RELATIONS IN THE TAUTOLOGICAL RING OF THE MODULI SPACE OF $K3$ SURFACES

RAHUL PANDHARIPANDE AND QIZHENG YIN

Abstract. We study the interplay of the moduli of curves and the moduli of $K3$ surfaces via the virtual class of the moduli spaces of stable maps. Using Getzler's relation in genus 1, we construct a universal decomposition of the diagonal in Chow in the third fiber product of the universal $K3$ surface. The decomposition has terms supported on Noether-Lefschetz loci which are not visible in the Beauville-Voisin decomposition for a fixed $K3$ surface. As a result of our universal decomposition, we prove the conjecture of Marian-Oprea-Pandharipande: the full tautological ring of the moduli space of $K3$ surfaces is generated in Chow by the classes of the Noether-Lefschetz loci. Explicit boundary relations are constructed for all κ classes.

More generally, we propose a connection between relations in the tautological ring of the moduli spaces of curves and relations in the tautological ring of the moduli space of $K3$ surfaces. The WDVV relation in genus 0 is used in our proof of the MOP conjecture.

Contents

Date: October 2016.

0. Introduction

0.1. κ **classes.** Let $\mathcal{M}_{2\ell}$ be the moduli space of quasi-polarized $K3$ surfaces (X, H) of degree $2\ell > 0$:

- X is a nonsingular, projective $K3$ surface over $\mathbb{C}$,
- $H \in \mathrm{Pic}(X)$ is a primitive and nef class satisfying

$$\langle H, H \rangle_X \,=\, \int_X H^2 \,=\, 2\ell\,.$$

The basics of quasi-polarized $K3$ surfaces and their moduli are reviewed in Section 1.

Consider the universal quasi-polarized $K3$ surface over the moduli space,

$$\pi : \mathcal{X} \to \mathcal{M}_{2\ell}\,.$$

We define a canonical divisor class on the universal surface,

$$\mathcal{H} \;\in\; \mathsf{A}^1(\mathcal{X}, \mathbb{Q})\,,$$

which restricts to H on the fibers of π by the following construction. Let $\overline{\mathsf{M}}_{0,1}(\pi, H)$ be the π-relative moduli space of stable maps: $\overline{\mathsf{M}}_{0,1}(\pi, H)$ parameterizes stable maps from genus 0 curves with 1 marked point to the fibers of π representing the fiberwise class H. Let

$$\epsilon : \overline{\mathsf{M}}_{0,1}(\pi, H) \to \mathcal{X}$$

be the evaluation morphism over $\mathcal{M}_{2\ell}$. The moduli space $\overline{\mathsf{M}}_{0,1}(\pi, H)$ carries a π-relative reduced obstruction theory with reduced virtual class of π-relative dimension 1. We define

$$\mathcal{H} \,=\, \frac{1}{N_0(\ell)} \,\cdot\, \epsilon_* \left[\overline{\mathsf{M}}_{0,1}(\pi, H)\right]^{\mathrm{red}} \;\in\; \mathsf{A}^1(\mathcal{X}, \mathbb{Q})\,,$$

where $N_0(\ell)$ is the genus 0 Gromov-Witten invariant[1]

$$N_0(\ell) = \int_{[\overline{\mathsf{M}}_{0,0}(X,H)]^{\mathrm{red}}} 1\,.$$

By the Yau-Zaslow formula[2], the invariant $N_0(\ell)$ is never 0 for $\ell \geq -1$,

$$\sum_{\ell=-1}^{\infty} q^\ell N_0(\ell) \,=\, \frac{1}{q} + 24 + 324q + 3200q^2 \ldots\,.$$

The construction of $\mathcal{H}$ is discussed further in Section 2.1.

The π-relative tangent bundle of $\mathcal{X}$,

$$\mathcal{T}_\pi \to \mathcal{X}\,,$$

[1] While $\ell > 0$ is required for the quasi-polarization (X, H), the reduced Gromov-Witten invariant $N_0(\ell)$ is well-defined for all $\ell \geq -1$.

[2] The formula was proposed in [29]. The first proofs in the primitive case can be found in [1, 8]. We will later require the full Yau-Zaslow formula for the genus 0 Gromov-Witten counts also in imprimitive classes proven in [14].

is of rank 2 and is canonically defined. Using $\mathcal{H}$ and $c_2(\mathcal{T}_\pi)$, we define the κ classes,

$$\kappa_{[a;b]} = \pi_* \left(\mathcal{H}^a \cdot c_2(\mathcal{T}_\pi)^b \right) \in \mathsf{A}^{a+2b-2}(\mathcal{M}_{2\ell}, \mathbb{Q}) .$$

Our definition follows [16, Section 4] *except for the canonical choice of* $\mathcal{H}$. The construction here requires *no* choices to be made in the definition of the κ classes.

0.2. **Strict tautological classes.** The Noether-Lefschetz loci also define classes in the Chow ring $\mathsf{A}^\star(\mathcal{M}_{2\ell}, \mathbb{Q})$. Let

$$\mathsf{NL}^\star(\mathcal{M}_{2\ell}) \subset \mathsf{A}^\star(\mathcal{M}_{2\ell}, \mathbb{Q})$$

be the subalgebra generated by the Noether-Lefschetz loci (of all codimensions). On the Noether-Lefschetz locus[3]

$$\mathcal{M}_\Lambda \to \mathcal{M}_{2\ell} ,$$

corresponding to the larger Picard lattice $\Lambda \supset (2\ell)$, richer κ classes may be defined by simultaneously using several elements of Λ.

We define *canonical* κ classes based on the lattice polarization Λ. A nonzero class $L \in \Lambda$ is *admissible* if

(i) $L = m \cdot \widetilde{L}$ with $\widetilde{L}$ primitive, $m > 0$, and $\langle \widetilde{L}, \widetilde{L} \rangle_\Lambda \geq -2$,
(ii) $\langle H, L \rangle_\Lambda \geq 0$,

and in case of equality in (ii), which forces equality in (i) by the Hodge index theorem,

(ii') L is effective.

Effectivity is *equivalent* to the condition

$$\langle H, L \rangle_\Lambda \geq 0$$

for *every* quasi-polarization $H \in \Lambda$ for a generic $K3$ surface parameterized by $\mathcal{M}_\Lambda$.

For $L \in \Lambda$ admissible, we define

$$\mathcal{L} = \frac{1}{N_0(L)} \cdot \epsilon_* \left[\overline{\mathsf{M}}_{0,1}(\pi_\Lambda, L) \right]^{\mathrm{red}} \in \mathsf{A}^1(\mathcal{X}_\Lambda, \mathbb{Q}) ,$$

where $\pi_\Lambda : \mathcal{X}_\Lambda \to \mathcal{M}_\Lambda$ is the universal $K3$ surface. The reduced Gromov-Witten invariant

$$N_0(L) = \int_{[\overline{\mathsf{M}}_{0,0}(X,L)]^{\mathrm{red}}} 1$$

is nonzero for all admissible classes by the full Yau-Zaslow formula proven in [14], see Section 1.4.

For $L_1, \ldots, L_k \in \Lambda$ admissible classes, we have canonically constructed divisors

$$\mathcal{L}_1, \ldots, \mathcal{L}_k \in \mathsf{A}^1(\mathcal{X}_\Lambda, \mathbb{Q}) .$$

[3]We view the Noether-Lefschetz loci as proper maps to $\mathcal{M}_{2\ell}$ instead of subspaces.

We define the richer κ classes on $\mathcal{M}_\Lambda$ by

$$\kappa_{[L_1^{a_1},\ldots,L_k^{a_k};b]} = \pi_{\Lambda *}\left(\mathcal{L}_1^{a_1}\cdots\mathcal{L}_k^{a_k}\cdot c_2(\mathcal{T}_{\pi_\Lambda})^b\right) \in \mathsf{A}^{\sum_i a_i+2b-2}(\mathcal{M}_\Lambda,\mathbb{Q})\,. \tag{1}$$

We will sometimes suppress the dependence on the L_i,

$$\kappa_{[L_1^{a_1},\ldots,L_k^{a_k};b]} = \kappa_{[a_1,\ldots,a_k;b]}\,.$$

We define the *strict tautological ring* of the moduli space of $K3$ surfaces,

$$\mathsf{R}^\star(\mathcal{M}_{2\ell}) \subset \mathsf{A}^\star(\mathcal{M}_{2\ell},\mathbb{Q})\,,$$

to be the subring generated by the push-forwards from the Noether-Lefschetz loci $\mathcal{M}_\Lambda$ of all products of the κ classes (1) obtained from admissible classes of Λ. By definition,

$$\mathsf{NL}^\star(\mathcal{M}_{2\ell}) \subset \mathsf{R}^\star(\mathcal{M}_{2\ell})\,.$$

There is no need to include a κ index for the first Chern class of $\mathcal{T}_\pi$ since

$$c_1(\mathcal{T}_\pi) = -\pi^*\lambda$$

where $\lambda = c_1(\mathbb{E})$ is the first Chern class of the Hodge line bundle

$$\mathbb{E} \to \mathcal{M}_{2\ell}$$

with fiber $H^0(X,K_X)$ over the moduli point $(X,H)\in\mathcal{M}_{2\ell}$. The Hodge class λ is known to be supported on Noether-Lefschetz divisors.[4]

A slightly different tautological ring of the moduli space of $K3$ surfaces was defined in [16]. A basic result conjectured in [18] and proven in [5] is the isomorphism

$$\mathsf{NL}^1(\mathcal{M}_{2\ell}) = \mathsf{A}^1(\mathcal{M}_{2\ell},\mathbb{Q})\,.$$

In fact, the Picard group of $\mathcal{M}_\Lambda$ is generated by the Noether-Lefschetz divisors of $\mathcal{M}_\Lambda$ for every lattice polarization Λ of rank ≤ 17 by [5]. As an immediate consequence, the strict tautological ring defined here is isomorphic to the tautological ring of [16] in all codimensions up to 17. Since the dimension of $\mathcal{M}_{2\ell}$ is 19, the differences in the two definitions are only possible in degrees 18 and 19.

We prefer to work with the strict tautological ring. A basic advantage is that the κ classes are defined canonically (and not *up to twist* as in [16]). Every class of the strict tautological ring $\mathsf{R}^\star(\mathcal{M}_{2\ell})$ is defined explicitly. A central result of the paper is the following generation property conjectured first in [16].

Theorem 1. *The strict tautological ring is generated by Noether-Lefschetz loci,*

$$\mathsf{NL}^\star(\mathcal{M}_{2\ell}) = \mathsf{R}^\star(\mathcal{M}_{2\ell})\,.$$

[4]By [6], λ on $\mathcal{M}_\Lambda$ is supported on Noether-Lefschetz divisors for every lattice polarization Λ. See also [17, Theorem 3.1] for a stronger statement: λ on $\mathcal{M}_{2\ell}$ is supported on any infinite collection of Noether-Lefschetz divisors.

Our construction also defines the strict tautological ring

$$\mathsf{R}^\star(\mathcal{M}_\Lambda) \subset \mathsf{A}^\star(\mathcal{M}_\Lambda, \mathbb{Q})$$

for every lattice polarization Λ. As before, the subring generated by the Noether-Lefschetz loci corresponding to lattices $\widetilde{\Lambda} \supset \Lambda$ is contained in the strict tautological ring,

$$\mathsf{NL}^\star(\mathcal{M}_\Lambda) \subset \mathsf{R}^\star(\mathcal{M}_\Lambda)\,.$$

In fact, we prove a generation result parallel to Theorem 1 for every lattice polarization,

$$\mathsf{NL}^\star(\mathcal{M}_\Lambda) = \mathsf{R}^\star(\mathcal{M}_\Lambda)\,.$$

While the definition of $\mathsf{R}^\star(\mathcal{M}_\Lambda)$ includes infinitely many generators, $\mathsf{NL}^\star(\mathcal{M}_\Lambda)$ is finite-dimensional as a $\mathbb{Q}$-vector space by [7].

0.3. **Fiber products of the universal surface.** Let $\mathcal{X}^n$ denote the n^{th} fiber product of the universal $K3$ surface over $\mathcal{M}_{2\ell}$,

$$\pi^n : \mathcal{X}^n \to \mathcal{M}_{2\ell}\,.$$

The strict tautological ring

$$\mathsf{R}^\star(\mathcal{X}^n) \subset \mathsf{A}^\star(\mathcal{X}^n, \mathbb{Q})$$

is defined to be the subring generated by the push-forwards to $\mathcal{X}^n$ from the Noether-Lefschetz loci

$$\pi^n_\Lambda : \mathcal{X}^n_\Lambda \to \mathcal{M}_\Lambda$$

of all products of

- the π^n_Λ-relative diagonals in $\mathcal{X}^n_\Lambda$,
- the pull-backs of $\mathcal{L} \in \mathsf{A}^1(\mathcal{X}_\Lambda, \mathbb{Q})$ via the n projections

$$\mathcal{X}^n_\Lambda \to \mathcal{X}_\Lambda$$

  for every admissible $L \in \Lambda$,
- the pull-backs of $c_2(\mathcal{T}_{\pi_\Lambda}) \in \mathsf{A}^2(\mathcal{X}_\Lambda, \mathbb{Q})$ via the n projections,
- the pull-backs of $\mathsf{R}^\star(\mathcal{M}_\Lambda)$ via π^{n*}_Λ.

The construction also defines the strict tautological ring

$$\mathsf{R}^\star(\mathcal{X}^n_\Lambda) \subset \mathsf{A}^\star(\mathcal{X}^n_\Lambda, \mathbb{Q})$$

for every lattice polarization Λ.

0.4. **Export construction.** Let $\overline{\mathsf{M}}_{g,n}(\pi_\Lambda, L)$ be the π_Λ-relative moduli space of stable maps representing the admissible class $L \in \Lambda$. The evaluation map at the n markings is

$$\epsilon^n : \overline{\mathsf{M}}_{g,n}(\pi_\Lambda, L) \to \mathcal{X}_\Lambda^n \,.$$

Conjecture 1. *The push-forward of the reduced virtual fundamental class lies in the strict tautological ring,*

$$\epsilon^n_* \left[\overline{\mathsf{M}}_{g,n}(\pi_\Lambda, L)\right]^{\mathrm{red}} \;\in\; \mathsf{R}^\star(\mathcal{X}_\Lambda^n) \,.$$

When Conjecture 1 is restricted to a fixed $K3$ surface X, another open question is obtained.

Conjecture 2. *The push-forward of the reduced virtual fundamental class,*

$$\epsilon^n_* \left[\overline{\mathsf{M}}_{g,n}(X, L)\right]^{\mathrm{red}} \;\in\; \mathsf{A}^\star(X^n, \mathbb{Q}) \,,$$

lies in the Beauville-Voisin ring of X^n generated by the diagonals and the pull-backs of $\mathrm{Pic}(X)$ *via the n projections.*

If Conjecture 1 could be proven also for descendents (and in an effective form), then we could export tautological relations on $\overline{\mathsf{M}}_{g,n}$ to $\mathcal{X}_\Lambda^n$ via the morphisms

$$\overline{\mathsf{M}}_{g,n} \;\xleftarrow{\tau}\; \overline{\mathsf{M}}_{g,n}(\pi_\Lambda, L) \;\xrightarrow{\epsilon^n_\Lambda}\; \mathcal{X}_\Lambda^n \,.$$

More precisely, given a relation Rel among tautological classes on $\overline{\mathsf{M}}_{g,n}$,

$$\epsilon^n_* \tau^*(\mathsf{Rel}) \;=\; 0 \;\in\; \mathsf{R}^\star(\mathcal{X}_\Lambda^n)$$

would then be a relation among strict tautological classes on $\mathcal{X}_\Lambda^n$.

We prove Theorem 1 as a consequence of the export construction for the WDVV relation in genus 0 and for Getzler's relation in genus 1. The required parts of Conjectures 1 and 2 are proven by hand.

0.5. **WDVV and Getzler.** We fix an admissible class $L \in \Lambda$ and the corresponding divisor $\mathcal{L} \in \mathsf{A}^1(\mathcal{X}_\Lambda, \mathbb{Q})$. For $i \in \{1, \ldots, n\}$, let

$$\mathcal{L}_{(i)} \;\in\; \mathsf{A}^1(\mathcal{X}_\Lambda^n, \mathbb{Q})$$

denote the pull-back of $\mathcal{L}$ via the i^{th} projection

$$\mathrm{pr}_{(i)} : \mathcal{X}_\Lambda^n \to \mathcal{X}_\Lambda \,.$$

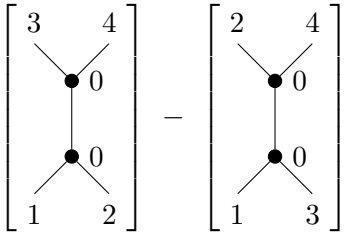
For $1 \le i < j \le n$, let

$$\Delta_{(ij)} \;\in\; \mathsf{A}^2(\mathcal{X}_\Lambda^n, \mathbb{Q})$$

be the π_Λ^n-relative diagonal where the i^{th} and j^{th} coordinates are equal. We write

$$\Delta_{(ijk)} \;=\; \Delta_{(ij)} \cdot \Delta_{(jk)} \;\in\; \mathsf{A}^4(\mathcal{X}_\Lambda^n, \mathbb{Q}) \,.$$

The Witten-Dijkgraaf-Verlinde-Verlinde relation in genus 0 is

(2) 

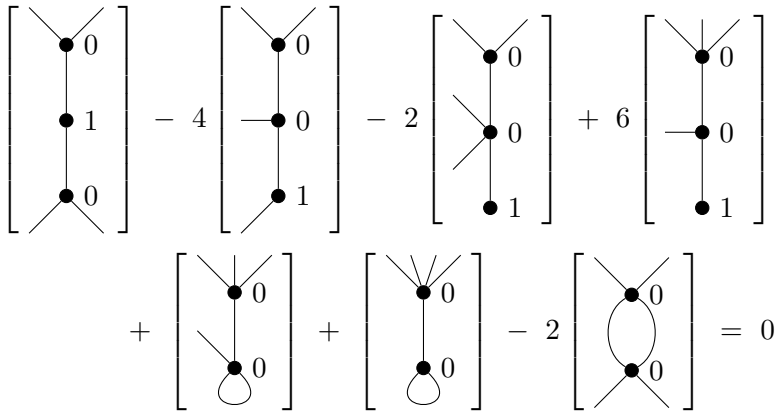
$= 0 \in \mathsf{A}^1(\overline{\mathsf{M}}_{0,4}, \mathbb{Q})\,.$

Theorem 2. *For all admissible $L \in \Lambda$, exportation of the WDVV relation yields*

$$(\dagger)\quad \mathcal{L}_{(1)}\mathcal{L}_{(2)}\mathcal{L}_{(3)}\Delta_{(34)} + \mathcal{L}_{(1)}\mathcal{L}_{(3)}\mathcal{L}_{(4)}\Delta_{(12)} - \mathcal{L}_{(1)}\mathcal{L}_{(2)}\mathcal{L}_{(3)}\Delta_{(24)} - \mathcal{L}_{(1)}\mathcal{L}_{(2)}\mathcal{L}_{(4)}\Delta_{(13)} + \ldots = 0 \in \mathsf{A}^5(\mathcal{X}^4_\Lambda, \mathbb{Q})\,,$$

where the dots stand for strict tautological classes supported over proper Noether-Lefschetz divisors of $\mathcal{M}_\Lambda$.

Getzler [11] in 1997 discovered a beautiful relation in the cohomology of $\overline{\mathsf{M}}_{1,4}$ which was proven to hold in Chow in [22]:

(3) 

$= 0 \in \mathsf{A}^2(\overline{\mathsf{M}}_{1,4}, \mathbb{Q})\,.$

Here, the strata are summed over all marking distributions and are taken in the stack sense (following the conventions of [11]).

Theorem 3. *For admissible $L \in \Lambda$ satisfying the condition $\langle L, L\rangle_\Lambda \geq 0$, exportation of Getzler's relation yields*

$$\begin{aligned}(\ddagger)\quad &\mathcal{L}_{(1)}\Delta_{(12)}\Delta_{(34)} + \mathcal{L}_{(3)}\Delta_{(12)}\Delta_{(34)} + \mathcal{L}_{(1)}\Delta_{(13)}\Delta_{(24)} + \mathcal{L}_{(2)}\Delta_{(13)}\Delta_{(24)} + \mathcal{L}_{(1)}\Delta_{(14)}\Delta_{(23)}\\ &+ \mathcal{L}_{(2)}\Delta_{(14)}\Delta_{(23)} - \mathcal{L}_{(1)}\Delta_{(234)} - \mathcal{L}_{(2)}\Delta_{(134)} - \mathcal{L}_{(3)}\Delta_{(124)} - \mathcal{L}_{(4)}\Delta_{(123)}\\ &- \mathcal{L}_{(1)}\Delta_{(123)} - \mathcal{L}_{(1)}\Delta_{(124)} - \mathcal{L}_{(1)}\Delta_{(134)} - \mathcal{L}_{(2)}\Delta_{(234)} + \ldots = 0 \in \mathsf{A}^5(\mathcal{X}^4_\Lambda, \mathbb{Q})\,,\end{aligned}$$

where the dots stand for strict tautological classes supported over proper Noether-Lefschetz loci of $\mathcal{M}_\Lambda$.

The statements of Theorems 2 and 3 contain only the *principal* terms of the relation (not supported over proper Noether-Lefschetz loci of $\mathcal{M}_\Lambda$). We will write all the terms represented by the dots in Sections 4 and 6.

The relation of Theorem 2 is obtained from the export construction after dividing by the genus 0 reduced Gromov-Witten invariant $N_0(L)$. The latter never vanishes for admissible classes. Similarly, for Theorem 3, the export construction has been divided by the genus 1 reduced Gromov-Witten invariant

$$N_1(L) = \int_{[\overline{\mathsf{M}}_{1,1}(X,L)]^{\mathrm{red}}} \mathrm{ev}^*(\mathsf{p})\,,$$

where $\mathsf{p} \in H^4(X,\mathbb{Q})$ is the class of a point on X. By a result of Oberdieck discussed in Section 1.5, $N_1(L)$ does not vanish for admissible classes satisfying $\langle L, L\rangle_\Lambda \geq 0$.

0.6. **Relations on $\mathcal{X}^3_\Lambda$.** As a Corollary of Getzler's relation, we have the following result. Let

$$\mathrm{pr}_{(123)} : \mathcal{X}^4_\Lambda \to \mathcal{X}^3_\Lambda$$

be the projection to the first 3 factors. Let $L = H$ and consider the operation

$$\mathrm{pr}_{(123)*}(\mathcal{H}_{(4)} \cdot -)$$

applied to the relation (‡). We obtain a universal decomposition of the diagonal $\Delta_{(123)}$ which generalizes the result of Beauville-Voisin [2] for a fixed $K3$ surface.[5]

Corollary 4. *The π^3_Λ-relative diagonal $\Delta_{(123)}$ admits a decomposition with principal terms*

$$\begin{aligned}(\ddagger')\quad 2\ell \cdot \Delta_{(123)} \;=\; & \mathcal{H}^2_{(1)}\Delta_{(23)} + \mathcal{H}^2_{(2)}\Delta_{(13)} + \mathcal{H}^2_{(3)}\Delta_{(12)} \\ & - \mathcal{H}^2_{(1)}\Delta_{(12)} - \mathcal{H}^2_{(1)}\Delta_{(13)} - \mathcal{H}^2_{(2)}\Delta_{(23)} + \ldots \;\in\; \mathsf{A}^4(\mathcal{X}^3_\Lambda, \mathbb{Q})\,,\end{aligned}$$

where the dots stand for strict tautological classes supported over proper Noether-Lefschetz loci of $\mathcal{M}_\Lambda$.

The diagonal $\Delta_{(123)}$ controls the behavior of the κ classes. For instance, we have

$$\kappa_{[a;b]} \;=\; \pi^3_*\left(\mathcal{H}^a_{(1)} \cdot \Delta^b_{(23)} \cdot \Delta_{(123)}\right) \;\in\; \mathsf{A}^{a+2b-2}(\mathcal{M}_{2\ell}, \mathbb{Q})\,.$$

The diagonal decomposition of Corollary 4 plays a fundamental role in the proof of Theorem 1.

[5] See also [26] for a related discussion.

0.7. **Cohomological results.** Bergeron and Li have an announced an independent approach to the generation (in most codimensions) of the tautological ring $\mathsf{RH}^\star(\mathcal{M}_\Lambda)$ by Noether-Lefschetz loci in cohomology. Petersen [25] has proven the vanishing[6]

$$\mathsf{RH}^{18}(\mathcal{M}_{2\ell}) = \mathsf{RH}^{19}(\mathcal{M}_{2\ell}) = 0\,.$$

We expect the above vanishing to hold also in Chow.

What happens in codimension 17 is a very interesting question. By a result of van der Geer and Katsura [10],

$$\mathsf{RH}^{17}(\mathcal{M}_{2\ell}) \neq 0\,.$$

We hope the stronger statement

$$\mathsf{RH}^{17}(\mathcal{M}_{2\ell}) = \mathbb{Q} \tag{4}$$

holds. If true, (4) would open the door to a numerical theory of proportionalities in the tautological ring. The evidence for (4) is rather limited at the moment. Careful calculations in the $\ell = 1$ and 2 cases would be very helpful here.

0.8. **Acknowledgments.** We are grateful to G. Farkas, G. van der Geer, D. Huybrechts, Z. Li, A. Marian, D. Maulik, G. Oberdieck, D. Oprea, D. Petersen, and J. Shen for many discussions about the moduli of $K3$ surfaces. The paper was completed at the conference *Curves on surfaces and threefolds* at the Bernoulli center in Lausanne in June 2016 attended by both authors.

R. P. was partially supported by SNF-200021143274, SNF-200020162928, ERC-2012-AdG-320368-MCSK, SwissMAP, and the Einstein Stiftung. Q. Y. was supported by the grant ERC-2012-AdG-320368-MCSK.

1. $K3$ surfaces

1.1. **Reduced Gromov-Witten theory.** Let X be a nonsingular, projective $K3$ surface over $\mathbb{C}$, and let

$$L \in \mathrm{Pic}(X) = H^2(X,\mathbb{Z}) \cap H^{1,1}(X,\mathbb{C})$$

be a nonzero effective class. The moduli space $\overline{\mathsf{M}}_{g,n}(X,L)$ of genus g stable maps with n marked points has expected dimension

$$\dim^{\mathrm{vir}}_{\mathbb{C}}\, \overline{\mathsf{M}}_{g,n}(X,\beta) = \int_L c_1(X) + (\dim_{\mathbb{C}}(X)-3)(1-g) + n = g - 1 + n\,.$$

However, as the obstruction theory admits a 1-dimensional trivial quotient, the virtual class $[\overline{\mathsf{M}}_{g,n}(X,L)]^{\mathrm{vir}}$ vanishes. The standard Gromov-Witten theory is trivial.

[6] We use the complex grading here.

Curve counting on $K3$ surfaces is captured instead by the *reduced* Gromov-Witten theory constructed first via the twistor family in [8]. An algebraic construction following [3] is given in [18]. The reduced class

$$\big[\overline{\mathsf{M}}_{g,n}(X,L)\big]^{\mathrm{red}} \ \in \ \mathsf{A}_{g+n}(\overline{\mathsf{M}}_{g,n}(X,L),\mathbb{Q})$$

has dimension $g+n$. The reduced Gromov-Witten integrals of X,

$$\Big\langle \tau_{a_1}(\gamma_1)\cdots\tau_{a_n}(\gamma_n)\Big\rangle_{g,L}^{X,\mathrm{red}} = \int_{[\overline{\mathsf{M}}_{g,n}(X,L)]^{\mathrm{red}}} \prod_{i=1}^{n} \mathrm{ev}_i^*(\gamma_i)\cup\psi_i^{a_i} \ \in \ \mathbb{Q}\,, \tag{5}$$

are well-defined. Here, $\gamma_i \in H^\star(X,\mathbb{Q})$ and ψ_i is the standard descendent class at the i^{th} marking. Under deformations of X for which L remains a $(1,1)$-class, the integrals (5) are invariant.

1.2. **Curve classes on $K3$ surfaces.** Let X be a nonsingular, projective $K3$ surface over $\mathbb{C}$. The second cohomology of X is a rank 22 lattice with intersection form

$$H^2(X,\mathbb{Z}) \cong U\oplus U\oplus U\oplus E_8(-1)\oplus E_8(-1)\,, \tag{6}$$

where

$$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and

$$E_8(-1) = \begin{pmatrix} -2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 \end{pmatrix}$$

is the (negative) Cartan matrix. The intersection form (6) is even.

The *divisibility* $m(L)$ is the largest positive integer which divides the lattice element $L\in H^2(X,\mathbb{Z})$. If the divisibility is 1, L is *primitive*. Elements with equal divisibility and norm square are equivalent up to orthogonal transformation of $H^2(X,\mathbb{Z})$, see [28].

1.3. **Lattice polarization.** A primitive class $H\in \mathrm{Pic}(X)$ is a *quasi-polarization* if

$$\langle H,H\rangle_X > 0 \quad \text{and} \quad \langle H,[C]\rangle_X \geq 0$$

for every curve $C\subset X$. A sufficiently high tensor power H^n of a quasi-polarization is base point free and determines a birational morphism

$$X\to \widetilde{X}$$

contracting A-D-E configurations of (-2)-curves on X. Therefore, every quasi-polarized $K3$ surface is algebraic.

Let Λ be a fixed rank r primitive[7] sublattice

$$\Lambda \subset U \oplus U \oplus U \oplus E_8(-1) \oplus E_8(-1)$$

with signature $(1, r-1)$, and let $v_1, \ldots, v_r \in \Lambda$ be an integral basis. The discriminant is

$$\Delta(\Lambda) = (-1)^{r-1} \det \begin{pmatrix} \langle v_1, v_1 \rangle & \cdots & \langle v_1, v_r \rangle \\ \vdots & \ddots & \vdots \\ \langle v_r, v_1 \rangle & \cdots & \langle v_r, v_r \rangle \end{pmatrix} .$$

The sign is chosen so $\Delta(\Lambda) > 0$.

A Λ-*polarization* of a $K3$ surface X is a primitive embedding

$$j : \Lambda \hookrightarrow \mathrm{Pic}(X)$$

satisfying two properties:

(i) the lattice pairs $\Lambda \subset U^3 \oplus E_8(-1)^2$ and $\Lambda \subset H^2(X, \mathbb{Z})$ are isomorphic via an isometry which restricts to the identity on Λ,
(ii) $\mathrm{Im}(j)$ contains a quasi-polarization.

By (ii), every Λ-polarized $K3$ surface is algebraic.

The period domain M of Hodge structures of type $(1, 20, 1)$ on the lattice $U^3 \oplus E_8(-1)^2$ is an analytic open subset of the 20-dimensional nonsingular isotropic quadric Q,

$$M \subset Q \subset \mathbb{P}\big((U^3 \oplus E_8(-1)^2) \otimes_{\mathbb{Z}} \mathbb{C}\big) .$$

Let $M_\Lambda \subset M$ be the locus of vectors orthogonal to the entire sublattice $\Lambda \subset U^3 \oplus E_8(-1)^2$.

Let Γ be the isometry group of the lattice $U^3 \oplus E_8(-1)^2$, and let

$$\Gamma_\Lambda \subset \Gamma$$

be the subgroup restricting to the identity on Λ. By global Torelli, the moduli space $\mathcal{M}_\Lambda$ of Λ-polarized $K3$ surfaces is the quotient

$$\mathcal{M}_\Lambda = M_\Lambda / \Gamma_\Lambda .$$

We refer the reader to [9] for a detailed discussion.

1.4. **Genus 0 invariants.** Let $L \in \mathrm{Pic}(X)$ be a nonzero and *admissible* class on a $K3$ surface X as defined in Section 0.2:

(i) $\frac{1}{m(L)^2} \cdot \langle L, L \rangle_X \geq -2$,
(ii) $\langle H, L \rangle_X \geq 0$.

In case of equalities in both (i) and (ii), we further require L to be effective.

[7] A sublattice is primitive if the quotient is torsion free.

Proposition 1. *The reduced genus* 0 *Gromov-Witten invariant*

$$N_0(L) = \int_{[\overline{\mathsf{M}}_{0,0}(X,L)]^{\mathrm{red}}} 1$$

is nonzero for all admissible classes L*.*

Proof. The result is a direct consequence of the full Yau-Zaslow formula (including multiple classes) proven in [14]. We define $N_0(\ell)$ for $\ell \geq -1$ by

$$\sum_{\ell=-1}^{\infty} q^{\ell} N_0(\ell) \;=\; \frac{1}{q\prod_{n=1}^{\infty}(1-q^n)^{24}} \;=\; \frac{1}{q} + 24 + 324q + 3200q^2 \ldots \,.$$

For $\ell < -1$, we set $N_0(\ell) = 0$. By the full Yau-Zaslow formula,

$$N_0(L) = \sum_{r|m(L)} \frac{1}{r^3} N_0\left(\frac{\langle L, L\rangle_X}{2r^2}\right). \tag{7}$$

Since all $N_0(\ell)$ for $\ell \geq -1$ are positive, the right side of (7) is positive. □

1.5. **Genus** 1 **invariants.** Let $L \in \mathrm{Pic}(X)$ be an admissible class on a $K3$ surface X. Let

$$N_1(L) = \int_{[\overline{\mathsf{M}}_{1,1}(X,L)]^{\mathrm{red}}} \mathrm{ev}^*(\mathsf{p})$$

be the reduced invariant virtually counting elliptic curves passing through a point of X. We define

$$\sum_{\ell=0}^{\infty} q^{\ell} N_1(\ell) \;=\; \frac{\sum_{k=1}^{\infty}\sum_{d|k} dkq^k}{q\prod_{n=1}^{\infty}(1-q^n)^{24}} \;=\; 1 + 30q + 480q^2 + 5460q^3 \ldots \,.$$

For $\ell \leq -1$, we set $N_1(\ell) = 0$. If L is primitive,

$$N_1(L) = N_1\left(\frac{\langle L, L\rangle_X}{2}\right)$$

by a result of [8]. In particular, $N_1(L) > 0$ for L admissible and primitive if $\langle L, L\rangle_X \geq 0$.

Proposition 2 (Oberdieck)**.** *The reduced genus* 1 *Gromov-Witten invariant* $N_1(L)$ *is nonzero for all admissible classes* L *satisfying* $\langle L, L\rangle_X \geq 0$*.*

Proof. The result is a direct consequence of the multiple cover formula for the reduced Gromov-Witten theory of $K3$ surfaces conjectured in [21]. By the multiple cover formula,

$$N_1(L) = \sum_{r|m(L)} r N_1\left(\frac{\langle L, L\rangle_X}{2r^2}\right). \tag{8}$$

Since all $N_1(\ell)$ for $\ell \geq 0$ are positive, the right side of (8) is positive.

To complete the argument, we must prove the multiple cover formula (8) in the required genus 1 case. We derive (8) from the genus 2 case of the Katz-Klemm-Vafa formula for imprimitive classes proven in [24]. Let

$$N_2(L) = \int_{[\overline{\mathsf{M}}_2(X,L)]^{\mathrm{red}}} \lambda_2\,,$$

where λ_2 is the pull-back of the second Chern class of the Hodge bundle on $\overline{\mathsf{M}}_2$. Using the well-known boundary expression[8] for λ_2 in the tautological ring of $\overline{\mathsf{M}}_2$, Pixton [19, Appendix] proves

$$N_2(L) = \frac{1}{10} N_1(L) + \frac{\langle L, L\rangle_X^2}{960} N_0(L)\,. \tag{9}$$

By [24], the multiple cover formula for $N_2(L)$ carries a factor of r. By the Yau-Zaslow formula for imprimitive classes [14], the term $\frac{\langle L,L\rangle_X^2}{960} N_0(L)$ also carries a factor of

$$(r^2)^2 \cdot \frac{1}{r^3} = r\,.$$

By (9), $N_1(L)$ must then carry a factor of r in the multiple cover formula exactly as claimed in (8). □

1.6. **Vanishing.** Let $L \in \mathrm{Pic}(X)$ be an inadmissible class on a $K3$ surface X. The following vanishing result holds.

Proposition 3. *For inadmissible L, the reduced virtual class is 0 in Chow,*

$$\big[\overline{\mathsf{M}}_{g,n}(X,L)\big]^{\mathrm{red}} \;=\; 0 \;\in\; \mathsf{A}_{g+n}(\overline{\mathsf{M}}_{g,n}(X,L),\mathbb{Q})\,.$$

Proof. Consider a 1-parameter family of $K3$ surfaces

$$\pi_C : \mathcal{X} \to (C,0) \tag{10}$$

with special fiber $\pi^{-1}(0) = X$ for which the class L is algebraic on all fibers. Let

$$\phi : \overline{\mathsf{M}}_{g,n}(\pi_C, L) \to C \tag{11}$$

be the universal moduli space of stable maps to the fibers of π_C. Let

$$\iota : 0 \hookrightarrow C$$

be the inclusion of the special point. By the construction of the reduced class,

$$[\overline{\mathsf{M}}_{g,n}(X,L)]^{\mathrm{red}} = \iota^![\overline{\mathsf{M}}_{g,n}(\pi_C,L)]^{\mathrm{red}}\,.$$

Using the argument of [18, Lemma 2] for elliptically fibered $K3$ surfaces with a section, such a family (10) can be found for which the fiber of ϕ is *empty* over a general point of C since L is not generically effective. The vanishing

$$\big[\overline{\mathsf{M}}_{g,n}(X,L)\big]^{\mathrm{red}} \;=\; 0 \;\in\; \mathsf{A}_{g+n}(\overline{\mathsf{M}}_{g,n}(X,L),\mathbb{Q}) \tag{12}$$

[8]See [20]. A more recent approach valid also for higher genus can be found in [13].

then follows: $\iota^!$ of *any* cycle which does not dominate C is 0.

If the family (10) consists of projective $K3$ surfaces, the argument stays within the Gromov-Witten theory of algebraic varieties. However, if the family consists of non-algebraic $K3$ surfaces (as may be the case since L is not ample), a few more steps are needed. First, we can assume *all* stable maps to the fiber of the family (10) lie over $0 \in C$ and map to the algebraic fiber X. There is no difficulty in constructing the moduli space of stable maps (11). In fact, all the geometry takes place over an Artinian neighborhood of $0 \in C$. Therefore the cones and intersection theory are all algebraic. We conclude the vanishing (12). □

2. Gromov-Witten theory for families of $K3$ surfaces

2.1. **The divisor $\mathcal{L}$.** Let $\mathcal{B}$ be any nonsingular base scheme, and let

$$\pi_{\mathcal{B}} : \mathcal{X}_{\mathcal{B}} \to \mathcal{B}$$

be a family of Λ-polarized $K3$ surfaces.[9] For $L \in \Lambda$ admissible, consider the moduli space

$$\overline{\mathsf{M}}_{g,n}(\pi_{\mathcal{B}}, L) \to \mathcal{B}\,. \tag{13}$$

The relationship between the $\pi_{\mathcal{B}}$-relative standard and reduced obstruction theory of $\overline{\mathsf{M}}_{g,n}(\pi_{\mathcal{B}}, L)$ yields

$$\big[\overline{\mathsf{M}}_{g,n}(\pi_{\mathcal{B}}, L)\big]^{\mathrm{vir}} = -\lambda \cdot \big[\overline{\mathsf{M}}_{g,n}(\pi_{\mathcal{B}}, L)\big]^{\mathrm{red}}$$

where λ is the pull-back via (13) of the Hodge bundle on $\mathcal{B}$. The reduced class is of $\pi_{\mathcal{B}}$-relative dimension $g+n$.

The canonical divisor class associated to an admissible $L \in \Lambda$ is

$$\mathcal{L} \;=\; \frac{1}{N_0(L)} \;\cdot\; \epsilon_* \big[\overline{\mathsf{M}}_{0,1}(\pi_{\mathcal{B}}, L)\big]^{\mathrm{red}} \;\in\; \mathsf{A}^1(\mathcal{X}_{\mathcal{B}}, \mathbb{Q})\,.$$

By Proposition 1, the reduced Gromov-Witten invariant

$$N_0(L) = \int_{[\overline{\mathsf{M}}_{0,0}(X,L)]^{\mathrm{red}}} 1$$

is not zero.

For a family of Λ-polarized $K3$ surfaces over any base scheme $\mathcal{B}$, we define

$$\mathcal{L} \;\in\; \mathsf{A}^1(\mathcal{X}_{\mathcal{B}}, \mathbb{Q})$$

[9] Since the quasi-polarization class may not be ample, $\mathcal{X}_{\mathcal{B}}$ may be a nonsingular algebraic space. There is no difficulty in defining the moduli space of stable maps and the associated virtual classes for such nonsingular algebraic spaces. Since the stable maps are to the fiber classes, the moduli spaces are of finite type. In the original paper on virtual fundamental classes by Behrend and Fantechi [3], the obstruction theory on the moduli space of stable maps was required to have a global resolution (usually obtained from an ample bundle on the target). However, the global resolution hypothesis was removed by Kresch in [15, Theorem 5.2.1].

by pull-back from the universal family over the nonsingular moduli stack $\mathcal{M}_\Lambda$.

2.2. **The divisor $\widehat{\mathcal{L}}$.** Let $\mathcal{X}_\Lambda$ denote the universal Λ-polarized $K3$ surface over $\mathcal{M}_\Lambda$,

$$\pi_\Lambda : \mathcal{X}_\Lambda \to \mathcal{M}_\Lambda\,.$$

For $L \in \Lambda$ admissible, Let $\overline{\mathsf{M}}_{0,0}(\pi_\Lambda, L)$ be the π_Λ-relative moduli space of genus 0 stable maps. Let

$$\phi : \overline{\mathsf{M}}_{0,0}(\pi_\Lambda, L) \to \mathcal{M}_\Lambda$$

be the proper structure map. The reduced virtual class $\left[\overline{\mathsf{M}}_{0,0}(\pi_\Lambda, L)\right]^{\mathrm{red}}$ is of ϕ-relative dimension 0 and satisfies

$$\phi_* \left[\overline{\mathsf{M}}_{0,0}(\pi_\Lambda, L)\right]^{\mathrm{red}} \;=\; N_0(L) \cdot [\mathcal{M}_\Lambda] \;\neq\; 0\,.$$

The universal curve over the moduli space of stable maps,

$$\mathsf{C} \to \overline{\mathsf{M}}_{0,0}(\pi_\Lambda, L)\,,$$

carries an evaluation morphism

$$\epsilon_{\overline{\mathsf{M}}} : \mathsf{C} \to \mathcal{X}_{\overline{\mathsf{M}}} \;=\; \phi^* \mathcal{X}_\Lambda$$

over $\mathcal{M}_\Lambda$. Via the Hilbert-Chow map, the image of $\epsilon_{\overline{\mathsf{M}}}$ determines a canonical Chow cohomology class

$$\widehat{\mathcal{L}} \;\in\; \mathsf{A}^1(\mathcal{X}_{\overline{\mathsf{M}}}, \mathbb{Q})\,.$$

Via pull-back, we also have the class

$$\mathcal{L} \;\in\; \mathsf{A}^1(\mathcal{X}_{\overline{\mathsf{M}}}, \mathbb{Q})$$

constructed in Section 2.1.

The classes $\widehat{\mathcal{L}}$ and $\mathcal{L}$ are are certainly equal when restricted to the fibers of

$$\pi_{\overline{\mathsf{M}}} : \mathcal{X}_{\overline{\mathsf{M}}} \;\to\; \overline{\mathsf{M}}_{0,0}(\pi_\Lambda, L)\,.$$

However, more is true. We define the reduced virtual class of $\mathcal{X}_{\overline{\mathsf{M}}}$ by flat pull-back,

$$[\mathcal{X}_{\overline{\mathsf{M}}}]^{\mathrm{red}} \;=\; \pi_{\overline{\mathsf{M}}}^* \left[\overline{\mathsf{M}}_{0,0}(\pi_\Lambda, L)\right]^{\mathrm{red}} \;\in\; \mathsf{A}_{\mathsf{d}(\Lambda)+2}(\mathcal{X}_{\overline{\mathsf{M}}}, \mathbb{Q})\,,$$

where $\mathsf{d}(\Lambda) = 20 - \mathrm{rank}(\Lambda)$ is the dimension of $\mathcal{M}_\Lambda$.

Theorem 5. *For $L \in \Lambda$ admissible,*

$$\widehat{\mathcal{L}} \cap [\mathcal{X}_{\overline{\mathsf{M}}}]^{\mathrm{red}} \;=\; \mathcal{L} \cap [\mathcal{X}_{\overline{\mathsf{M}}}]^{\mathrm{red}} \;\in\; \mathsf{A}_{\mathsf{d}(\Lambda)+1}(\mathcal{X}_{\overline{\mathsf{M}}}, \mathbb{Q})\,.$$

The proof of Theorem 5 will be given in Section 5.

3. Basic push-forwards in genus 0 and 1

3.1. **Push-forwards of reduced classes.** Let $L \in \Lambda$ be a nonzero class. As discussed in Section 0.4, the export construction requires knowing the push-forward of the reduced virtual class $\big[\overline{\mathsf{M}}_{g,n}(\pi_\Lambda, L)\big]^{\mathrm{red}}$ via the evaluation map

$$\epsilon^n : \overline{\mathsf{M}}_{g,n}(\pi_\Lambda, L) \to \mathcal{X}_\Lambda^n \,.$$

Fortunately, to export the WDVV and Getzler relations, we only need to analyze three simple cases.

3.2. **Case $g = 0$, $n \geq 1$.** Consider the push-forward class in genus 0,

$$\epsilon_*^n \big[\overline{\mathsf{M}}_{0,n}(\pi_\Lambda, L)\big]^{\mathrm{red}} \;\in\; \mathsf{A}^n(\mathcal{X}_\Lambda^n, \mathbb{Q}) \,.$$

For $n = 1$ and $L \in \Lambda$ admissible, we have by definition

$$\epsilon_* \big[\overline{\mathsf{M}}_{0,1}(\pi_\Lambda, L)\big]^{\mathrm{red}} \;=\; N_0(L) \cdot \mathcal{L} \,.$$

Proposition 4. *For all $n \geq 1$, we have*

$$\epsilon_*^n \big[\overline{\mathsf{M}}_{0,n}(\pi_\Lambda, L)\big]^{\mathrm{red}} = \begin{cases} N_0(L) \cdot \mathcal{L}_{(1)} \cdots \mathcal{L}_{(n)} & \text{if } L \in \Lambda \text{ is admissible}, \\ 0 & \text{if not}. \end{cases}$$

Here $\mathcal{L}_{(i)}$ is the pull-back of $\mathcal{L}$ via the i^{th} projection.

Proof. Consider first the case where the class $L \in \Lambda$ is admissible. The evaluation map ϵ^n factors as

$$\overline{\mathsf{M}}_{0,n}(\pi_\Lambda, L) \;\xrightarrow{\epsilon^n_{\overline{\mathsf{M}}}}\; \mathcal{X}^n_{\overline{\mathsf{M}}} \;\xrightarrow{\rho^n}\; \mathcal{X}^n_\Lambda$$

where $\epsilon^n_{\overline{\mathsf{M}}}$ is the lifted evaluation map and ρ^n is the projection. We have

$$\begin{aligned} \epsilon_*^n \big[\overline{\mathsf{M}}_{0,n}(\pi_\Lambda, L)\big]^{\mathrm{red}} &= \rho^n_* \epsilon^n_{\overline{\mathsf{M}}*} \big[\overline{\mathsf{M}}_{0,n}(\pi_\Lambda, L)\big]^{\mathrm{red}} \\ &= \rho^n_* \left(\widehat{\mathcal{L}}_{(1)} \cdots \widehat{\mathcal{L}}_{(n)} \cap [\mathcal{X}^n_{\overline{\mathsf{M}}}]^{\mathrm{red}} \right) \\ &= \rho^n_* \left(\mathcal{L}_{(1)} \cdots \mathcal{L}_{(n)} \cap [\mathcal{X}^n_{\overline{\mathsf{M}}}]^{\mathrm{red}} \right) \\ &= N_0(L) \cdot \mathcal{L}_{(1)} \cdots \mathcal{L}_{(n)} \cap [\mathcal{X}^n_\Lambda] \,, \end{aligned}$$

where the third equality is a consequence of Theorem 5.

Next, consider the case where $L \in \Lambda$ is inadmissible. By Proposition 3 and a spreading out argument [27, 1.1.2], the reduced class $\big[\overline{\mathsf{M}}_{0,n}(\pi_\Lambda, L)\big]^{\mathrm{red}}$ is supported over a proper subset of $\mathcal{M}_\Lambda$. Since $K3$ surfaces are not ruled, the support of

$$\epsilon_*^n \big[\overline{\mathsf{M}}_{0,n}(\pi_\Lambda, L)\big]^{\mathrm{red}} \;\in\; \mathsf{A}^n(\mathcal{X}_\Lambda^n, \mathbb{Q})$$

has codimension at least $n+1$ and therefore vanishes. □

3.3. **Case $g = 1$, $n = 1$.** The push-forward class

$$\epsilon_* \left[\overline{\mathsf{M}}_{1,1}(\pi_\Lambda, L)\right]^{\mathrm{red}} \in \mathsf{A}^0(\mathcal{X}_\Lambda, \mathbb{Q})$$

is a multiple of the fundamental class of $\mathcal{X}_\Lambda$.

Proposition 5. *We have*

$$\epsilon_* \left[\overline{\mathsf{M}}_{1,1}(\pi_\Lambda, L)\right]^{\mathrm{red}} = \begin{cases} N_1(L) \cdot [\mathcal{X}_\Lambda] & \text{if } L \in \Lambda \text{ is admissible and } \langle L, L\rangle_\Lambda \geq 0\,, \\ 0 & \text{if not}. \end{cases}$$

Proof. The multiple of the fundamental class $[\mathcal{X}_\Lambda]$ can be computed fiberwise: it is the genus 1 Gromov-Witten invariant

$$N_1(L) = \int_{[\overline{\mathsf{M}}_{1,1}(X,L)]^{\mathrm{red}}} \mathrm{ev}^*(\mathsf{p})\,.$$

The invariant vanishes for $L \in \mathrm{Pic}(X)$ inadmissible as well as for L admissible and $\langle L, L\rangle_X < 0$. □

3.4. **Case $g = 1$, $n = 2$.** The push-forward class is a divisor,

$$\epsilon_*^2 \left[\overline{\mathsf{M}}_{1,2}(\pi_\Lambda, L)\right]^{\mathrm{red}} \in \mathsf{A}^1(\mathcal{X}_\Lambda^2, \mathbb{Q})\,.$$

Proposition 6. *We have*

$$\epsilon_*^2 \left[\overline{\mathsf{M}}_{1,2}(\pi_\Lambda, L)\right]^{\mathrm{red}} = \begin{cases} N_1(L) \cdot \Big(\mathcal{L}_{(1)} + \mathcal{L}_{(2)} + Z(L)\Big) & \text{if } L \in \Lambda \text{ is admissible and } \langle L, L\rangle_\Lambda \geq 0\,, \\ 0 & \text{if not}. \end{cases}$$

Here $Z(L)$ is a divisor class in $\mathsf{A}^1(\mathcal{M}_\Lambda, \mathbb{Q})$ depending on L.[10]

In Section 7.2, we will compute $Z(L)$ explicitly in terms of Noether-Lefschetz divisors in the moduli space $\mathcal{M}_\Lambda$.

Proof. Consider first the case where the class $L \in \Lambda$ is admissible and $\langle L, L\rangle_\Lambda \geq 0$. If L is a multiple of the quasi-polarization H, we may assume $\Lambda = (2\ell)$. Then, the relative Picard group

$$\mathrm{Pic}(\mathcal{X}_\Lambda / \mathcal{M}_\Lambda)$$

has rank 1. Since the reduced class $\left[\overline{\mathsf{M}}_{1,2}(\pi_\Lambda, L)\right]^{\mathrm{red}}$ is $\mathfrak{S}_2$-invariant, the push-forward takes the form

$$\epsilon_*^2 \left[\overline{\mathsf{M}}_{1,2}(\pi_\Lambda, L)\right]^{\mathrm{red}} = c(L) \cdot \big(\mathcal{L}_{(1)} + \mathcal{L}_{(2)}\big) + \widetilde{Z}(L) \in \mathsf{A}^1(\mathcal{X}_\Lambda^2, \mathbb{Q})\,, \tag{14}$$

where $c(L) \in \mathbb{Q}$ and $\widetilde{Z}(L)$ is (the pull-back of) a divisor class in $\mathsf{A}^1(\mathcal{M}_\Lambda, \mathbb{Q})$.

[10] We identify $\mathsf{A}^\star(\mathcal{M}_\Lambda, \mathbb{Q})$ as a subring of $\mathsf{A}^\star(\mathcal{X}_\Lambda^n, \mathbb{Q})$ via π_Λ^{n*}.

The constant $c(L)$ can be computed fiberwise: by the divisor equation[11], we have

$$c(L) = N_1(L)\,.$$

Since $N_1(L) \neq 0$ by Proposition 2, we can rewrite (14) as

$$\epsilon_*^2\left[\overline{\mathsf{M}}_{1,2}(\pi_\Lambda, L)\right]^{\mathrm{red}} \;=\; N_1(L)\cdot\Big(\mathcal{L}_{(1)} + \mathcal{L}_{(2)} + Z(L)\Big) \;\in\; \mathsf{A}^1(\mathcal{X}_\Lambda^2, \mathbb{Q})\,,$$

where $Z(L) \in \mathsf{A}^1(\mathcal{M}_\Lambda, \mathbb{Q})$.

If $L \neq m\cdot H$, we may assume Λ to be a rank 2 lattice with $H, L \in \Lambda$. Then, the push-forward class takes the form

$$\begin{aligned}(15)\quad \epsilon_*^2\left[\overline{\mathsf{M}}_{1,2}(\pi_\Lambda, L)\right]^{\mathrm{red}} \;=\; & c_H(L)\cdot\big(\mathcal{H}_{(1)} + \mathcal{H}_{(2)}\big) + c_L(L)\cdot\big(\mathcal{L}_{(1)} + \mathcal{L}_{(2)}\big) \\ & + \widetilde{Z}(L) \;\in\; \mathsf{A}^1(\mathcal{X}_\Lambda^2, \mathbb{Q})\,,\end{aligned}$$

where $c_H(L), c_L(L) \in \mathbb{Q}$ and $\widetilde{Z}(L) \in \mathsf{A}^1(\mathcal{M}_\Lambda, \mathbb{Q})$. By applying the divisor equation with respect to

$$\langle L, L\rangle_\Lambda \cdot H - \langle H, L\rangle_\Lambda \cdot L\,,$$

we find

$$c_H(L)\Big(2\ell\langle L, L\rangle_\Lambda - \langle H, L\rangle_\Lambda^2\Big) = 0\,.$$

Since $2\ell\langle L, L\rangle_\Lambda - \langle H, L\rangle_\Lambda^2 < 0$ by the Hodge index theorem, we have $c_H(L) = 0$. Moreover, by applying the divisor equation with respect to H, we find

$$c_L(L) = N_1(L)\,.$$

Since $N_1(L) \neq 0$ by Proposition 2, we can rewrite (15) as

$$\epsilon_*^2\left[\overline{\mathsf{M}}_{1,2}(\pi_\Lambda, L)\right]^{\mathrm{red}} \;=\; N_1(L)\cdot\Big(\mathcal{L}_{(1)} + \mathcal{L}_{(2)} + Z(L)\Big) \;\in\; \mathsf{A}^1(\mathcal{X}_\Lambda^2, \mathbb{Q})\,,$$

where $Z(L) \in \mathsf{A}^1(\mathcal{M}_\Lambda, \mathbb{Q})$.

Next, consider the case where the class $L \in \Lambda$ is inadmissible. As before, by Proposition 3 and a spreading out argument, the reduced class $\left[\overline{\mathsf{M}}_{1,2}(\pi_\Lambda, L)\right]^{\mathrm{red}}$ is supported over a proper subset of $\mathcal{M}_\Lambda$. Since $K3$ surfaces are not elliptically connected[12], the support of the push-forward class

$$\epsilon_*^2\left[\overline{\mathsf{M}}_{1,2}(\pi_\Lambda, L)\right]^{\mathrm{red}} \;\in\; \mathsf{A}^1(\mathcal{X}_\Lambda^2, \mathbb{Q})$$

has codimension at least 2. Hence, the push-forward class vanishes.

[11]Since L is a multiple of the quasi-polarization, $\langle L, L\rangle_\Lambda > 0$.

[12]A nonsingular projective variety Y is said to be *elliptically connected* if there is a genus 1 curve passing through two general points of Y. In dimension ≥ 2, elliptically connected varieties are uniruled, see [12, Proposition 6.1].

Finally, for $L \in \Lambda$ admissible and $\langle L, L\rangle_\Lambda < 0$, the reduced class $\left[\overline{\mathsf{M}}_{1,2}(\pi_\Lambda, L)\right]^{\mathrm{red}}$ is fiberwise supported on the products of finitely many curves in the $K3$ surface.[13] This implies the support of the push-forward class $\epsilon_*^2\left[\overline{\mathsf{M}}_{1,2}(\pi_\Lambda, L)\right]^{\mathrm{red}}$ has codimension 2 in $\mathcal{X}_\Lambda^2$. Hence, the push-forward class vanishes. □

4. Exportation of the WDVV relation

4.1. **Exportation.** Let $L \in \Lambda$ be an admissible class. Consider the morphisms

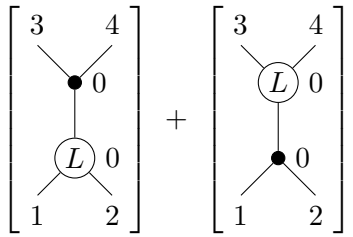
$$\overline{\mathsf{M}}_{0,4} \xleftarrow{\tau} \overline{\mathsf{M}}_{0,4}(\pi_\Lambda, L) \xrightarrow{\epsilon^4} \mathcal{X}_\Lambda^4 .$$

Following the notation of Section 0.4, we export here the WDVV relation with respect to the curve class L,

$$\epsilon_*^4 \tau^*(\mathsf{WDVV}) = 0 \in \mathsf{A}^5(\mathcal{X}_\Lambda^4, \mathbb{Q}) . \tag{16}$$

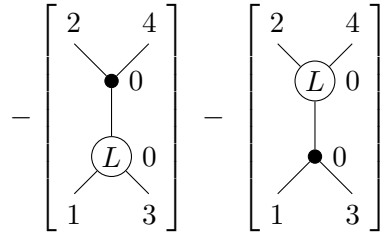
We will compute $\epsilon_*^4\tau^*(\mathsf{WDVV})$ by applying the splitting axiom of Gromov-Witten theory to the two terms of the WDVV relation (2). The splitting axiom requires a distribution of the curve class to each vertex of each graph appearing in (2).

4.2. **WDVV relation: unsplit contributions.** The unsplit contributions are obtained from curve class distributions which do *not* split L. The first unsplit contributions come from the first graph of (2):



$$N_0(L) \cdot \Big(\mathcal{L}_{(1)}\mathcal{L}_{(2)}\mathcal{L}_{(3)}\Delta_{(34)} + \mathcal{L}_{(1)}\mathcal{L}_{(3)}\mathcal{L}_{(4)}\Delta_{(12)} \Big) .$$

The unsplit contributions from the second graph of (2) are:



[13]The proof exactly follows the argument of Proposition 3. We find a (possibly non-algebraic) 1-parameter family of $K3$ surfaces for which the class L is generically a multiple of a (-2)-curve. The open moduli space of stable maps to the $K3$ fibers which are not supported on the family of (-2)-curves (and its limit curve in the special fiber) is constrained to lie over the special point in the base of the family. The specialization argument of Proposition 3 then shows the virtual class is 0 when restricted to the open moduli space of stable maps to the special fiber which are not supported on the limit curve.

$$-N_0(L)\cdot\Big(\mathcal{L}_{(1)}\mathcal{L}_{(2)}\mathcal{L}_{(3)}\Delta_{(24)}+\mathcal{L}_{(1)}\mathcal{L}_{(2)}\mathcal{L}_{(4)}\Delta_{(13)}\Big)\,.$$

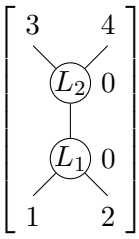
The curve class 0 vertex is not reduced and yields the usual intersection form (which explains the presence of diagonal $\Delta_{(ij)}$). The curve class L vertex is reduced. We have applied Proposition 4 to compute the push-forward to $\mathcal{X}^4_\Lambda$. All terms are of relative codimension 5 (codimension 1 each for the factors $\mathcal{L}_{(i)}$ and codimension 2 for the diagonal $\Delta_{(ij)}$). The four unsplit terms (divided by $N_0(L)$) exactly constitute the principal part of Theorem 2.

4.3. **WDVV relation: split contributions.** The split contributions are obtained from non-trivial curve class distributions to the vertices

$$L=L_1+L_2\,,\quad L_1\,,\,L_2\neq 0\,.$$

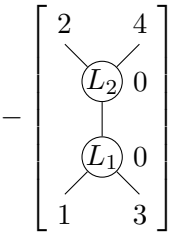
By Proposition 4, we need only consider distributions where *both* L_1 and L_2 are admissible classes. Let $\widetilde{\Lambda}$ be the saturation[14] of the span of L_1, L_2, and Λ. There are two types.

• If $\mathrm{rank}(\widetilde{\Lambda})=\mathrm{rank}(\Lambda)+1$, the split contributions are pushed forward from $\mathcal{X}^4_{\widetilde{\Lambda}}$ via the map $\mathcal{X}^4_{\widetilde{\Lambda}}\to\mathcal{X}^4_\Lambda$. Both vertices carry the reduced class by the obstruction calculation of [18, Lemma 1]. The split contributions are:



$$N_0(L_1)N_0(L_2)\langle L_1,L_2\rangle_{\widetilde{\Lambda}}\cdot\mathcal{L}_{1,(1)}\mathcal{L}_{1,(2)}\mathcal{L}_{2,(3)}\mathcal{L}_{2,(4)}\,,$$

$$-$$

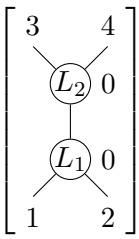


$$-N_0(L_1)N_0(L_2)\langle L_1,L_2\rangle_{\widetilde{\Lambda}}\cdot\mathcal{L}_{1,(1)}\mathcal{L}_{1,(3)}\mathcal{L}_{2,(2)}\mathcal{L}_{2,(4)}\,.$$

All terms are of relative codimension 5 (codimension 1 for the Noether-Lefschetz condition and codimension 1 each for the factors $\mathcal{L}_{a,(i)}$).

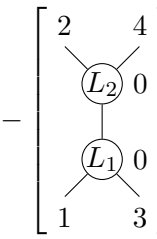
[14]We work only with primitive sublattices of $U^3\oplus E_8(-1)^2$.

• If $\widetilde{\Lambda} = \Lambda$, there is no obstruction cancellation as above. The extra reduction yields a factor of $-\lambda$. The split contributions are:



$$N_0(L_1)N_0(L_2)\langle L_1, L_2\rangle_{\widetilde{\Lambda}} \cdot (-\lambda)\mathcal{L}_{1,(1)}\mathcal{L}_{1,(2)}\mathcal{L}_{2,(3)}\mathcal{L}_{2,(4)}\,,$$

$$-$$



$$-N_0(L_1)N_0(L_2)\langle L_1, L_2\rangle_{\widetilde{\Lambda}} \cdot (-\lambda)\mathcal{L}_{1,(1)}\mathcal{L}_{1,(3)}\mathcal{L}_{2,(2)}\mathcal{L}_{2,(4)}\,.$$

All terms are of relative codimension 5 (codimension 1 for $-\lambda$ and codimension 1 each for the factors $\mathcal{L}_{a,(i)}$).

4.4. **Proof of Theorem 2.** The complete exported relation (16) is obtained by adding the unsplit contributions to the summation over all split contributions

$$L = L_1 + L_2$$

of both types. Split contributions of the first type are explicitly supported over the Noether-Lefschetz locus corresponding to

$$\widetilde{\Lambda} \subset U^3 \oplus E_8^2\,.$$

Split contributions of the second type all contain the factor $-\lambda$. The class λ is known to be a linear combination of proper Noether-Lefschetz divisors of $\mathcal{M}_\Lambda$ by [6, Theorem 1.2]. Hence, we view the split contributions of the second type also as being supported over Noether-Lefschetz loci. For the formula of Theorem 2, we normalize the relation by dividing by $N_0(L)$. □

5. Proof of Theorem 5

5.1. **Overview.** Let $L \in \Lambda$ be an admissible class, and let $\overline{\mathsf{M}}_{0,0}(\pi_\Lambda, L)$ be the π_Λ-relative moduli space of genus 0 stable maps,

$$\phi : \overline{\mathsf{M}}_{0,0}(\pi_\Lambda, L) \to \mathcal{M}_\Lambda\,.$$

Let $\mathcal{X}_{\overline{\mathsf{M}}}$ be the universal Λ-polarized $K3$ surface over $\overline{\mathsf{M}}_{0,0}(\pi_\Lambda, L)$,

$$\pi_{\overline{\mathsf{M}}} : \mathcal{X}_{\overline{\mathsf{M}}} \to \overline{\mathsf{M}}_{0,0}(\pi_\Lambda, L)\,.$$

In Sections 2.1 and 2.2, we have constructed two divisor classes

$$\widehat{\mathcal{L}}\,,\ \mathcal{L}\ \in\ \mathsf{A}^1(\mathcal{X}_{\overline{\mathsf{M}}}, \mathbb{Q})\,.$$

We define the κ classes with respect to $\widehat{\mathcal{L}}$ by

$$\widehat{\kappa}_{[L^a;b]}\ =\ \pi_{\overline{\mathsf{M}}*}\left(\widehat{\mathcal{L}}^a \cdot c_2(\mathcal{T}_{\pi_{\overline{\mathsf{M}}}})^b\right)\ \in\ \mathsf{A}^{a+2b-2}\left(\overline{\mathsf{M}}_{0,0}(\pi_\Lambda, L), \mathbb{Q}\right)\,.$$

Since $\widehat{\mathcal{L}}$ and $\mathcal{L}$ are equal on the fibers of $\pi_{\overline{\mathsf{M}}}$, the difference $\widehat{\mathcal{L}} - \mathcal{L}$ is the pull-back[15] of a divisor class in $\mathsf{A}^1\left(\overline{\mathsf{M}}_{0,0}(\pi_\Lambda, L), \mathbb{Q}\right)$. In fact, the difference is equal[16] to

$$\frac{1}{24}\ \cdot\ \left(\widehat{\kappa}_{[L;1]} - \kappa_{[L;1]}\right)\ \in\ \mathsf{A}^1\left(\overline{\mathsf{M}}_{0,0}(\pi_\Lambda, L), \mathbb{Q}\right)\,.$$

Therefore,

$$\widehat{\mathcal{L}} - \frac{1}{24}\ \cdot\ \widehat{\kappa}_{[L;1]}\ =\ \mathcal{L} - \frac{1}{24}\ \cdot\ \kappa_{[L;1]}\ \in\ \mathsf{A}^1(\mathcal{X}_{\overline{\mathsf{M}}}, \mathbb{Q})\,. \tag{17}$$

Our strategy for proving Theorem 5 is to export the WDVV relation via the morphisms

$$\overline{\mathsf{M}}_{0,4}\ \xleftarrow{\tau}\ \overline{\mathsf{M}}_{0,4}(\pi_\Lambda, L)\ \xrightarrow{\epsilon^4_{\overline{\mathsf{M}}}}\ \mathcal{X}^4_{\overline{\mathsf{M}}}\,.$$

We deduce the following identity from the exported relation

$$\epsilon^4_{\overline{\mathsf{M}}*}\tau^*(\mathsf{WDVV})\ =\ 0\ \in\ \mathsf{A}_{\mathsf{d}(\Lambda)+3}(\mathcal{X}^4_{\overline{\mathsf{M}}}, \mathbb{Q})\,, \tag{18}$$

where $\mathsf{d}(\Lambda) = 20 - \mathrm{rank}(\Lambda)$ is the dimension of $\mathcal{M}_\Lambda$.

Proposition 7. *For $L \in \Lambda$ admissible,*

$$\widehat{\kappa}_{[L;1]} \cap \left[\overline{\mathsf{M}}_{0,0}(\pi_\Lambda, L)\right]^{\mathrm{red}}\ =\ \kappa_{[L;1]} \cap \left[\overline{\mathsf{M}}_{0,0}(\pi_\Lambda, L)\right]^{\mathrm{red}}\ \in\ \mathsf{A}_{\mathsf{d}(\Lambda)-1}\left(\overline{\mathsf{M}}_{0,0}(\pi_\Lambda, L), \mathbb{Q}\right)\,.$$

Equation (17) and Proposition 7 together yield

$$\widehat{\mathcal{L}} \cap [\mathcal{X}_{\overline{\mathsf{M}}}]^{\mathrm{red}}\ =\ \mathcal{L} \cap [\mathcal{X}_{\overline{\mathsf{M}}}]^{\mathrm{red}}\ \in\ \mathsf{A}_{\mathsf{d}(\Lambda)+1}(\mathcal{X}_{\overline{\mathsf{M}}}, \mathbb{Q})\,,$$

thus proving Theorem 5.

The exportation process is almost identical to the one in Section 4. However, since we work over $\overline{\mathsf{M}}_{0,0}(\pi_\Lambda, L)$ instead of $\mathcal{M}_\Lambda$, we do *not* require Proposition 4 (whose proof uses Theorem 5).

[15] We use here the vanishing $H^1(X, \mathcal{O}_X) = 0$ for $K3$ surfaces X and the base change theorem.

[16] We keep the same notation for the pull-backs of the κ classes via the structure map ϕ. Also, we identify $\mathsf{A}^\star\left(\overline{\mathsf{M}}_{0,0}(\pi_\Lambda, L), \mathbb{Q}\right)$ as a subring of $\mathsf{A}^\star(\mathcal{X}^n_{\overline{\mathsf{M}}}, \mathbb{Q})$ via $\pi^{n*}_{\overline{\mathsf{M}}}$.

5.2. **Exportation.** We briefly describe the exportation (18) of the WDVV relation with respect to the curve class L. As in Section 4, the outcome of $\epsilon^4_{\overline{\mathsf{M}}*}\tau^*(\mathsf{WDVV})$ consists of unsplit and split contributions:

- For the unsplit contributions, the difference is that one should replace $\mathcal{L}$ by the corresponding $\widehat{\mathcal{L}}$. Moreover, since we do *not* push-forward to $\mathcal{X}^4_\Lambda$, there is no overall coefficient $N_0(L)$.

- For the split contributions corresponding to the admissible curve class distributions

$$L = L_1 + L_2\,,$$

one again replaces $\mathcal{L}_i$ by the corresponding $\widehat{\mathcal{L}}_i$ and removes the coefficient $N_0(L_i)$. As before, the terms are either supported over proper Noether-Lefschetz divisors of $\mathcal{M}_\Lambda$, or multiplied by (the pull-back of) $-\lambda$.

We obtain the following analog of Theorem 2.

Proposition 8. *For admissible $L \in \Lambda$, exportation of the WDVV relation yields*

$$\begin{aligned}(19)\quad \Big(\widehat{\mathcal{L}}_{(1)}\widehat{\mathcal{L}}_{(2)}\widehat{\mathcal{L}}_{(3)}\Delta_{(34)} + \widehat{\mathcal{L}}_{(1)}\widehat{\mathcal{L}}_{(3)}\widehat{\mathcal{L}}_{(4)}\Delta_{(12)} - \widehat{\mathcal{L}}_{(1)}\widehat{\mathcal{L}}_{(2)}\widehat{\mathcal{L}}_{(3)}\Delta_{(24)} \\ - \widehat{\mathcal{L}}_{(1)}\widehat{\mathcal{L}}_{(2)}\widehat{\mathcal{L}}_{(4)}\Delta_{(13)} + \ldots\Big) \cap [\mathcal{X}^4_{\overline{\mathsf{M}}}]^{\mathrm{red}} \;=\; 0 \;\in\; \mathsf{A}_{\mathsf{d}(\Lambda)+3}(\mathcal{X}^4_{\overline{\mathsf{M}}},\mathbb{Q})\,,\end{aligned}$$

where the dots stand for (Gromov-Witten) tautological classes supported over proper Noether-Lefschetz divisors of $\mathcal{M}_\Lambda$.

Here, the Gromov-Witten tautological classes on $\mathcal{X}^n_{\overline{\mathsf{M}}}$ are defined by replacing $\mathcal{L}$ by $\widehat{\mathcal{L}}$ in Section 0.3.

5.3. **Proof of Proposition 7.** We distinguish two cases.

Case $\langle L, L\rangle_\Lambda \neq 0$.

First, we rewrite (17) as

$$\widehat{\kappa}_{[L;1]} - \kappa_{[L;1]} \;=\; 24\cdot(\widehat{\mathcal{L}} - \mathcal{L}) \;\in\; \mathsf{A}^1(\mathcal{X}_{\overline{\mathsf{M}}},\mathbb{Q})\,.$$

By the same argument, we also have

$$\widehat{\kappa}_{[L^3;0]} - \kappa_{[L^3;0]} \;=\; 3\langle L, L\rangle_\Lambda\cdot(\widehat{\mathcal{L}} - \mathcal{L}) \;\in\; \mathsf{A}^1(\mathcal{X}_{\overline{\mathsf{M}}},\mathbb{Q})\,.$$

By combining the above equations, we find

$$(20)\quad \langle L, L\rangle_\Lambda\cdot\widehat{\kappa}_{[L;1]} - 8\cdot\widehat{\kappa}_{[L^3;0]} \;=\; \langle L, L\rangle_\Lambda\cdot\kappa_{[L;1]} - 8\cdot\kappa_{[L^3;0]} \;\in\; \mathsf{A}^1\left(\overline{\mathsf{M}}_{0,0}(\pi_\Lambda, L),\mathbb{Q}\right)\,.$$

Next, we apply (19) with respect to L and insert $\Delta_{(12)}\Delta_{(34)} \in \mathsf{A}^4(\mathcal{X}^4_{\overline{\mathsf{M}}},\mathbb{Q})$. The relation

$$\Delta_{(12)}\Delta_{(34)} \cap \epsilon^4_{\overline{\mathsf{M}}*}\tau^*(\mathsf{WDVV}) \;=\; 0 \;\in\; \mathsf{A}_{\mathsf{d}(\Lambda)-1}(\mathcal{X}^4_{\overline{\mathsf{M}}},\mathbb{Q})$$

pushes down via

$$\pi^4_{\overline{\mathsf{M}}} : \mathcal{X}^4_{\overline{\mathsf{M}}} \to \overline{\mathsf{M}}_{0,0}(\pi_\Lambda, L)$$

to yield the result

$$\Big(2\langle L, L\rangle_\Lambda \cdot \widehat{\kappa}_{[L;1]} - 2\cdot \widehat{\kappa}_{[L^3;0]}\Big) \cap \big[\overline{\mathsf{M}}_{0,0}(\pi_\Lambda, L)\big]^{\mathrm{red}} \in \phi^* \mathsf{NL}^1(\mathcal{M}_\Lambda, \mathbb{Q}) \cap \big[\overline{\mathsf{M}}_{0,0}(\pi_\Lambda, L)\big]^{\mathrm{red}} . \tag{21}$$

Since $\langle L, L\rangle_\Lambda \neq 0$, a combination of (20) and (21) yields

$$\widehat{\kappa}_{[L;1]} \cap \big[\overline{\mathsf{M}}_{0,0}(\pi_\Lambda, L)\big]^{\mathrm{red}} \in \phi^* \mathsf{A}^1(\mathcal{M}_\Lambda, \mathbb{Q}) \cap \big[\overline{\mathsf{M}}_{0,0}(\pi_\Lambda, L)\big]^{\mathrm{red}} .$$

In other words, there is a divisor class $D \in \mathsf{A}^1(\mathcal{M}_\Lambda, \mathbb{Q})$ for which

$$\widehat{\kappa}_{[L;1]} \cap \big[\overline{\mathsf{M}}_{0,0}(\pi_\Lambda, L)\big]^{\mathrm{red}} = \phi^*(D) \cap \big[\overline{\mathsf{M}}_{0,0}(\pi_\Lambda, L)\big]^{\mathrm{red}} \in \mathsf{A}_{\mathsf{d}(\Lambda)-1}\big(\overline{\mathsf{M}}_{0,0}(\pi_\Lambda, L), \mathbb{Q}\big) .$$

Then, by the projection formula, we find

$$\phi_*\Big(\widehat{\kappa}_{[L;1]} \cap \big[\overline{\mathsf{M}}_{0,0}(\pi_\Lambda, L)\big]^{\mathrm{red}}\Big) = N_0(L)\cdot \kappa_{[L;1]} = N_0(L)\cdot D \in \mathsf{A}^1(\mathcal{M}_\Lambda, \mathbb{Q}) .$$

Hence $D = \kappa_{[L;1]}$, which proves Proposition 7 in case $\langle L, L\rangle_\Lambda \neq 0$.

Case $\langle L, L\rangle_\Lambda = 0$.

Let $H \in \Lambda$ be the quasi-polarization and let

$$\mathcal{H} \in \mathsf{A}^1(\mathcal{X}_{\overline{\mathsf{M}}}, \mathbb{Q})$$

be the pull-back of the class $\mathcal{H} \in \mathsf{A}^1(\mathcal{X}_\Lambda, \mathbb{Q})$. We define the κ classes

$$\widehat{\kappa}_{[H^{a_1}, L^{a_2}; b]} = \pi_{\overline{\mathsf{M}}*}\Big(\mathcal{H}^{a_1}\cdot \widehat{\mathcal{L}}^{a_2}\cdot c_2(\mathcal{T}_{\pi_{\overline{\mathsf{M}}}})^b\Big) \in \mathsf{A}^{a_1+a_2+2b-2}\big(\overline{\mathsf{M}}_{0,0}(\pi_\Lambda, L), \mathbb{Q}\big) .$$

First, by the same argument used to prove (17), we have

$$\widehat{\kappa}_{[H,L^2;0]} - \kappa_{[H,L^2;0]} = 2\langle H, L\rangle_\Lambda \cdot (\widehat{\mathcal{L}} - \mathcal{L}) \in \mathsf{A}^1(\mathcal{X}_{\overline{\mathsf{M}}}, \mathbb{Q}) .$$

By combining the above equation with (17), we find

$$\langle H, L\rangle_\Lambda \cdot \widehat{\kappa}_{[L;1]} - 12\cdot \widehat{\kappa}_{[H,L^2;0]} = \langle H, L\rangle_\Lambda \cdot \kappa_{[L;1]} - 12\cdot \kappa_{[H,L^2;0]} \in \mathsf{A}^1\big(\overline{\mathsf{M}}_{0,0}(\pi_\Lambda, L), \mathbb{Q}\big) . \tag{22}$$

Next, we apply (19) with respect to L and insert $\mathcal{H}_{(1)}\mathcal{H}_{(2)}\Delta_{(34)} \in \mathsf{A}^4(\mathcal{X}^4_{\overline{\mathsf{M}}}, \mathbb{Q})$. The relation

$$\mathcal{H}_{(1)}\mathcal{H}_{(2)}\Delta_{(34)} \cap \epsilon^4_{\overline{\mathsf{M}}*}\tau^*(\mathsf{WDVV}) = 0 \in \mathsf{A}_{\mathsf{d}(\Lambda)-1}(\mathcal{X}^4_{\overline{\mathsf{M}}}, \mathbb{Q})$$

pushes down via $\pi^4_{\overline{\mathsf{M}}}$ to yield the result

$$\Big(\langle H, L\rangle^2_\Lambda \cdot \widehat{\kappa}_{[L;1]} - 2\langle H, L\rangle_\Lambda \cdot \widehat{\kappa}_{[H,L^2;0]}\Big) \cap \big[\overline{\mathsf{M}}_{0,0}(\pi_\Lambda, L)\big]^{\mathrm{red}} \in \phi^* \mathsf{NL}^1(\mathcal{M}_\Lambda, \mathbb{Q}) \cap \big[\overline{\mathsf{M}}_{0,0}(\pi_\Lambda, L)\big]^{\mathrm{red}} . \tag{23}$$

Since $\langle H, L\rangle_\Lambda \neq 0$ by the Hodge index theorem, a combination of (22) and (23) yields

$$\widehat{\kappa}_{[L;1]} \cap \left[\overline{\mathsf{M}}_{0,0}(\pi_\Lambda, L)\right]^{\mathrm{red}} \ \in \ \phi^* \, \mathsf{A}^1(\mathcal{M}_\Lambda, \mathbb{Q}) \cap \left[\overline{\mathsf{M}}_{0,0}(\pi_\Lambda, L)\right]^{\mathrm{red}} .$$

As in the previous case, we conclude

$$\widehat{\kappa}_{[L;1]} \cap \left[\overline{\mathsf{M}}_{0,0}(\pi_\Lambda, L)\right]^{\mathrm{red}} \ = \ \kappa_{[L;1]} \cap \left[\overline{\mathsf{M}}_{0,0}(\pi_\Lambda, L)\right]^{\mathrm{red}} \ \in \ \mathsf{A}_{\mathsf{d}(\Lambda)-1}\left(\overline{\mathsf{M}}_{0,0}(\pi_\Lambda, L), \mathbb{Q}\right) .$$

The proof of Proposition 7 (and thus Theorem 5) is complete. □

6. Exportation of Getzler's relation

6.1. **Exportation.** Let $L \in \Lambda$ be an admissible class satisfying $\langle L, L\rangle_\Lambda \geq 0$. Consider the morphisms

$$\overline{\mathsf{M}}_{1,4} \ \xleftarrow{\tau} \ \overline{\mathsf{M}}_{1,4}(\pi_\Lambda, L) \ \xrightarrow{\epsilon^4} \ \mathcal{X}^4_\Lambda .$$

Following the notation of Section 0.4, we export here Getzler's relation with respect to the curve class L,

$$\epsilon^4_* \tau^*(\mathsf{Getzler}) \ = \ 0 \ \in \ \mathsf{A}^5(\mathcal{X}^4_\Lambda, \mathbb{Q}) . \tag{24}$$

We will compute $\epsilon^4_* \tau^*(\mathsf{Getzler})$ by applying the splitting axiom of Gromov-Witten theory to the 7 terms of Getzler's relation (3). The splitting axiom requires a distribution of the curve class to each vertex of each graph appearing in (3).

6.2. **Curve class distributions.** To export Getzler's relation with respect to the curve class L, we will use the following properties for the graphs which arise:

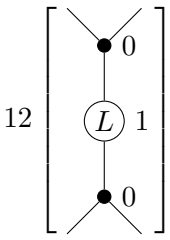
(i) Only distributions of admissible classes contribute.
(ii) A genus 1 vertex with valence[17] 2 or a genus 0 vertex with valence at least 4 must carry a nonzero class.
(iii) A genus 1 vertex with valence 1 cannot be adjacent to a genus 0 vertex with a nonzero class.
(iv) A genus 1 vertex with valence 2 cannot be adjacent to two genus 0 vertices with nonzero classes.

Property (i) is a consequence of Propositions 4, 5, and 6. For Property (ii), the moduli of contracted 2-pointed genus 1 curve produces a positive dimensional fiber of the push-forward to $\mathcal{X}^4_\Lambda$ (and similarly for contracted 4-point genus 0 curves). Properties (iii) and (iv) are consequences of positive dimensional fibers of the push-forward to $\mathcal{X}^4_\Lambda$ obtained from the elliptic component. We leave the elementary details to the reader.

[17] The valence counts all incident half-edges (both from edges and markings).

6.3. **Getzler's relation: unsplit contributions.** We begin with the unsplit contributions. The strata appearing in Getzler's relation are ordered as in (3).

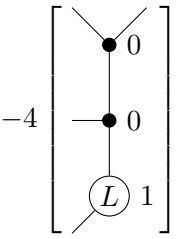
Stratum 1.



$$\begin{aligned}12N_1(L)\cdot\Big(&\mathcal{L}_{(1)}\Delta_{(12)}\Delta_{(34)}+\mathcal{L}_{(3)}\Delta_{(12)}\Delta_{(34)}+\mathcal{L}_{(1)}\Delta_{(13)}\Delta_{(24)}\\&+\mathcal{L}_{(2)}\Delta_{(13)}\Delta_{(24)}+\mathcal{L}_{(1)}\Delta_{(14)}\Delta_{(23)}+\mathcal{L}_{(2)}\Delta_{(14)}\Delta_{(23)}\Big)\\&+12N_1(L)\cdot Z(L)\Big(\Delta_{(12)}\Delta_{(34)}+\Delta_{(13)}\Delta_{(24)}+\Delta_{(14)}\Delta_{(23)}\Big)\end{aligned}$$

By Property (ii), the genus 1 vertex must carry the curve class L in the unsplit case. The contribution is then calculated using Propositions 4 and 6.

Stratum 2.

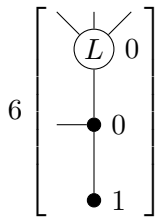


$$\begin{aligned}-12N_1(L)\cdot\Big(&\mathcal{L}_{(1)}\Delta_{(234)}+\mathcal{L}_{(2)}\Delta_{(134)}+\mathcal{L}_{(3)}\Delta_{(124)}+\mathcal{L}_{(4)}\Delta_{(123)}\\&+\mathcal{L}_{(1)}\Delta_{(123)}+\mathcal{L}_{(1)}\Delta_{(124)}+\mathcal{L}_{(1)}\Delta_{(134)}+\mathcal{L}_{(2)}\Delta_{(234)}\Big)\\&-12N_1(L)\cdot Z(L)\Big(\Delta_{(123)}+\Delta_{(124)}+\Delta_{(134)}+\Delta_{(234)}\Big)\end{aligned}$$

Again by Property (ii), the genus 1 vertex must carry the curve class L in the unsplit case. The contribution is then calculated using Propositions 4 and 6.

Stratum 3. No contribution by Properties (ii) and (iii).

Stratum 4.



$$N_0(L) \cdot \lambda \mathcal{L}_{(1)} \mathcal{L}_{(2)} \mathcal{L}_{(3)} \mathcal{L}_{(4)}$$

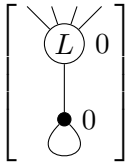
The genus 0 vertex of valence 4 must carry the curve class L in the unsplit case. The contracted genus 1 vertex contributes the virtual class

$$\epsilon_*[\overline{\mathsf{M}}_{1,1}(\pi_\Lambda, 0)]^{\mathrm{vir}} = \frac{1}{24} \cdot \lambda \in \mathsf{A}^1(\mathcal{X}^1_\Lambda, \mathbb{Q}) \,. \tag{25}$$

The coefficient 6 together with the 4 graphs which occur cancel the 24 in the denominator of (25). Proposition 4 is then applied to the genus 0 vertex of valence 4.

Stratum 5. No contribution by Property (ii) since there are two genus 0 vertices of valence 4.

Stratum 6.



$$\frac{1}{2} N_0(L) \cdot \kappa_{[L;1]} \mathcal{L}_{(1)} \mathcal{L}_{(2)} \mathcal{L}_{(3)} \mathcal{L}_{(4)}$$

The genus 0 vertex of valence 4 must carry the curve class L in the unsplit case. Proposition 4 is applied to the genus 0 vertex of valence 4. The self-edge of the contracted genus 0 vertex yields a factor of $c_2(\mathcal{T}_{\pi_\Lambda})$. The contribution of the contracted genus 0 vertex is

$$\frac{1}{2} \cdot \kappa_{[L;1]}$$

where the factor of $\frac{1}{2}$ is included since the self-edge is not oriented.

Stratum 7. No contribution by Property (ii) since there are two genus 0 vertices of valence 4.

We have already seen that λ is expressible in term of the Noether-Lefschetz divisors of $\mathcal{M}_\Lambda$. Since we will later express $Z(L)$ and $\kappa_{[L;1]}$ in terms of the Noether-Lefschetz divisors of $\mathcal{M}_\Lambda$, the principal terms in the above analysis only occur in Strata 1 and 2.

The principal parts of Strata 1 and 2 (divided[18] by $12N_1(L)$) exactly constitute the principal part of Theorem 3.

6.4. **Getzler's relation: split contributions.** The split contributions are obtained from non-trivial curve class distributions to the vertices. By Property (i), we need only consider distributions of admissible classes.

Case A. The class L is divided into two nonzero parts

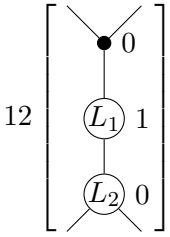
$$L = L_1 + L_2\,.$$

Let $\widetilde{\Lambda}$ be the saturation of the span of L_1, L_2, and Λ.

- If $\mathrm{rank}(\widetilde{\Lambda}) = \mathrm{rank}(\Lambda) + 1$, the contributions are pushed forward from $\mathcal{X}^4_{\widetilde{\Lambda}}$ via the map $\mathcal{X}^4_{\widetilde{\Lambda}} \to \mathcal{X}^4_{\Lambda}$.
- If $\widetilde{\Lambda} = \Lambda$, the contributions are multiplied by $-\lambda$.

With the above rules, the formulas below address both the $\mathrm{rank}(\widetilde{\Lambda}) = \mathrm{rank}(\Lambda) + 1$ and the $\mathrm{rank}(\widetilde{\Lambda}) = \mathrm{rank}(\Lambda)$ cases simultaneously.

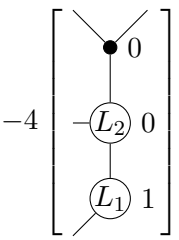
Stratum 1.



$$12N_1(L_1)N_0(L_2)\langle L_1, L_2\rangle_{\widetilde{\Lambda}} \cdot \Big(\mathcal{L}_{2,(1)}\mathcal{L}_{2,(2)}\Delta_{(34)} + \mathcal{L}_{2,(3)}\mathcal{L}_{2,(4)}\Delta_{(12)} + \mathcal{L}_{2,(1)}\mathcal{L}_{2,(3)}\Delta_{(24)} + \mathcal{L}_{2,(2)}\mathcal{L}_{2,(4)}\Delta_{(13)} + \mathcal{L}_{2,(1)}\mathcal{L}_{2,(4)}\Delta_{(23)} + \mathcal{L}_{2,(2)}\mathcal{L}_{2,(3)}\Delta_{(14)}\Big)$$

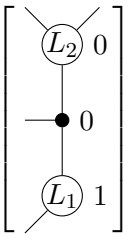
By Property (ii), the genus 1 vertex must carry a nonzero curve class. The contribution is calculated using Propositions 4 and 6.

Stratum 2.



[18]The admissibility of L together with condition $\langle L, L\rangle_\Lambda \geq 0$ implies $N_1(L) \neq 0$ by Proposition 2.

$$-4N_1(L_1)N_0(L_2)\langle L_1, L_2\rangle_{\widetilde{\Lambda}} \cdot \Big(\mathcal{L}_{2,(1)}\mathcal{L}_{2,(2)}\Delta_{(23)} + \mathcal{L}_{2,(1)}\mathcal{L}_{2,(2)}\Delta_{(24)} + \mathcal{L}_{2,(1)}\mathcal{L}_{2,(3)}\Delta_{(34)} + \mathcal{L}_{2,(1)}\mathcal{L}_{2,(2)}\Delta_{(13)} + \mathcal{L}_{2,(1)}\mathcal{L}_{2,(2)}\Delta_{(14)} + \mathcal{L}_{2,(2)}\mathcal{L}_{2,(3)}\Delta_{(34)} + \mathcal{L}_{2,(1)}\mathcal{L}_{2,(3)}\Delta_{(12)} + \mathcal{L}_{2,(1)}\mathcal{L}_{2,(3)}\Delta_{(14)} + \mathcal{L}_{2,(2)}\mathcal{L}_{2,(3)}\Delta_{(24)} + \mathcal{L}_{2,(1)}\mathcal{L}_{2,(4)}\Delta_{(12)} + \mathcal{L}_{2,(1)}\mathcal{L}_{2,(4)}\Delta_{(13)} + \mathcal{L}_{2,(2)}\mathcal{L}_{2,(4)}\Delta_{(23)}\Big)$$

-4 

$$-12N_1(L_1)N_0(L_2) \cdot \Big(\mathcal{L}_{1,(1)}\mathcal{L}_{2,(2)}\mathcal{L}_{2,(3)}\mathcal{L}_{2,(4)} + \mathcal{L}_{1,(2)}\mathcal{L}_{2,(1)}\mathcal{L}_{2,(3)}\mathcal{L}_{2,(4)} + \mathcal{L}_{1,(3)}\mathcal{L}_{2,(1)}\mathcal{L}_{2,(2)}\mathcal{L}_{2,(4)} + \mathcal{L}_{1,(4)}\mathcal{L}_{2,(1)}\mathcal{L}_{2,(2)}\mathcal{L}_{2,(3)}\Big)$$

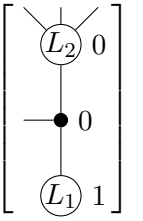
$$-4N_1(L_1)N_0(L_2)\cdot\Big(\mathcal{L}_{1,(1)}\mathcal{L}_{2,(1)}\mathcal{L}_{2,(2)}\mathcal{L}_{2,(3)}+\mathcal{L}_{1,(1)}\mathcal{L}_{2,(1)}\mathcal{L}_{2,(2)}\mathcal{L}_{2,(4)}+\mathcal{L}_{1,(1)}\mathcal{L}_{2,(1)}\mathcal{L}_{2,(3)}\mathcal{L}_{2,(4)} + \mathcal{L}_{1,(2)}\mathcal{L}_{2,(1)}\mathcal{L}_{2,(2)}\mathcal{L}_{2,(3)} + \mathcal{L}_{1,(2)}\mathcal{L}_{2,(1)}\mathcal{L}_{2,(2)}\mathcal{L}_{2,(4)} + \mathcal{L}_{1,(2)}\mathcal{L}_{2,(2)}\mathcal{L}_{2,(3)}\mathcal{L}_{2,(4)} + \mathcal{L}_{1,(3)}\mathcal{L}_{2,(1)}\mathcal{L}_{2,(2)}\mathcal{L}_{2,(3)} + \mathcal{L}_{1,(3)}\mathcal{L}_{2,(1)}\mathcal{L}_{2,(3)}\mathcal{L}_{2,(4)} + \mathcal{L}_{1,(3)}\mathcal{L}_{2,(2)}\mathcal{L}_{2,(3)}\mathcal{L}_{2,(4)} + \mathcal{L}_{1,(4)}\mathcal{L}_{2,(1)}\mathcal{L}_{2,(2)}\mathcal{L}_{2,(3)} + \mathcal{L}_{1,(4)}\mathcal{L}_{2,(1)}\mathcal{L}_{2,(3)}\mathcal{L}_{2,(4)} + \mathcal{L}_{1,(4)}\mathcal{L}_{2,(2)}\mathcal{L}_{2,(3)}\mathcal{L}_{2,(4)}\Big)$$

$$- 12N_1(L_1)N_0(L_2) \cdot Z(L_1)\Big(\mathcal{L}_{2,(1)}\mathcal{L}_{2,(2)}\mathcal{L}_{2,(3)} + \mathcal{L}_{2,(1)}\mathcal{L}_{2,(2)}\mathcal{L}_{2,(4)} + \mathcal{L}_{2,(1)}\mathcal{L}_{2,(3)}\mathcal{L}_{2,(4)} + \mathcal{L}_{2,(2)}\mathcal{L}_{2,(3)}\mathcal{L}_{2,(4)}\Big)$$

By Property (ii), the genus 1 vertex must carry a nonzero curve class. There are two possibilities for the distribution. Both contributions are calculated using Propositions 4 and 6.

Stratum 3. No contribution by Properties (ii) and (iii).

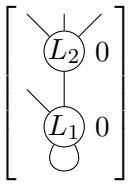
Stratum 4.

6 

$$24N_1(L_1)N_0(L_2) \cdot \mathcal{L}_{2,(1)}\mathcal{L}_{2,(2)}\mathcal{L}_{2,(3)}\mathcal{L}_{2,(4)}$$

By Property (iii), the genus 0 vertex in the middle can not carry a nonzero curve class. The contribution is calculated using Propositions 4 and 5.

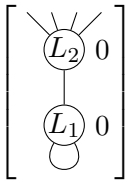
Stratum 5.



$$\frac{1}{2}N_0(L_1)N_0(L_2)\langle L_1,L_1\rangle_{\widetilde{\Lambda}}\langle L_1,L_2\rangle_{\widetilde{\Lambda}}\cdot\Big(\mathcal{L}_{1,(1)}\mathcal{L}_{2,(2)}\mathcal{L}_{2,(3)}\mathcal{L}_{2,(4)}+\mathcal{L}_{1,(2)}\mathcal{L}_{2,(1)}\mathcal{L}_{2,(3)}\mathcal{L}_{2,(4)}$$
$$+\mathcal{L}_{1,(3)}\mathcal{L}_{2,(1)}\mathcal{L}_{2,(2)}\mathcal{L}_{2,(4)}+\mathcal{L}_{1,(4)}\mathcal{L}_{2,(1)}\mathcal{L}_{2,(2)}\mathcal{L}_{2,(3)}\Big)$$

The factor $\frac{1}{2}\langle L_1,L_1\rangle_{\widetilde{\Lambda}}$ is obtained from the self-edge. The contribution is calculated using Proposition 4.

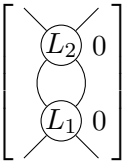
Stratum 6.



$$\frac{1}{2}N_0(L_1)N_0(L_2)\langle L_1,L_1\rangle_{\widetilde{\Lambda}}\langle L_1,L_2\rangle_{\widetilde{\Lambda}}\cdot\mathcal{L}_{2,(1)}\mathcal{L}_{2,(2)}\mathcal{L}_{2,(3)}\mathcal{L}_{2,(4)}$$

The factor $\frac{1}{2}\langle L_1,L_1\rangle_{\widetilde{\Lambda}}$ is obtained from the self-edge. The contribution is calculated using Proposition 4.

Stratum 7.

$$-2\left[\quad\right.$$


$$\left.\quad\right]$$

$$-N_0(L_1)N_0(L_2)\langle L_1,L_2\rangle^2_{\widetilde{\Lambda}}\cdot\Big(\mathcal{L}_{1,(1)}\mathcal{L}_{1,(2)}\mathcal{L}_{2,(3)}\mathcal{L}_{2,(4)}+\mathcal{L}_{2,(1)}\mathcal{L}_{2,(2)}\mathcal{L}_{1,(3)}\mathcal{L}_{1,(4)}$$
$$+\mathcal{L}_{1,(1)}\mathcal{L}_{1,(3)}\mathcal{L}_{2,(2)}\mathcal{L}_{2,(4)}+\mathcal{L}_{2,(1)}\mathcal{L}_{2,(3)}\mathcal{L}_{1,(2)}\mathcal{L}_{1,(4)}$$
$$+\mathcal{L}_{1,(1)}\mathcal{L}_{1,(4)}\mathcal{L}_{2,(2)}\mathcal{L}_{2,(3)}+\mathcal{L}_{2,(1)}\mathcal{L}_{2,(4)}\mathcal{L}_{1,(2)}\mathcal{L}_{1,(3)}\Big)$$

The factor $-2\left(\frac{1}{2}\langle L_1,L_2\rangle^2_{\widetilde{\Lambda}}\right)$ is obtained from two middle edges (the $\frac{1}{2}$ comes from the symmetry of the graph). The contribution is calculated using Proposition 4.

Case B. The class L is divided into three nonzero parts

$$L = L_1 + L_2 + L_3\,.$$

Let $\widetilde{\Lambda}$ be the saturation of the span of L_1, L_2, L_3, and Λ. By Properties (ii)-(iv), only Stratum 2 contributes.

- If $\mathrm{rank}(\widetilde{\Lambda}) = \mathrm{rank}(\Lambda) + 2$, the contributions are pushed forward from $\mathcal{X}^4_{\widetilde{\Lambda}}$ via the map $\mathcal{X}^4_{\widetilde{\Lambda}} \to \mathcal{X}^4_{\Lambda}$.
- If $\mathrm{rank}(\widetilde{\Lambda}) = \mathrm{rank}(\Lambda) + 1$, the contributions are pushed forward from $\mathcal{X}^4_{\widetilde{\Lambda}}$ via the map $\mathcal{X}^4_{\widetilde{\Lambda}} \to \mathcal{X}^4_{\Lambda}$ *and* multiplied by $-\lambda$.
- If $\widetilde{\Lambda} = \Lambda$, the contributions are multiplied by $(-\lambda)^2$.

With the above rules, the formula below addresses all three cases

$$\mathrm{rank}(\widetilde{\Lambda}) = \mathrm{rank}(\Lambda) + 2\,, \quad \mathrm{rank}(\widetilde{\Lambda}) = \mathrm{rank}(\Lambda) + 1\,, \quad \mathrm{rank}(\widetilde{\Lambda}) = \mathrm{rank}(\Lambda)$$

simultaneously.

Stratum 2.

$$-4 \left[\begin{array}{c} L_3 \;\; 0 \\ | \\ L_2 \;\; 0 \\ | \\ L_1 \;\; 1 \end{array} \right]$$

$$\begin{aligned} -4N_1(L_1)N_0(L_2)N_0(L_3)\langle L_1, L_2\rangle_{\widetilde{\Lambda}}\langle L_2, L_3\rangle_{\widetilde{\Lambda}} \cdot \Big(& \mathcal{L}_{2,(1)}\mathcal{L}_{3,(2)}\mathcal{L}_{3,(3)} + \mathcal{L}_{2,(1)}\mathcal{L}_{3,(2)}\mathcal{L}_{3,(4)} \\ & + \mathcal{L}_{2,(1)}\mathcal{L}_{3,(3)}\mathcal{L}_{3,(4)} + \mathcal{L}_{2,(2)}\mathcal{L}_{3,(1)}\mathcal{L}_{3,(3)} + \mathcal{L}_{2,(2)}\mathcal{L}_{3,(1)}\mathcal{L}_{3,(4)} + \mathcal{L}_{2,(2)}\mathcal{L}_{3,(3)}\mathcal{L}_{3,(4)} \\ & + \mathcal{L}_{2,(3)}\mathcal{L}_{3,(1)}\mathcal{L}_{3,(2)} + \mathcal{L}_{2,(3)}\mathcal{L}_{3,(1)}\mathcal{L}_{3,(4)} + \mathcal{L}_{2,(3)}\mathcal{L}_{3,(2)}\mathcal{L}_{3,(4)} \\ & + \mathcal{L}_{2,(4)}\mathcal{L}_{3,(1)}\mathcal{L}_{3,(2)} + \mathcal{L}_{2,(4)}\mathcal{L}_{3,(1)}\mathcal{L}_{3,(3)} + \mathcal{L}_{2,(4)}\mathcal{L}_{3,(2)}\mathcal{L}_{3,(3)} \Big) \end{aligned}$$

The contribution is calculated using Propositions 4 and 6.

6.5. **Proof of Theorem 3.** The complete exported relation (24) is obtained by adding all the unsplit contributions of Section 6.3 to all the split contributions of Section 6.4. Using the Noether-Lefschetz support[19] of

$$\lambda\,, \quad \kappa_{[L;1]}\,, \quad Z(L)$$

the only principal contributions are unsplit and obtained from Strata 1 and 2. For the formula of Theorem 3, we normalize the relation by dividing by $12N_1(L)$. □

[19]To be proven in Section 7.2.

6.6. **Higher genus relations.** In genus 2, there is a basic relation among tautological classes in codimension 2 on $\overline{\mathcal{M}}_{2,3}$, see [4]. However, to export in genus 2, we would first have to prove genus 2 analogues of the push-forward results in genus 0 and 1 of Section 3. To build a theory which allows the exportation of all the known tautological relations[20] on the moduli space of curves to the moduli space of $K3$ surfaces is an interesting direction of research. Fortunately, to prove the Noether-Lefschetz generation of Theorem 1, only the relations in genus 0 and 1 are required.

7. Noether-Lefschetz generation

7.1. **Overview.** We present here the proof of Theorem 1: the strict tautological ring is generated by Noether-Lefschetz loci,

$$\mathsf{NL}^\star(\mathcal{M}_\Lambda) = \mathsf{R}^\star(\mathcal{M}_\Lambda)\,.$$

We will use the exported WDVV relation (†) of Theorem 2, the exported Getzler's relation (‡) of Theorem 3, the diagonal decomposition ($\ddagger'$) of Corollary 4, and an induction on codimension.

For (‡), we will require not only the principal terms which appear in the statement of Theorem 3, but the entire formula proven in Section 6. In particular, for (‡) we will *not* divide by the factor $12N_1(L)$.

7.2. **Codimension 1.** The base of the induction on codimension consists of all of the *divisorial* κ classes:

$$\kappa_{[L^3;0]}\,,\ \kappa_{[L;1]}\,,\ \kappa_{[L_1^2,L_2;0]}\,,\ \kappa_{[L_1,L_2,L_3;0]}\ \in\ \mathsf{R}^1(\mathcal{M}_\Lambda)\,, \tag{26}$$

for $L, L_1, L_2, L_3 \in \Lambda$ admissible. Our first goal is to prove the divisorial κ classes (26) are expressible in terms of Noether-Lefschetz divisors in $\mathcal{M}_\Lambda$. In addition, we will determine the divisor $Z(L)$ defined in Proposition 6 for all $L \in \Lambda$ admissible and $\langle L, L\rangle_\Lambda \geq 0$.

Let $L, L_1, L_2, L_3 \in \Lambda$ be admissible, and let $H \in \Lambda$ be the quasi-polarization with

$$\langle H, H\rangle_\Lambda = 2\ell > 0\,.$$

Case A. $\kappa_{[L^3;0]}$, $\kappa_{[L;1]}$, and $Z(L)$ for $\langle L, L\rangle_\Lambda > 0$.

- We apply (†) with respect to L and insert $\Delta_{(12)}\Delta_{(34)} \in \mathsf{R}^4(\mathcal{X}^4_\Lambda)$. The relation

$$\epsilon^4_*\tau^*(\mathsf{WDVV}) \cup \Delta_{(12)}\Delta_{(34)}\ =\ 0\ \in\ \mathsf{R}^9(\mathcal{X}^4_\Lambda)$$

pushes down via

$$\pi^4_\Lambda : \mathcal{X}^4_\Lambda \to \mathcal{M}_\Lambda$$

[20]For a survey of Pixton's relations, see [23].

to yield the result

$$2\langle L, L\rangle_\Lambda \cdot \kappa_{[L;1]} - 2 \cdot \kappa_{[L^3;0]} \ \in \ \mathsf{NL}^1(\mathcal{M}_\Lambda)\,. \tag{27}$$

• We apply (‡) with respect to L and insert $\mathcal{L}_{(1)}\mathcal{L}_{(2)}\mathcal{L}_{(3)}\mathcal{L}_{(4)} \in \mathsf{R}^4(\mathcal{X}^4_\Lambda)$. The relation

$$\epsilon^4_*\tau^*(\mathsf{Getzler}) \cup \mathcal{L}_{(1)}\mathcal{L}_{(2)}\mathcal{L}_{(3)}\mathcal{L}_{(4)} \ = \ 0 \ \in \ \mathsf{R}^9(\mathcal{X}^4_\Lambda)$$

pushes down via π^4_Λ to yield the result

$$\begin{aligned} 72N_1(L)\langle L, L\rangle_\Lambda \cdot \kappa_{[L^3;0]} + 36N_1(L)\langle L, L\rangle^2_\Lambda \cdot Z(L) \\ - 48N_1(L)\langle L, L\rangle_\Lambda \cdot \kappa_{[L^3;0]} + \frac{1}{2}N_0(L)\langle L, L\rangle^4_\Lambda \cdot \kappa_{[L;1]} \ \in \ \mathsf{NL}^1(\mathcal{M}_\Lambda)\,. \end{aligned}$$

The divisors $Z(L)$ and $\kappa_{[L;1]}$ are obtained from the unsplit contributions of Strata 1, 2, and 6. After combining terms, we find

$$24N_1(L) \cdot \kappa_{[L^3;0]} + \frac{1}{2}N_0(L)\langle L, L\rangle^3_\Lambda \cdot \kappa_{[L;1]} + 36N_1(L)\langle L, L\rangle_\Lambda \cdot Z(L) \ \in \ \mathsf{NL}^1(\mathcal{M}_\Lambda)\,. \tag{28}$$

• We apply (‡) with respect to L and insert $\mathcal{L}_{(1)}\mathcal{L}_{(2)}\Delta_{(34)} \in \mathsf{R}^4(\mathcal{X}^4_\Lambda)$. After push-down via π^4_Λ to $\mathcal{M}_\Lambda$, we obtain

$$\begin{aligned} 288N_1(L) \cdot \kappa_{[L^3;0]} + 12N_1(L)\langle L, L\rangle_\Lambda \cdot \kappa_{[L;1]} + 48N_1(L) \cdot \kappa_{[L^3;0]} \\ + 288N_1(L)\langle L, L\rangle_\Lambda \cdot Z(L) + 24N_1(L)\langle L, L\rangle_\Lambda \cdot Z(L) \\ - 24N_1(L)\langle L, L\rangle_\Lambda \cdot \kappa_{[L;1]} - 24N_1(L) \cdot \kappa_{[L^3;0]} - 24N_1(L) \cdot \kappa_{[L^3;0]} \\ - 24N_1(L)\langle L, L\rangle_\Lambda \cdot Z(L) + \frac{1}{2}N_0(L)\langle L, L\rangle^3_\Lambda \cdot \kappa_{[L;1]} \ \in \ \mathsf{NL}^1(\mathcal{M}_\Lambda)\,. \end{aligned}$$

After combining terms, we find

$$\begin{aligned} 288N_1(L) \cdot \kappa_{[L^3;0]} - \Big(12N_1(L)\langle L, L\rangle_\Lambda - \frac{1}{2}N_0(L)\langle L, L\rangle^3_\Lambda\Big) \cdot \kappa_{[L;1]} \\ + 288N_1(L)\langle L, L\rangle_\Lambda \cdot Z(L) \ \in \ \mathsf{NL}^1(\mathcal{M}_\Lambda)\,. \end{aligned} \tag{29}$$

• We apply (‡) with respect to L and insert $\Delta_{(12)}\Delta_{(34)} \in \mathsf{R}^4(\mathcal{X}^4_\Lambda)$. After push-down via π^4_Λ to $\mathcal{M}_\Lambda$, we obtain

$$\begin{aligned} 576N_1(L) \cdot \kappa_{[L;1]} + 48N_1(L) \cdot \kappa_{[L;1]} + 6912N_1(L) \cdot Z(L) + 576N_1(L) \cdot Z(L) \\ - 48N_1(L) \cdot \kappa_{[L;1]} - 48N_1(L) \cdot \kappa_{[L;1]} - 1152N_1(L) \cdot Z(L) \\ + \frac{1}{2}N_0(L)\langle L, L\rangle^2_\Lambda \cdot \kappa_{[L;1]} \ \in \ \mathsf{NL}^1(\mathcal{M}_\Lambda)\,. \end{aligned}$$

After combining terms, we find

$$\Big(528N_1(L) + \frac{1}{2}N_0(L)\langle L, L\rangle^2_\Lambda\Big) \cdot \kappa_{[L;1]} + 6336N_1(L) \cdot Z(L) \ \in \ \mathsf{NL}^1(\mathcal{M}_\Lambda)\,. \tag{30}$$

The system of equations (27), (28), (29), and (30) yields the matrix

$$\begin{pmatrix} -2 & 2\langle L,L\rangle_\Lambda & 0 \\ 24N_1(L) & \frac{1}{2}N_0(L)\langle L,L\rangle_\Lambda^3 & 36N_1(L)\langle L,L\rangle_\Lambda \\ 288N_1(L) & -12N_1(L)\langle L,L\rangle_\Lambda + \frac{1}{2}N_0(L)\langle L,L\rangle_\Lambda^3 & 288N_1(L)\langle L,L\rangle_\Lambda \\ 0 & 528N_1(L) + \frac{1}{2}N_0(L)\langle L,L\rangle_\Lambda^2 & 6336N_1(L) \end{pmatrix}. \tag{31}$$

Since $N_0(L), N_1(L) \neq 0$, straightforward linear algebra[21] shows the matrix (31) to have maximal rank 3. We have therefore proven

$$\kappa_{[L^3;0]},\ \kappa_{[L;1]},\ Z(L)\ \in\ \mathsf{NL}^1(\mathcal{M}_\Lambda)$$

and completed the analysis of Case A.

Case B. $\kappa_{[H^2,L;0]}$ for $\langle L,L\rangle_\Lambda > 0$.

We apply $(\ddagger')$ with insertion $\mathcal{L}_{(1)}\mathcal{L}_{(2)}\mathcal{L}_{(3)} \in \mathsf{R}^3(\mathcal{X}_\Lambda^3)$, and push-down via π_Λ^3 to $\mathcal{M}_\Lambda$. Since

$$\kappa_{[H;1]}\,,\ Z(H)\ \in\ \mathsf{NL}^1(\mathcal{M}_\Lambda)$$

by Case A, we find

$$2\ell\cdot\kappa_{[L^3;0]} - 3\langle L,L\rangle_\Lambda\cdot\kappa_{[H^2,L;0]}\ \in\ \mathsf{NL}^1(\mathcal{M}_\Lambda)\,.$$

Since $\kappa_{[L^3;0]} \in \mathsf{NL}^1(\mathcal{M}_\Lambda)$ by Case A, we have

$$\kappa_{[H^2,L;0]}\ \in\ \mathsf{NL}^1(\mathcal{M}_\Lambda)\,.$$

Case B is complete.

Case C. $\kappa_{[L^3;0]}$, $\kappa_{[H^2,L;0]}$, and $\kappa_{[L;1]}$ for $\langle L,L\rangle_\Lambda < 0$.

• We apply $(\ddagger')$ with insertion $\mathcal{L}_{(1)}\mathcal{L}_{(2)}\mathcal{L}_{(3)} \in \mathsf{R}^3(\mathcal{X}_\Lambda^3)$, and push-down via π_Λ^3 to $\mathcal{M}_\Lambda$. Since

$$\kappa_{[H;1]}\,,\ Z(H)\ \in\ \mathsf{NL}^1(\mathcal{M}_\Lambda)$$

by Case A, we find

$$2\ell\cdot\kappa_{[L^3;0]} - 3\langle L,L\rangle_\Lambda\cdot\kappa_{[H^2,L;0]}\ \in\ \mathsf{NL}^1(\mathcal{M}_\Lambda)\,. \tag{32}$$

[21]One may even consider λ as a 4^{th} variable in the equations (27), (28), (29), and (30). For $\Lambda = (2\ell)$ and $L = H$, the only λ terms are obtained from the unsplit contribution of Stratum 4 to $(\ddagger)$. We find the matrix

$$\begin{pmatrix} -2 & 2(2\ell) & 0 & 0 \\ 24N_1(\ell) & \frac{1}{2}N_0(\ell)(2\ell)^3 & 36N_1(\ell)(2\ell) & N_0(\ell)(2\ell)^3 \\ 288N_1(\ell) & -12N_1(\ell)(2\ell) + \frac{1}{2}N_0(\ell)(2\ell)^3 & 288N_1(\ell)(2\ell) & N_0(\ell)(2\ell)^3 \\ 0 & 528N_1(\ell) + \frac{1}{2}N_0(\ell)(2\ell)^2 & 6336N_1(\ell) & N_0(\ell)(2\ell)^2 \end{pmatrix}$$

whose determinant is easily seen to be nonzero. In particular, we obtain a geometric proof of the fact

$$\lambda\ \in\ \mathsf{NL}^1(\mathcal{M}_{2\ell})\,.$$

The determinant of the 4×4 matrix is likely nonzero for every Λ and H (in which case additional λ terms appear). We plan to carry out more detailed computation in the future.

• We apply ($\ddagger'$) with insertion $\mathcal{H}_{(1)}\mathcal{L}_{(2)}\mathcal{L}_{(3)} \in \mathsf{R}^3(\mathcal{X}^3_\Lambda)$, and push-down via π^3_Λ to $\mathcal{M}_\Lambda$. Since $\kappa_{[H^3;0]} \in \mathsf{NL}^1(\mathcal{M}_\Lambda)$ by Case A, we find

$$2\ell \cdot \kappa_{[H,L^2;0]} - 2\langle H, L\rangle_\Lambda \cdot \kappa_{[H^2,L;0]} \ \in \ \mathsf{NL}^1(\mathcal{M}_\Lambda)\,. \tag{33}$$

• We apply (†) with respect to L, insert $\mathcal{H}_{(1)}\mathcal{H}_{(2)}\mathcal{L}_{(3)}\mathcal{L}_{(4)} \in \mathsf{R}^4(\mathcal{X}^4_\Lambda)$, and push-down via π^4_Λ to $\mathcal{M}_\Lambda$. We find

$$\langle H, L\rangle^2 \cdot \kappa_{[L^3;0]} + \langle L, L\rangle^2 \cdot \kappa_{[H^2,L;0]} - 2\langle H, L\rangle\langle L, L\rangle \cdot \kappa_{[H,L^2;0]} \ \in \ \mathsf{NL}^1(\mathcal{M}_\Lambda)\,. \tag{34}$$

• We apply (†) with respect to L, insert $\Delta_{(12)}\Delta_{(34)} \in \mathsf{R}^4(\mathcal{X}^4_\Lambda)$, and push-down via π^4_Λ to $\mathcal{M}_\Lambda$. We find

$$2\langle L, L\rangle_\Lambda \cdot \kappa_{[L;1]} - 2 \cdot \kappa_{[L^3;0]} \ \in \ \mathsf{NL}^1(\mathcal{M}_\Lambda)\,. \tag{35}$$

The system of equations (32), (33), and (34) for

$$\kappa_{[L^3;0]}\,, \quad \kappa_{[H,L^2;0]}\,, \quad \kappa_{[H^2,L;0]}$$

yields the matrix

$$\begin{pmatrix} 2\ell & 0 & -3\langle L, L\rangle_\Lambda \\ 0 & 2\ell & -2\langle H, L\rangle_\Lambda \\ \langle H, L\rangle^2_\Lambda & -2\langle H, L\rangle_\Lambda\langle L, L\rangle_\Lambda & \langle L, L\rangle^2_\Lambda \end{pmatrix}$$

with determinant

$$2\ell\langle L, L\rangle_\Lambda\Big(2\ell\langle L, L\rangle_\Lambda - \langle H, L\rangle^2_\Lambda\Big) > 0$$

by the Hodge index theorem applied to the second factor. Therefore,

$$\kappa_{[L^3;0]}\,, \ \kappa_{[H,L^2;0]}\,, \ \kappa_{[H^2,L;0]} \ \in \ \mathsf{NL}^1(\mathcal{M}_\Lambda)\,,$$

and by (35), we have $\kappa_{[L;1]} \in \mathsf{NL}^1(\mathcal{M}_\Lambda)$. Case C is complete.

Case D. $\kappa_{[L^3;0]}$, $\kappa_{[H^2,L;0]}$, $\kappa_{[L;1]}$, and $Z(L)$ for $\langle L, L\rangle_\Lambda = 0$.

• We apply ($\ddagger'$) with insertion $\mathcal{L}_{(1)}\mathcal{L}_{(2)}\mathcal{L}_{(3)} \in \mathsf{R}^3(\mathcal{X}^3_\Lambda)$, and push-down via π^3_Λ to $\mathcal{M}_\Lambda$. Since

$$\kappa_{[H;1]}\,, \ Z(H) \ \in \ \mathsf{NL}^1(\mathcal{M}_\Lambda)$$

by Case A, we find

$$2\ell \cdot \kappa_{[L^3;0]} - 3\langle L, L\rangle_\Lambda \cdot \kappa_{[H^2,L;0]} \ \in \ \mathsf{NL}^1(\mathcal{M}_\Lambda)\,,$$

hence[22] $\kappa_{[L^3;0]} \in \mathsf{NL}^1(\mathcal{M}_\Lambda)$.

[22]A direct argument using elliptically fibered $K3$ surfaces shows $\kappa_{[L^3;0]} = 0$ for $\langle L, L\rangle_\Lambda = 0$.

• We apply ($\ddagger'$) with insertion $\mathcal{H}_{(1)}\mathcal{L}_{(2)}\mathcal{L}_{(3)} \in \mathsf{R}^3(\mathcal{X}^3_\Lambda)$, and push-down via π^3_Λ to $\mathcal{M}_\Lambda$. We find

$$2\ell \cdot \kappa_{[H,L^2;0]} - 2\langle H, L\rangle_\Lambda \cdot \kappa_{[H^2,L;0]} \ \in \ \mathsf{NL}^1(\mathcal{M}_\Lambda)\,. \tag{36}$$

• We apply ($\dagger$) with respect to L, insert $\mathcal{H}_{(1)}\mathcal{H}_{(2)}\Delta_{(34)} \in \mathsf{R}^4(\mathcal{X}^4_\Lambda)$, and push-down via π^4_Λ to $\mathcal{M}_\Lambda$. We find

$$\langle H, L\rangle^2_\Lambda \cdot \kappa_{[L;1]} - 2\langle H, L\rangle_\Lambda \cdot \kappa_{[H,L^2;0]} \ \in \ \mathsf{NL}^1(\mathcal{M}_\Lambda)\,.$$

Since $\langle H, L\rangle_\Lambda \neq 0$ by the Hodge index theorem, we have

$$\langle H, L\rangle_\Lambda \cdot \kappa_{[L;1]} - 2 \cdot \kappa_{[H,L^2;0]} \ \in \ \mathsf{NL}^1(\mathcal{M}_\Lambda)\,. \tag{37}$$

• We apply ($\ddagger$) with respect to L, insert $\mathcal{H}_{(1)}\mathcal{H}_{(2)}\mathcal{H}_{(3)}\mathcal{L}_{(4)} \in \mathsf{R}^4(\mathcal{X}^4_\Lambda)$, and push-down via π^4_Λ to $\mathcal{M}_\Lambda$. We find

$$\begin{aligned} 36N_1(L)\langle H, L\rangle_\Lambda \cdot \kappa_{[H^2,L;0]} + 36N_1(L)(2\ell) \cdot \kappa_{[H,L^2;0]} + 36N_1(L)(2\ell)\langle H, L\rangle_\Lambda \cdot Z(L) \\ - 36N_1(L)\langle H, L\rangle_\Lambda \cdot \kappa_{[H^2,L;0]} \ \in \ \mathsf{NL}^1(\mathcal{M}_\Lambda)\,. \end{aligned}$$

Since $N_1(L) \neq 0$, we have

$$\kappa_{[H,L^2;0]} + \langle H, L\rangle_\Lambda \cdot Z(L) \ \in \ \mathsf{NL}^1(\mathcal{M}_\Lambda)\,. \tag{38}$$

• We apply ($\ddagger$) with respect to L, insert $\mathcal{H}_{(1)}\mathcal{H}_{(2)}\Delta_{(34)} \in \mathsf{R}^4(\mathcal{X}^4_\Lambda)$, and push-down via π^4_Λ to $\mathcal{M}_\Lambda$. We find

$$\begin{aligned} 288N_1(L) \cdot \kappa_{[H^2,L;0]} + 12N_1(L)(2\ell) \cdot \kappa_{[L;1]} + 48N_1(L) \cdot \kappa_{[H^2,L;0]} \\ + 288N_1(L)(2\ell) \cdot Z(L) + 24N_1(L)(2\ell) \cdot Z(L) \\ - 24N_1(L) \cdot \kappa_{[H^2,L;0]} - 24N_1(L) \cdot \kappa_{[H^2,L;0]} - 24N_1(L)(2\ell) \cdot Z(L) \ \in \ \mathsf{NL}^1(\mathcal{M}_\Lambda)\,. \end{aligned}$$

After combining terms, we obtain

$$24 \cdot \kappa_{[H^2,L;0]} + 2\ell \cdot \kappa_{[L;1]} + 24(2\ell) \cdot Z(L) \ \in \ \mathsf{NL}^1(\mathcal{M}_\Lambda)\,. \tag{39}$$

We multiply (39) by $\langle H, L\rangle_\Lambda$, and make substitutions using (36), (37), and (38), which yields

$$(12 + 2 - 24)(2\ell) \cdot \kappa_{[H,L^2;0]} \ \in \ \mathsf{NL}^1(\mathcal{M}_\Lambda)\,.$$

Therefore, $\kappa_{[H,L^2;0]} \in \mathsf{NL}^1(\mathcal{M}_\Lambda)$. Then, again by (36), (37), and (38),

$$\kappa_{[H^2,L;0]}\,,\ \kappa_{[L;1]}\,,\ Z(L) \ \in \ \mathsf{NL}^1(\mathcal{M}_\Lambda)\,.$$

Case D is complete.

Case E. $\kappa_{[L_1,L_2,L_3;0]}$ for arbitrary $L_1, L_2, L_3 \in \Lambda$.

We apply $(\ddagger')$ with insertion $\mathcal{L}_{1,(1)}\mathcal{L}_{2,(2)}\mathcal{L}_{3,(3)} \in \mathsf{R}^3(\mathcal{X}_\Lambda^3)$, and push-down via π_Λ^3 to $\mathcal{M}_\Lambda$. The result expresses $2\ell \cdot \kappa_{[L_1,L_2,L_3;0]}$ in terms of Noether-Lefschetz divisors and κ divisors treated in the previous cases. Therefore,

$$\kappa_{[L_1,L_2,L_3;0]} \ \in \ \mathsf{NL}^1(\mathcal{M}_\Lambda)\,.$$

Case E is complete.

Cases A-E together cover all divisorial κ classes and prove the divisorial case of Theorem 1.

Proposition 9. *The strict tautological ring in codimension* 1 *is generated by Noether-Lefschetz loci,*

$$\mathsf{NL}^1(\mathcal{M}_\Lambda) = \mathsf{R}^1(\mathcal{M}_\Lambda)\,.$$

In fact, by the result of [5], $\mathsf{NL}^1(\mathcal{M}_\Lambda)$ generates *all* of $\mathsf{A}^1(\mathcal{M}_\Lambda)$ for $\mathrm{rank}(\Lambda) \leq 17$. We have given a direct proof of Proposition 9 using exported relations which is valid for every lattice polarization Λ without rank restriction. The same method will be used to prove the full statement of Theorem 1.

7.3. **Second Chern class.** The next step is to eliminate the $c_2(\mathcal{T}_{\pi_\Lambda})$ index in the class $\kappa_{[L_1^{a_1},\ldots,L_k^{a_k};b]}$ and reduce to the case

$$\kappa_{[L_1^{a_1},\ldots,L_k^{a_k};0]}\,.$$

Our strategy is to express $c_2(\mathcal{T}_{\pi_\Lambda}) \in \mathsf{R}^2(\mathcal{X}_\Lambda)$ in terms of simpler strict tautological classes.

From now on, we will require only the decomposition $(\ddagger')$.

• We apply $(\ddagger')$ with insertion $\mathcal{H}_{(1)}\mathcal{H}_{(2)}\Delta_{(23)} \in \mathsf{R}^4(\mathcal{X}_\Lambda^3)$, and push-down via π_Λ^3 to $\mathcal{M}_\Lambda$. As a result, we find

$$2\ell \cdot \kappa_{[H^2;1]} - \kappa_{[H^3;0]}\kappa_{[H;1]} - 2 \cdot \kappa_{[H^4;0]} + 2 \cdot \kappa_{[H^4;0]} \ \in \ \mathsf{NL}^2(\mathcal{M}_\Lambda)\,,$$

where we have used Proposition 9 for all the non-principal terms corresponding to larger lattices. By Proposition 9 for Λ, we have $\kappa_{[H^3;0]}$, $\kappa_{[H;1]} \in \mathsf{NL}^1(\mathcal{M}_\Lambda)$. We conclude

$$\kappa_{[H^2;1]} \ \in \ \mathsf{NL}^2(\mathcal{M}_\Lambda)\,.$$

• We apply $(\ddagger')$ with insertion $\Delta_{(12)} \in \mathsf{R}^2(\mathcal{X}_\Lambda^3)$, and push-forward to $\mathcal{X}_\Lambda$ via the third projection

$$\mathrm{pr}_{(3)} : \mathcal{X}_\Lambda^3 \to \mathcal{X}_\Lambda\,.$$

We find

$$\begin{aligned} 2\ell \cdot c_2(\mathcal{T}_{\pi_\Lambda}) &= 2 \cdot \mathcal{H}^2 + 24 \cdot \mathcal{H}^2 - \kappa_{[H^2;1]} - 2 \cdot \mathcal{H}^2 + \dots \\ &= 24 \cdot \mathcal{H}^2 - \kappa_{[H^2;1]} + \dots \ \in \ \mathsf{R}^2(\mathcal{X}_\Lambda)\,, \end{aligned}$$

where the dots stand for strict tautological classes supported over proper Noether-Lefschetz loci of $\mathcal{M}_\Lambda$.

We have already proven $\kappa_{[H^2;1]} \in \mathsf{NL}^2(\mathcal{M}_\Lambda)$. Therefore, up to strict tautological classes supported over proper Noether-Lefschetz loci of $\mathcal{M}_\Lambda$, we may replace $c_2(\mathcal{T}_{\pi_\Lambda})$ by

$$\frac{24}{2\ell} \cdot \mathcal{H}^2 \ \in \ \mathsf{R}^2(\mathcal{X}_\Lambda)\,.$$

The replacement lowers the $c_2(\mathcal{T}_{\pi_\Lambda})$ index of κ classes. By induction, we need only prove Theorem 1 for κ classes with trivial $c_2(\mathcal{T}_{\pi_\Lambda})$ index.

7.4. **Proof of Theorem 1.** The κ classes with trivial $c_2(\mathcal{T}_{\pi_\Lambda})$ index can be written as

$$\kappa_{[H^a,L_1,\dots,L_k;0]} \ \in \ \mathsf{R}^{a+k-2}(\mathcal{M}_\Lambda)\,,$$

where the $L_i \in \Lambda$ are admissible classes (not necessarily distinct) that are different from the quasi-polarization H.

Codimension 2.

In codimension 2, the complete list of κ classes (with trivial $c_2(\mathcal{T}_{\pi_\Lambda})$ index) is:

$$\kappa_{[H^4;0]}\,,\ \kappa_{[H^3,L;0]}\,,\ \kappa_{[H^2,L_1,L_2;0]}\,,\ \kappa_{[H,L_1,L_2,L_3;0]}\,,\ \kappa_{[L_1,L_2,L_3,L_4;0]} \ \in \ \mathsf{R}^2(\mathcal{M}_\Lambda)\,.$$

• For $\kappa_{[H^4;0]}$, we apply $(\ddagger')$ with insertion $\mathcal{H}^2_{(1)}\Delta_{(23)} \in \mathsf{R}^4(\mathcal{X}^3_\Lambda)$, and push-down via π^3_Λ to $\mathcal{M}_\Lambda$. We find

$$2\ell \cdot \kappa_{[H^2;1]} - 24 \cdot \kappa_{[H^4;0]} - 2 \cdot \kappa_{[H^4;0]} + 2 \cdot \kappa_{[H^4;0]} + 2\ell \cdot \kappa_{[H^2;1]} \ \in \ \mathsf{NL}^2(\mathcal{M}_\Lambda)\,,$$

where we have used Proposition 9 for all the non-principal terms corresponding to larger lattices. Since $\kappa_{[H^2;1]} \in \mathsf{NL}^2(\mathcal{M}_\Lambda)$ by Section 7.3, we have $\kappa_{[H^4;0]} \in \mathsf{NL}^2(\mathcal{M}_\Lambda)$.

• For $\kappa_{[H^3,L;0]}$, we apply $(\ddagger')$ with insertion $\mathcal{H}^2_{(1)}\mathcal{H}_{(2)}\mathcal{L}_{(3)} \in \mathsf{R}^4(\mathcal{X}^3_\Lambda)$, and push-down via π^3_Λ to $\mathcal{M}_\Lambda$. We find

$$2\ell \cdot \kappa_{[H^3,L;0]} - \langle H, L\rangle_\Lambda \cdot \kappa_{[H^4;0]} - 2 \cdot \kappa_{[H^3;0]}\kappa_{[H^2,L;0]} + 2\ell \cdot \kappa_{[H^3,L;0]} \ \in \ \mathsf{NL}^2(\mathcal{M}_\Lambda)\,,$$

hence $\kappa_{[H^3,L;0]} \in \mathsf{NL}^2(\mathcal{M}_\Lambda)$.

• For $\kappa_{[H^2,L_1,L_2;0]}$, we apply $(\ddagger')$ with insertion $\mathcal{H}^2_{(1)}\mathcal{L}_{1,(2)}\mathcal{L}_{2,(3)} \in \mathsf{R}^4(\mathcal{X}^3_\Lambda)$, and push-down via π^3_Λ to $\mathcal{M}_\Lambda$. We find

$$2\ell \cdot \kappa_{[H^2,L_1,L_2;0]} - \langle L_1, L_2\rangle_\Lambda \cdot \kappa_{[H^4;0]} \\ - 2 \cdot \kappa_{[H^2,L_1;0]}\kappa_{[H^2,L_2;0]} + 2\ell \cdot \kappa_{[H^2,L_1,L_2;0]} \ \in \ \mathsf{NL}^2(\mathcal{M}_\Lambda)\,,$$

hence $\kappa_{[H^2,L_1,L_2;0]} \in \mathsf{NL}^2(\mathcal{M}_\Lambda)$.

• For $\kappa_{[H,L_1,L_2,L_3;0]}$, we apply $(\ddagger')$ with insertion $\mathcal{H}_{(1)}\mathcal{L}_{1,(1)}\mathcal{L}_{2,(2)}\mathcal{L}_{3,(3)} \in \mathsf{R}^4(\mathcal{X}^3_\Lambda)$, and push-down via π^3_Λ to $\mathcal{M}_\Lambda$. We find

$$2\ell \cdot \kappa_{[H,L_1,L_2,L_3;0]} - \langle L_2, L_3\rangle_\Lambda \cdot \kappa_{[H^3,L_1;0]} - \kappa_{[H^2,L_2;0]}\kappa_{[H,L_1,L_3;0]} \\ - \kappa_{[H^2,L_3;0]}\kappa_{[H,L_1,L_2;0]} + \langle H, L_1\rangle_\Lambda \cdot \kappa_{[H^2,L_2,L_3;0]} \ \in \ \mathsf{NL}^2(\mathcal{M}_\Lambda)\,,$$

hence $\kappa_{[H,L_1,L_2,L_3;0]} \in \mathsf{NL}^2(\mathcal{M}_\Lambda)$.

• For $\kappa_{[L_1,L_2,L_3,L_4;0]}$, we apply $(\ddagger')$ with insertion $\mathcal{L}_{1,(1)}\mathcal{L}_{2,(1)}\mathcal{L}_{3,(2)}\mathcal{L}_{4,(3)} \in \mathsf{R}^4(\mathcal{X}^3_\Lambda)$, and push-down via π^3_Λ to $\mathcal{M}_\Lambda$. We find

$$2\ell \cdot \kappa_{[L_1,L_2,L_3,L_4;0]} - \langle L_3, L_4\rangle_\Lambda \cdot \kappa_{[H^2,L_1,L_2;0]} - \kappa_{[H^2,L_3;0]}\kappa_{[L_1,L_2,L_4;0]} \\ - \kappa_{[H^2,L_4;0]}\kappa_{[L_1,L_2,L_3;0]} + \langle L_1, L_2\rangle_\Lambda \cdot \kappa_{[H^2,L_3,L_4;0]} \ \in \ \mathsf{NL}^2(\mathcal{M}_\Lambda)\,,$$

hence $\kappa_{[L_1,L_2,L_3,L_4;0]} \in \mathsf{NL}^2(\mathcal{M}_\Lambda)$.

Codimension ≥ 3.

Our strategy in codimension $c \geq 3$ involves an induction on codimension together with a second induction on the H index a of the kappa class

$$\kappa_{[H^a,L_1,\ldots,L_k;0]} \ \in \ \mathsf{R}^{a+k-2}(\mathcal{M}_\Lambda)\,.$$

For the induction on c, we assume the Noether-Lefschetz generation for all *lower* codimension. The base case is Proposition 9. For the induction on a, we assume the Noether-Lefschetz generation for all *higher* H index.

• For the base of the induction on H index, consider the class

$$\kappa_{[H^a;0]} \ \in \ \mathsf{R}^{a-2}(\mathcal{M}_\Lambda)\,.$$

We apply $(\ddagger')$, insert

$$\mathcal{H}^{a-3}_{(1)}\mathcal{H}^2_{(2)}\mathcal{H}_{(3)} \ \in \ \mathsf{R}^a(\mathcal{X}^3_\Lambda) \quad \text{with } a-2=c\,,$$

and push-down via π^3_Λ to $\mathcal{M}_\Lambda$. By the induction on codimension, we obtain

$$2\ell \cdot \kappa_{[H^a;0]} - 2 \cdot \kappa_{[H^3;0]}\kappa_{[H^{a-1};0]} - \kappa_{[H^4;0]}\kappa_{[H^{a-2};0]} \\ + 2\ell \cdot \kappa_{[H^a;0]} + \kappa_{[H^5;0]}\kappa_{[H^{a-3};0]} \ \in \ \mathsf{NL}^{a-2}(\mathcal{M}_\Lambda)\,. \tag{40}$$

For both[23] $a = 5$ and $a > 5$, the coefficient of $\kappa_{[H^a;0]}$ is positive and the other terms in (40) are products of κ classes of lower codimension. Therefore, by the induction hypothesis,

$$\kappa_{[H^a;0]} \in \mathsf{NL}^{a-2}(\mathcal{M}_\Lambda)\,.$$

- If $a > 0$ and $k > 0$, we apply $(\ddagger')$, insert

$$\mathcal{H}_{(1)}^{a-1}\mathcal{L}_{1,(1)}\cdots\mathcal{L}_{k-1,(1)}\mathcal{H}_{(2)}\mathcal{L}_{k,(3)} \in \mathsf{R}^{a+k}(\mathcal{X}_\Lambda^3) \quad \text{with } a+k-2=c\,,$$

and push-down via π_Λ^3 to $\mathcal{M}_\Lambda$. By the induction on codimension, we obtain

$$\begin{aligned}(41)\quad & 2\ell\cdot\kappa_{[H^a,L_1,\ldots,L_k;0]} - \langle H, L_k\rangle_\Lambda\cdot\kappa_{[H^{a+1},L_1,\ldots,L_{k-1};0]} \\ & - \kappa_{[H^3;0]}\kappa_{[H^{a-1},L_1,\ldots,L_{k-1},L_k;0]} - \kappa_{[H^2,L_k;0]}\kappa_{[H^a,L_1,\ldots,L_{k-1};0]} \\ & + \kappa_{[H^3,L_k;0]}\kappa_{[H^{a-1},L_1,\ldots,L_{k-1};0]} \in \mathsf{NL}^{a+k-2}(\mathcal{M}_\Lambda)\,.\end{aligned}$$

Since the last three terms of (41) are products of κ classes of lower codimension (since $a+k \geq 5$), using the induction hypothesis again yields

$$2\ell\cdot\kappa_{[H^a,L_1,\ldots,L_k;0]} - \langle H, L_k\rangle_\Lambda\cdot\kappa_{[H^{a+1},L_1,\ldots,L_{k-1};0]} \in \mathsf{NL}^{a+k-2}(\mathcal{M}_\Lambda)\,,$$

which allows us to raise the H index.

- If $a = 0$, we apply $(\ddagger')$, insert

$$\mathcal{L}_{1,(1)}\cdots\mathcal{L}_{k-2,(1)}\mathcal{L}_{k-1,(2)}\mathcal{L}_{k,(3)} \in \mathsf{R}^{k}(\mathcal{X}_\Lambda^3) \quad \text{with } k-2=c\,,$$

and push-down via π_Λ^3 to $\mathcal{M}_\Lambda$. By the induction on codimension, we obtain

$$\begin{aligned}(42)\quad & 2\ell\cdot\kappa_{[L_1,\ldots,L_k;0]} - \langle L_{k-1}, L_k\rangle_\Lambda\cdot\kappa_{[H^2,L_1,\ldots,L_{k-2};0]} \\ & - \kappa_{[H^2,L_{k-1};0]}\kappa_{[L_1,\ldots,L_{k-2},L_k;0]} - \kappa_{[H^2,L_k;0]}\kappa_{[L_1,\ldots,L_{k-2},L_{k-1};0]} \\ & + \kappa_{[H^2,L_{k-1},L_k;0]}\kappa_{[L_1,\ldots,L_{k-2};0]} \in \mathsf{NL}^{k-2}(\mathcal{M}_\Lambda)\,.\end{aligned}$$

Since the last three terms of (42) are products of κ classes of lower codimension (since $k \geq 5$), using the induction hypothesis again yields

$$2\ell\cdot\kappa_{[L_1,\ldots,L_k;0]} - \langle L_{k-1}, L_k\rangle_\Lambda\cdot\kappa_{[H^2,L_1,\ldots,L_{k-2};0]} \in \mathsf{NL}^{k-2}(\mathcal{M}_\Lambda)\,,$$

which allows us to raise the H index.

The induction argument on codimension and H index is complete. The Noether-Lefschetz generation of Theorem 1 is proven. □

[23]Since $a-2=c\geq 3$, $a\geq 5$.

References

[1] A. Beauville, *Counting rational curves on $K3$ surfaces*, Duke Math. J. **97** (1999), 99–108.

[2] A. Beauville and C. Voisin, *On the Chow ring of a $K3$ surface*, J. Algebraic Geom. **13** (2004), 417–426.

[3] K. Behrend and B. Fantechi, *The intrinsic normal cone*, Invent. Math. **128** (1997), 45–88.

[4] P. Belorousski and R. Pandharipande, *A descendent relation in genus* 2, Ann. Scuola Norm. Sup. Pisa Cl. Sci. **29** (2000), 171–191.

[5] N. Bergeron, Z. Li, J. Millson, and C. Moeglin, *The Noether-Lefschetz conjecture and generalizations*, Invent. Math. (to appear).

[6] R. E. Borcherds, L. Katzarkov, T. Pantev, and N. I. Shepherd-Barron, *Families of $K3$ surfaces*, J. Algebraic Geom. **7** (1998), 183–193.

[7] J. H. Bruinier, M. Westerholt-Raum, *Kudla's modularity conjecture and formal Fourier-Jacobi series*, Forum Math. Pi **3** (2015), 30 pp.

[8] J. Bryan and N.-C. Leung, *The enumerative geometry of K3 surfaces and modular forms*, J. Amer. Math. Soc. **13** (2000), 371–410.

[9] I. Dolgachev and S. Kondō, *Moduli of $K3$ surfaces and complex ball quotients*, Arithmetic and geometry around hypergeometric functions, 43–100, Progr. Math. **260**, Birkhäuser, Basel, 2007.

[10] G. van der Geer and T. Katsura, *Note on tautological classes of moduli of $K3$ surfaces*, Mosc. Math. J. **5** (2005), 775–779.

[11] E. Getzler, *Intersection theory on $\overline{\mathcal{M}}_{1,4}$ and elliptic Gromov-Witten invariants*, J. Amer. Math. Soc. **10** (1997), 973–998.

[12] F. Gounelas, *Free curves on varieties*, Doc. Math. **21** (2016), 287–308.

[13] F. Janda, R. Pandharipande, A. Pixton, and D. Zvonkine, *Double ramification cycles on the moduli spaces of curves*, `arXiv:1602.04705`.

[14] A. Klemm, D. Maulik, R. Pandharipande, and E. Scheidegger, *Noether-Lefschetz theory and the Yau-Zaslow conjecture*, J. Amer. Math. Soc. **23** (2010), 1013–1040.

[15] A. Kresch, *Cycle groups of Artin stacks*, Invent. Math. **138** (1999), 495–536.

[16] A. Marian, D. Oprea, and R. Pandharipande, *Segre classes and Hilbert schemes of points*, Ann. Sci. Éc. Norm. Supér. (to appear).

[17] D. Maulik, *Supersingular $K3$ surfaces for large primes*, Duke Math. J. **163** (2014), 2357–2425.

[18] D. Maulik and R. Pandharipande, *Gromov-Witten theory and Noether-Lefschetz theory*, A celebration of algebraic geometry, 469–507, Clay Math. Proc. **18**, Amer. Math. Soc., Providence, RI, 2013.

[19] D. Maulik, R. Pandharipande, and R. P. Thomas, *Curves on $K3$ surfaces and modular forms*, with an appendix by A. Pixton, J. Topol. **3** (2010), 937–996.

[20] D. Mumford, *Towards an enumerative geometry of the moduli space of curves*, Arithmetic and geometry, Vol. II, 271–328, Progr. Math. **36**, Birkhäuser, Boston, MA, 1983.

[21] G. Oberdieck and R. Pandharipande, *Curve counting on $K3 \times E$, the Igusa cusp form χ_{10}, and descendent integration*, $K3$ surfaces and their moduli, 245–278, Progr. Math. **315**, Birkhäuser, Basel, 2016.

[22] R. Pandharipande, *A geometric construction of Getzler's elliptic relation*, Math. Ann. **313** (1999), 715–729.

[23] R. Pandharipande, *A calculus for the moduli space of curves*, Proceedings of Algebraic Geometry – Salt Lake City 2015, Proc. Sympos. Pure Math. (to appear).

[24] R. Pandharipande and R. P. Thomas, *The Katz-Klemm-Vafa conjecture for $K3$ surfaces*, Forum Math. Pi **4** (2016), 111 pp.

[25] D. Petersen, *A vanishing result for tautological classes on the moduli of $K3$ surfaces*, `arXiv:1606.06716`.

[26] C. Voisin, *Chow rings and decomposition theorems for families of $K3$ surfaces and Calabi-Yau hypersurfaces*, Geom. Topol. **16** (2012), 433–473.

[27] C. Voisin, *Chow rings, decomposition of the diagonal, and the topology of families*, Annals of Mathematics Studies **187**, Princeton University Press, Princeton, NJ, 2014, viii+163 pp.

[28] C. T. C. Wall, *On the orthogonal groups of unimodular quadratic forms*, Math. Ann. **147** (1962), 328–338.
[29] S.-T. Yau and E. Zaslow, *BPS states, string duality, and nodal curves on $K3$*, Nuclear Phys. B **471** (1996), 503–512.

Department of Mathematics, ETH Zürich
E-mail address: rahul@math.ethz.ch

Department of Mathematics, ETH Zürich
E-mail address: qizheng.yin@math.ethz.ch