"] \\
& (X, D) &
\end{tikzcd}$$

γ is a composition of simple blowups

$$\boxed{\log CH^*(X,D) \overset{\text{def}}{=} \lim_{\substack{\longrightarrow \\ (\tilde{X},\tilde{D}) \in \beta(X,D)}} CH^*(\tilde{X})}$$

$bCH^*(X)$ has the same definition except that blowups along all nonsingular varieties are allowed.

A nice exercise : $bCH^*(X)$ is generated by divisors

See Molcho-Schmitt-P 2020

The main point here for us :

$$DR_{g,A} \in CH^g(\bar{M}_{g,n})$$

$$\begin{array}{c} \log CH(X,\Delta) \\ \downarrow \pi_* \\ CH(X) \end{array}$$

naturally lifts to

$$DR^{\log}_{g,A} \in \log CH^g(\bar{M}_{g,n},\Delta)$$

In fact $DR^{\log}_{g,A}$ is more natural

than $DR_{g,A}$ from several perspectives.

Example: Let $A = (a_1, \dots, a_n)$ $\Sigma a_i = \Sigma b_i = 0$

and $B = (b_1, \dots, b_n)$

given any $M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \in SL_2(\mathbb{Z})$

we obtain new vectors

$$M A = m_{11} A + m_{21} B$$

$$M B = m_{12} A + m_{22} B$$

SL-invariance also for more vectors

Theorem (Holmes - Pixton - Schmitt 2017)

$$DR^{\log}_{g,A} \cdot DR^{\log}_{g,B} = DR^{\log}_{g,MA} \cdot DR^{\log}_{g,MB}$$

in $\log CH^g(\bar{M}_{g,n}, \Delta)$

Computation (Buryak-Rossi 2019):

$$\int_{\overline{\mathcal{M}}_{g,3}} \pi_* \left(DR^{\log}_{g,A} \cdot DR^{\log}_{g,B} \cdot DR^{\log}_{g,C} \right)$$

$$= \frac{\delta^{2g}}{2^{3g}\, g!\, (2g+1)!!}$$

by left multiplication

What is δ ? Must be an SL_3-invariant

of the 3×3 matrix $\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$.

Can't be det (since det $=0$).

δ = GCD of all 2×2 minors of

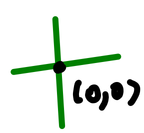
Sign doesn't matter!

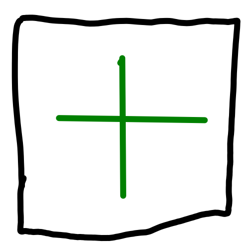
Localization in log GW (Graber 2021)

X toric $\supset$ D nc toric divisor

The localization formula for moduli space of log maps $\overline{M}_g(X/D)_n$ requires knowledge of log DR cycle.

Consider $X = \mathbb{P}^1 \times \mathbb{P}^1$

$D =$ 



Vertex at $(0,0)$ precisely involves

$$DR^{\log}_{g,A} \cdot DR^{\log}_{g,B} \in \log CH^{2g}(\overline{M}_{g,n})$$

Parallel to theory of Hodge integrals in the absolute case

Graber, Ranganathan

III How does $DR_{g,A}$ lift to $DR^{\log}_{g,A}$?

Approach of Holmes and Markus-Wise :

Let g and $A = (a_1, \ldots, a_n)$, $\sum a_i = 0$ be fixed.

There is a semi-canonical element

$$(\bar{\mathcal{M}}^{\diamond}_{g,n}, \Delta^{\diamond}) \xrightarrow{\phi^{\diamond}} (\bar{\mathcal{M}}_{g,n}, \Delta)$$

of the category $B(\bar{\mathcal{M}}_{g,n}, \Delta)$

called the diamond space.

defined by explicit sequence of simple blow-ups

The diamond space has the following property:

Let $\mathrm{Jac}_0 \to \overline{\mathcal{M}}_{g,n}$ be the universal Jacobian of multidegree 0 line bundle.

Via pull-back by $\phi^\diamond$, we have

$$\mathrm{Jac}_0^\diamond \to \overline{\mathcal{M}}_{g,n}^\diamond$$

Let $\mathcal{U}^\diamond \subset \overline{\mathcal{M}}_{g,n}^\diamond$ be the maximal open set where the rational map AJ_A extends,

$$AJ_A : \overline{\mathcal{M}}_{g,n}^\diamond \dashrightarrow \mathrm{Jac}_0^\diamond$$

defined on nonsingular curves by

$$(C, p_1, \ldots, p_n) \mapsto \mathcal{O}_C\left(\sum a_i p_i\right)$$

Then the property is

$$\boxed{AJ_A^{-1}(0_J^\diamond) \subset \mathcal{U}^\diamond \text{ is proper}}$$

Here : $AJ_A : \mathcal{U}^\diamond \to \mathrm{Jac}_0^\diamond$ is a morphism

$0_J^\diamond \subset \mathrm{Jac}_0^\diamond$ is the 0-section

Using the property of the diamond space

We define a class

$$AJ_A^{\star\,\text{refined}}(0_J^\diamond) \in CH^g(AJ_A^{-1}(0_J^\diamond))$$

and push-forward to $\bar{\mathcal{M}}_{g,n}^\diamond$

to define $DR_{g,A}^{\log} \in \log CH^g(\bar{\mathcal{M}}_{g,n}, \Delta)$

IV Why is it harder to compute $DR^{\log}_{g,A}$?

- Construction on $\overline{\mathcal{M}}^{\lozenge}_{g,n}$ uses the open set

$$\mathcal{U}^{\lozenge} \subset \overline{\mathcal{M}}^{\lozenge}_{g,n} .$$

We can apply the universal DR formula over $\mathcal{U}^{\lozenge}$,

but there is no push-forward to $\overline{\mathcal{M}}^{\lozenge}_{g,n}$.

also not canonical

- We must search for other geometric models to apply the universal DR formula.

Basic Questions :

(A) How will we decide how much to blow-up $\overline{\mathcal{M}}_{g,n}$?

(B) If we do find nice proper families of curves on a blow-up of $\overline{M}_{g,n}$, how will we know the universal DR formula. Calculates $DR_{g,n}^{\log}$?

(C) If we manage to solve (A) + (B), how will we express the answer? In what language?

In fact, questions (A), (B), (C) all have simple answers!

Answers :

Caporaso 93
P 94

Esteves, Melo,
Viviani, others

Kass-Pagani 19
Abreu-Pacini 20,21

(A) We will use moduli spaces of line bundles on curves with respect to certain stability conditions. Such moduli spaces will precisely determine blow-ups of $\overline{\mathcal{M}}_{g,n}$ and admit applications of the universal DR formula.

(B) The match with $DR^{log}_{g,n}$ is guaranteed by the twistability condition of Holmes-Schwarz 2021

(c) The answer is expressed in the language of piecewise polynomials on the cone complex of

$$(\overline{\mathcal{M}}_{g,n}, \Delta)$$

Brion, Payne, and Ranganathan

Piecewise polynomials specifically introduced in the context here by D. R.

An outcome is that $DR^{\log}_{g,n}$ is tautological in every sense as proven earlier by Ranganathan-Molcho Holmes-Schwarz

I will discuss (A).

David will cover topics (B) and (C) tomorrow.

V stability circa ~1990 Caporaso

I will describe the idea from the perspective of my paper from that period.

We are interested in a proper moduli space of line bundles on nodal curves.

We have a canonical stability condition θ on stable curves parameterized by $\bar{M}_{g,n}$:

For $(C, p_1, \ldots, p_n) \in \bar{M}_{g,n}$

and an irreducible component $D \subset C$

$$\theta(D) = 2g_D - 2 + \mathrm{val}_D$$ ← includes markings

Extend θ additively to subcurves $S \subset C$

Using θ, we can construct a moduli space:

$$\mathcal{P}ic^{\theta} \to \overline{M}_{g,n}$$

of θ-stable torsion free sheaves of rank 1 on stable curves by GIT.

$$\mathcal{L} \to (C, p_1, \dots, p_n) \text{ is } \theta\text{-stable}$$

we are interested in $\deg(\mathcal{L})=0$ case

$$\Updownarrow$$

for every subcurve $S \subset C$, $0 \to \mathcal{F}_S \to \mathcal{L} \to \mathcal{L}|_S \to 0$

$$\frac{\chi(\mathcal{F}_S)}{\theta(S)} < \frac{\chi(\mathcal{L})}{2g-2+n}$$

[Issues of non-locally free θ-stable sheaves solved by 1-step destabilization of C]

We obtain the universal Picard constructed by Caporaso, later P

Unfortunately, there are strictly semistable sheaves here.

Return to the subject almost 30 years later: Kass-Pagani, Abreu-Pacini

Idea is to study all possible stability conditions (not just θ).

We can perturb θ by finding a rule $\mathcal{E}$ which assigns a rational number to every component of every

Stable curve $(C, p_1, \dots, p_n) \in \bar{M}_{g,n}$

with the additive property under smoothing

and $\varepsilon(C) = 0$.

If ε is small, $\hat{\theta} = \theta + \varepsilon$

is positive on all subcurves, and

we obtain a moduli space as before by GIT

$$P_{ic}^{\hat{\theta}} \to \bar{M}_{g,n}$$

Abreu-Pacini construct such ε.

For generic choices $\Rightarrow$ No semistable elements!

The answer to (A):

Select a small and generic ε

Then we have

$$\begin{array}{c} Pic^{\hat\theta} \\ \downarrow \\ \overline{\mathcal{M}}_{g,n} \end{array}$$

Can be done explicitly, but there is a choice

$\hat\theta$ determines canonically a blow-up

$$\begin{tikzcd} \overline{\mathcal{M}}_{g,n}^{\hat\theta} \arrow[r, "AJ"] \arrow[dr] & Pic^{\hat\theta} \arrow[d] \\ & \overline{\mathcal{M}}_{g,n} \arrow[u, dashed, bend left, "AJ"] \end{tikzcd}$$

on which the Abel-Jacobi map defined by A extends

By pulling back the universal family over $\mathcal{P}ic^{\hat{\theta}}$ to $\bar{\mathcal{M}}_{g,n}^{\hat{\theta}}$, we can apply the universal DR formula.

The properties of the moduli space $\mathcal{P}ic^{\hat{\theta}}$ imply an affirmative answer to Question (B).

The final result (a version of Pixton's formula) can be efficiently expressed in language of piecewise polynomials (C).

VI Topics for David

- explain the twistability condition and how the geometry of $\mathfrak{P}ic^{\hat{\theta}}$ is sufficient for (B)

- Write the Pixton formula for $DR^{\log}_{g,A}$ in the language of piecewise polynomials (C)

- If time permits, discuss the very interesting and nontrivial dependence on the stability condition $\hat{\theta}$.

VII Aaron's calculation in $\overline{\mathcal{M}}_{g,4}$

$$\pi_\star \left(DR^{\log}_{g,(-2,2,0,0)} \cdot DR^{\log}_{g,(0,0,-2,2)} \right)$$

$$= DR_{g,(-2,2,0,0)} \cdot DR_{g,(0,0,-2,2)}$$

$$+ \text{Correction}$$

Where $\pi_\star : \log CH(\overline{\mathcal{M}}_{g,n}) \to CH(\overline{\mathcal{M}}_{g,n})$.

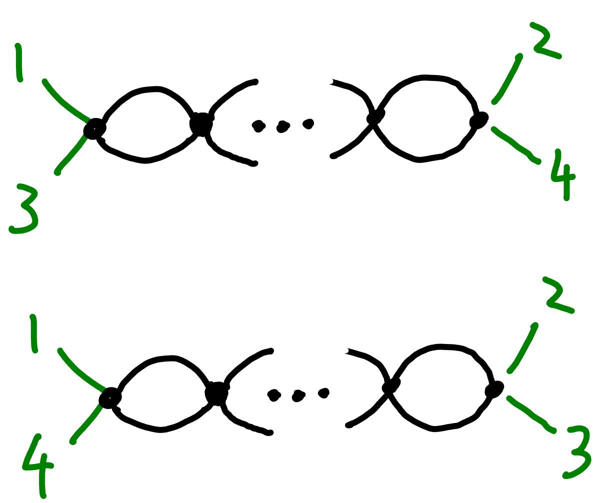
The entire theory of the lecture is used to calculate the correction.

Correction term is :

$$\sum_{\substack{1\le m\le g \\ g_0+\cdots+g_m = g-m}} \frac{(-1)^m}{2^m}\,(\xi_* + \hat{\xi}_*)\left[\begin{array}{c} DR_{g_0}(-2,0,1,1) \\ \cdot \\ DR_{g_0}(0,-2,1,1) \end{array} \otimes DR_{g_1}(-1,-1,1,1)^2 \otimes DR_{g_2}(-1,-1,1,1)^2 \otimes \cdots \otimes DR_{g_{m-1}}(-1,-1,1,1)^2 \otimes \begin{array}{c} DR_{g_m}(-1,-1,2,0) \\ \cdot \\ DR_{g_m}(-1,-1,0,2) \end{array}\right]$$

Where $\xi, \hat{\xi} : \prod_{i=0}^{m} \overline{M}_{g_i,4} \to \overline{M}_{g,4}$

via the graphs



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