Virasoro Constraints for moduli spaces of sheaves

Ezra = 60

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24 March 2022

* Photo by Jakob Creutz [unsplash]

- Let X be a nonsingular projective algebraic variety over $\mathbb{C}$

- Let M be a moduli space of sheaves on X

Basic requirements
- $\dim_{\mathbb{C}} X \leq 3$,
- M carries a universal sheaf and a natural virtual fundamental class $[M]^{vir} \in H_*(M)$

- We can construct descendent insertions

$$\mathrm{ch}_k(\gamma) \in H^*(M)$$

$k \in \mathbb{Z}_{\geq 0}$

$\gamma \in H^*(X)$

universal sheaf $\rightarrow$

$$
\begin{tikzcd}
 & \mathcal{E} \arrow[d] & \\
 & \mathcal{M} \times X \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
\mathcal{M} & & X
\end{tikzcd}
$$

$\gamma \in H^{\delta}(X)$

moduli space of sheaves $\nearrow$

$r = \dim_{\mathbb{C}} X$

$$\mathrm{ch}_k(\gamma) = \pi_{1*}\Big(\mathrm{ch}_k(\mathcal{E}) \cdot \pi_2^*(\gamma)\Big)$$

$$\in H^{2k+\delta-2r}(I)$$

descendent in sheaf theory $\nearrow$

- We define descendent integrals

$$\Big\langle \mathrm{ch}_{k_1}(\gamma_1) \cdots \mathrm{ch}_{k_\ell}(\gamma_\ell) \Big\rangle^{X}_{\mathcal{M}} = \int_{[\mathcal{M}]^{vir}} \mathrm{ch}_{k_1}(\gamma_1) \cdots \mathrm{ch}_{k_\ell}(\gamma_\ell)$$

Examples of descendents in sheaf theory

- X is a nonsingular projective curve

  $\mathcal{L} \to X$ is a line bundle

$$\mathcal{U}_{X,2,\mathcal{L}} = \text{moduli space of rank 2 stable bundles on } X \text{ with fixed det} = \mathcal{L}$$

deg $\mathcal{L} = 1$

[no semistables]

Descendents defined via the universal bundle

$$\mathcal{E} \to \mathcal{U}_{X,2,\mathcal{L}}$$

Theorem: $H^*(\mathcal{U}_{X,2,\mathcal{L}})$ generated by descendents. [Mumford, Kirwan, Zagier also found relations]

• X is a surface

Exactly parallel construction for moduli of sheaves on a surface

$\Rightarrow$ used in the theory of Donaldson invariants

• X curve or surface

$$\mathrm{Quot}_X(\mathbb{C}^n, \beta)$$

Hilbert polynomial

for X a curve $\Rightarrow$ $\mathrm{Quot}_X(\mathbb{C}^n, \beta)$ always carries 2-term obstruction theory $\Rightarrow$ vfc

for X a surface $\Rightarrow$ $\mathrm{Quot}_X(\mathbb{C}^n, \beta)$ often carries a vfc

- X is a 3 fold

  $\beta \in H_2(X, \mathbb{Z})$,

  $n \in \mathbb{Z}$

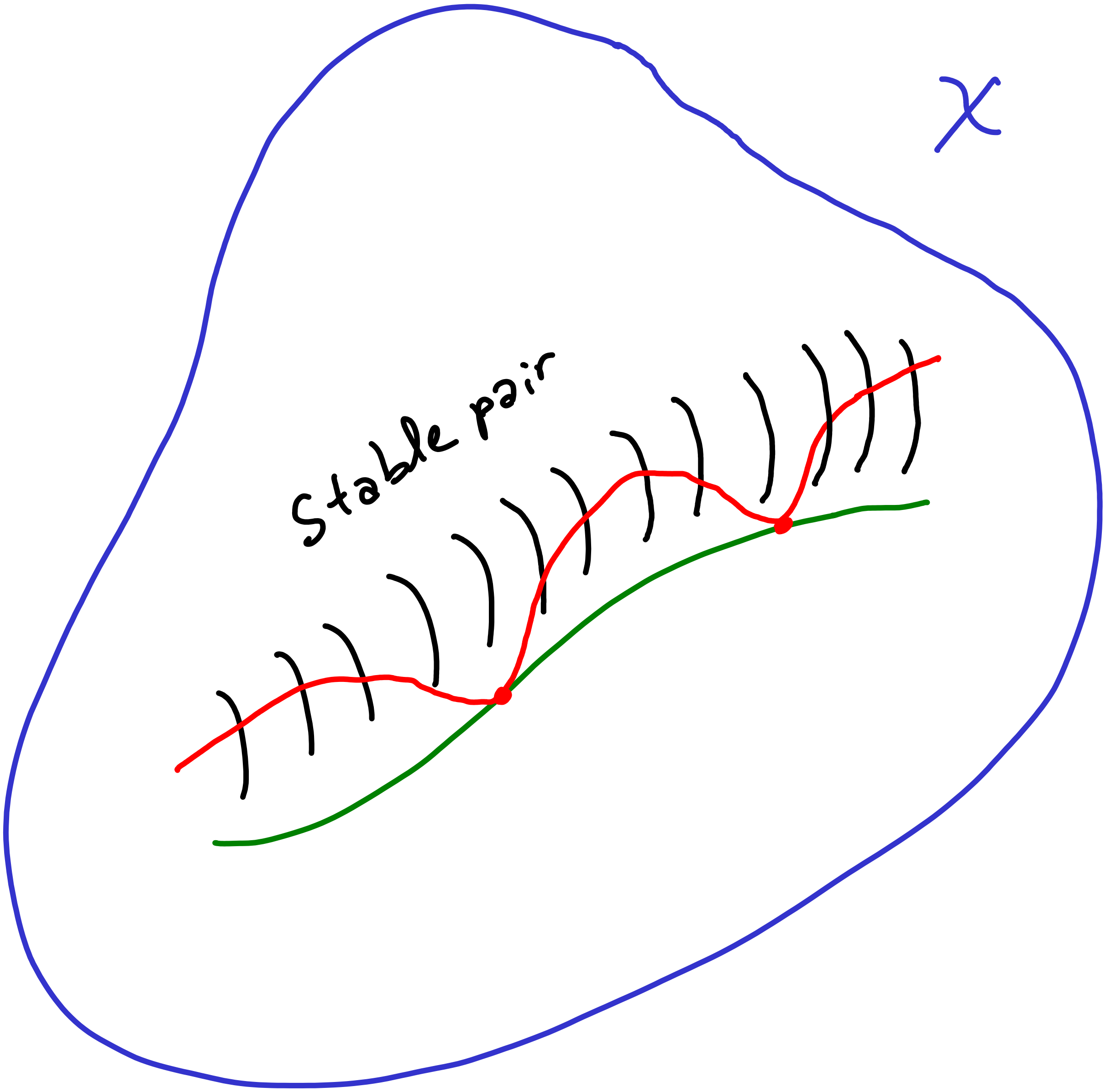
$P_n(X, \beta)$ is the moduli of stable pairs:

$$[\mathcal{F}, s] \in P_n(X, \beta)$$

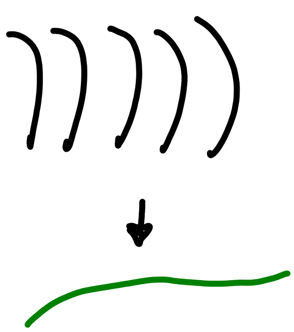
- $\mathcal{F}$ is pure sheaf of dimension 1
- $\mathcal{O}_X \xrightarrow{s} \mathcal{F}$ is a section with coker of dimension 0

Construction of $P_n(X, \beta)$: use Le Potier,

See papers by P– R. Thomas



$\mathcal{F}$



sheaf

$n = \chi(\mathcal{F})$

$\downarrow$

$\mathrm{Supp}(\mathcal{F})$

$\beta = [\mathrm{Supp}(\mathcal{F})]$

Example: $X = \mathbb{P}^3$

Then $P_n(X, d) \supset$ classical locus

which parameterizes ideal objects

$C \subset \mathbb{P}^3$ nonsingular irreducible curve of degree d

$F \to C$ line bundle of degree ℓ

$s \in H^0(C, F)$ a nonzero section

$$n = \ell - \text{genus}(C) + 1$$

Of course $P_n(X, d)$ also parameterizes more degenerate objects

We view $\mathbb{I} = [\mathcal{O}_X \xrightarrow{s} F]$ as an object in $D^b_{Coh}(X)$. Then

$$\mathrm{Def} = \mathrm{Ext}^1_0(\mathbb{I}, \mathbb{I})$$
$$\mathrm{Obs} = \mathrm{Ext}^2_0(\mathbb{I}, \mathbb{I})$$

higher Ext_0's vanish

We have a virtual fundamental class

$[P_n(X, \beta)]^{vir}$ of dimension $\int_\beta c_1(X)$

See "Counting curves via stable pairs" with R. Thomas

$$\begin{tikzcd} & \mathbb{I} \arrow[d] & \\ & P_n(X,\beta)\times X \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ P_n(X,\beta) & & X \end{tikzcd}$$

universal object

Virasoro constraints will take the form of universal relations among descendent invariants

$$\left\langle \mathrm{ch}_{k_1}(\gamma_1)\cdots\mathrm{ch}_{k_r}(\gamma_r)\right\rangle^X_{n,\beta}$$

$$\|$$

$$\int_{[P_n(X,\beta)]^{vir}} \mathrm{ch}_{k_1}(\gamma_1)\cdots\mathrm{ch}_{k_r}(\gamma_r) \in \mathbb{Q}$$

- Constraints are conjectural in most cases

Moreira OOP 2020

Theorem: Stationary constraints hold for X toric.

- The formulas here assume only (p,p) cohomology for X.

Moreira 2020 ⇒ Proposes parallel Virasoro constraints for all simply connected 3 folds X

Moreira 2020

Theorem: Virasoro constraints hold for descendent integrals on $\mathrm{Hilb}^n(S)$ for simply connected surfaces S

Hilbert scheme of points

Newer Results:

Dirk van Bree
Utrecht 2021

Virasoro constraints formulated and checked in many examples for moduli of stable bundles on surfaces

Moreira 2021 : Theorem for rank 2 bundles on curves

Woonam Lim
Moreira
Arkadij Bojko
ETHZ 2022

$\Rightarrow$ Explosion of examples

- Quot schemes on curves/surfaces
- Proofs for stable bundles on surfaces in some cases
- interplay with wall crossing

Algebraic constructions

Let $\mathbb{D}^X$ be the commutative $\mathbb{Q}$-algebra with generators

$$\left\{ ch_i(\gamma) \;\middle|\; i \geq 0,\ \gamma \in H^*(X) \right\}$$

subject to the basic relations

$$ch_i(\lambda \cdot \gamma) = \lambda \cdot ch_i(\gamma), \quad \lambda \in \mathbb{Q}$$

$$ch_i(\gamma + \hat{\gamma}) = ch_i(\gamma) + ch_i(\hat{\gamma}), \quad \gamma, \hat{\gamma} \in H^*(X)$$

In order to define the Virasoro constraints, we require three constructions in $\mathbb{D}^X$:

(i) Define $\mathbb{Q}$-derivations for $k \geq -1$

$$R_k : \mathbb{D}^X \to \mathbb{D}^X$$

by action on the generators

$$R_k(ch_i(\gamma)) = \prod_{n=0}^{k} (i + d(\gamma) - 3 + n) \, ch_{i+k}(\gamma)$$

is complex degree $\gamma \in H^{2d(\gamma)}(X)$

$$R_{-1}(ch_i(\gamma)) = ch_{i-1}(\gamma)$$

Convention $ch_{j<0}(\gamma) = 0$

(ii) Define $ch_a ch_b(\gamma) \in \mathbb{D}^X$ by the following formula

$$ch_a ch_b(\gamma) = \sum_i ch_a(\gamma_i^L)\, ch_b(\gamma_i^R)$$

where $\sum_i \gamma_i^L \otimes \gamma_i^R$ is the Künneth decomposition of

$$\underset{\substack{\uparrow \\ \text{diagonal}}}{\gamma \cdot \Delta} \in H^*(X \times X)$$

The notation

$$(-1)^{d^L d^R}(a+d^L-3)!\,(b+d^R-3)!\;ch_a ch_b(\gamma)$$

will be used for

$$\sum_i (-1)^{d(\gamma_i^L)d(\gamma_i^R)}\cdot(a+d(\gamma_i^L)-3)!\,(b+d(\gamma_i^R)-3)!$$
$$\cdot\; ch_a(\gamma_i^L)\,ch_b(\gamma_i^R)$$

factorials with negative arguments are defined to vanish.

d is always the complex degree

(iii) Define the operator

$$T_k : \mathbb{D}^x \to \mathbb{D}^x$$

by multiplication by the element

$$T_k = -\frac{1}{2} \sum_{a+b=k+2} (-1)^{d^L d^R} (a+d^L-3)!\,(b+d^R-3)!\,\mathrm{ch}_a \mathrm{ch}_b (c_1)$$

$$+ \frac{1}{24} \sum_{a+b=k} a!\, b!\, \mathrm{ch}_a \mathrm{ch}_b (c_1 c_2)$$

- in sums, we require $a, b \geq 0$
- $c_1, c_2 \in H^*(X)$ are the Chern classes of T_X

Virasoro Constraints

Define the constraint operator

$$\mathcal{L}_k = T_k + R_k + (k+1)!\, R_{-1}\, \mathrm{ch}_{k+1}(p)$$

for $k \geq -1$

Virasoro Conjecture [Moreira 00P]

X has only (p,p) cohomology

$\beta \in H_2(X, \mathbb{Z})$ curve class

$D \in \mathbb{D}^X$ is any element

Then, $\langle \mathcal{L}_k(D) \rangle^X_{n,\beta} = 0$ for $k \geq -1$.

Example : $X = \mathbb{P}^3$

$$L_1(D) = \left(-4\, ch_3(H) + R_1 + 2\, ch_2(p)\, R_{-1}\right) D$$

Try $D = ch_3(p)$ and β = Line class L

Then, we obtain

$$-4 \left\langle ch_3(H)\, ch_3(p) \right\rangle^{\mathbb{P}^3}_L$$

$$+\, 12 \left\langle ch_4(p) \right\rangle^{\mathbb{P}^3}_L$$

$$+\, 2 \left\langle ch_2(p)\, ch_2(p) \right\rangle^{\mathbb{P}^3}_L$$

$$\parallel$$

$$0$$

Check

$$-3q + 6q^2 - 3q^3$$

$$+\, q - 10q^2 + q^3$$

$$+\, 2q + 4q^2 + 2q^3$$

$$\parallel$$

$$0$$

Theorem (Moreira OOP 2020)

Nonsingular Projective

Let X be a toric 3fold.

For all $D \in \mathbb{D}_+^X$, ← stationary case

the Virasoro Constraints hold

$$\left\langle L_k(D) \right\rangle_\beta^X = 0 \quad \text{for } k \geq -1.$$

Define $\mathbb{D}_+^X \subset \mathbb{D}^X$ subalgebra

generated by

Stationary descendents

$$\left\{ \mathrm{ch}_i(\gamma) \;\middle|\; i \geq 0,\ \gamma \in H^{>0}(X) \right\}$$

Path of proof : X is a toric 3fold

GW Virasoro constraints hold

Semisimple / Givental-Teleman 2010

GW/Pairs descendent correspondence formula in the stationary case

Pixton-P 2012

OOP 2019

Lose control of descendents of 1 here

Transfer Virasoro constraints from GW theory to stable pairs

Moreira OOP 2020

Actually, we would like to run the whole argument in the other direction.

Main Challenge: Prove the Virasoro constraints for stable pairs directly using the geometry of $P_n(X,\beta)$.

Sub challenge: Control the descendents of $1 \in H^*(X)$.

$\mathrm{ch}_k(1)$ insertions

for the GW/descendent Correspondence:

subject to the natural relations

$$\begin{aligned} \tau_i(\lambda \cdot \gamma) &= \lambda\, \tau_i(\gamma)\,, \\ \tau_i(\gamma + \widehat{\gamma}) &= \tau_i(\gamma) + \tau_i(\widehat{\gamma}) \end{aligned}$$

for $\lambda \in \mathbb{Q}$ and $\gamma, \widehat{\gamma} \in H^*(X)$. The subalgebra $\mathbb{D}^{X+}_{\mathrm{GW}} \subset \mathbb{D}^{X}_{\mathrm{GW}}$ of stationary descendents is generated by

$$\{\, \tau_i(\gamma) \,|\, i \geqslant 0\,,\, \gamma \in H^{>0}(X, \mathbb{Q}) \,\}\,.$$

We will use Getzler's renormalization $\mathfrak{a}_k$ of the Gromov-Witten descendents[7]:

On the GW side

$$\{\tau_k(\gamma)\} \leftrightarrow \{\mathfrak{a}_{k+1}(\gamma)\}$$

$$(9) \qquad \sum_{n=-\infty}^{\infty} z^n \tau_n = \mathsf{Z}^0 + \sum_{n>0} \frac{(\imath u z)^{n-1}}{(1+zc_1)_n} \mathfrak{a}_n + \frac{1}{c_1} \sum_{n<0} \frac{(\imath u z)^{n-1}}{(1+zc_1)_n} \mathfrak{a}_n\,,$$

$$\mathsf{Z}^0 = \frac{z^{-2}u^{-2}}{\mathcal{S}\left(\frac{zu}{\theta}\right)} - z^{-2}u^{-2},$$

where we use standard notation for the Pochhammer symbol

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}\,.$$

For example[8],

$$(10) \qquad \tau_0(\gamma) = \mathfrak{a}_1(\gamma) + \frac{1}{24}\int_X \gamma c_2\,,$$

$$(11) \qquad \tau_1(\gamma) = \frac{\imath u}{2}\mathfrak{a}_2(\gamma) - \mathfrak{a}_1(\gamma \cdot c_1)\,.$$

For $k \geqslant 2$ and $\gamma \in H^{>0}(X)$, we have the general formula

$$(12) \qquad \tau_k(\gamma) = \frac{(\imath u)^k}{(k+1)!}\mathfrak{a}_{k+1}(\gamma) - \frac{(\imath u)^{k-1}}{k!}\left(\sum_{i=1}^{k} \frac{1}{i}\right)\mathfrak{a}_k(\gamma \cdot c_1) + \frac{(\imath u)^{k-2}}{(k-1)!}\left(\sum_{i=1}^{k-1}\frac{1}{i^2} + \sum_{1 \leqslant i < j \leqslant k-1} \frac{1}{ij}\right)\mathfrak{a}_{k-1}(\gamma \cdot c_1^2)\,.$$

0.6. **The GW/PT correspondence for essential descendents.** The subalgebra

$$\mathbb{D}^{X\star}_{\mathrm{PT}} \subset \mathbb{D}^{X+}_{\mathrm{PT}}$$

of *essential descendents* is generated by

$$\{\, \widetilde{\mathsf{ch}}_i(\gamma) \,|\, (i \geqslant 3, \gamma \in H^{>0}(X, \mathbb{Q})) \text{ or } (i = 2, \gamma \in H^{>2}(X, \mathbb{Q})) \,\}\,.$$

While closed formulas for the full GW/PT descendent transformation of [25] are not known in full generality, the stationary theory is much better understood [17].[9] The transformation takes the simplest form when restricted to essential descendents.

[7] We use $\imath$ for the square root of -1. The genus variable u will usually occur together with $\imath$.
[8] The constant term $\frac{1}{24}\int_X \gamma c_2$ in the formula does not contribute unless $\gamma \in H^2(X)$.
[9] See [13, 14] for an earlier view of descendents and descendent transformations.

Stationary GW/Pairs descendent correspondence

The GW/PT transformation restricted to the essential descendents is a linear map

$$\mathfrak{C}^\bullet : \mathbb{D}_{\mathrm{PT}}^{X\bigstar} \to \mathbb{D}_{\mathrm{GW}}^{X}$$

satisfying

$$\mathfrak{C}^\bullet(1) = 1$$

and is defined on monomials by

$$\mathfrak{C}^\bullet\Big(\widetilde{\mathsf{ch}}_{k_1}(\gamma_1)\ldots\widetilde{\mathsf{ch}}_{k_m}(\gamma_m)\Big) = \sum_{P \text{ set partition of } \{1,\ldots,m\}} \prod_{S\in P} \mathfrak{C}^\circ\Big(\prod_{i\in S}\widetilde{\mathsf{ch}}_{k_i}(\gamma_i)\Big).$$

The operations $\mathfrak{C}^\circ$ on $\mathbb{D}_{\mathrm{PT}}^{X\bigstar}$ are

$$\begin{aligned}(13)\quad \mathfrak{C}^\circ\Big(\widetilde{\mathsf{ch}}_{k_1+2}(\gamma)\Big) &= \frac{1}{(k_1+1)!}\mathfrak{a}_{k_1+1}(\gamma) + \frac{(\imath u)^{-1}}{k_1!}\sum_{|\mu|=k_1-1}\frac{\mathfrak{a}_{\mu_1}\mathfrak{a}_{\mu_2}(\gamma\cdot c_1)}{\mathrm{Aut}(\mu)} \\ &+ \frac{(\imath u)^{-2}}{k_1!}\sum_{|\mu|=k_1-2}\frac{\mathfrak{a}_{\mu_1}\mathfrak{a}_{\mu_2}(\gamma\cdot c_1^2)}{\mathrm{Aut}(\mu)} + \frac{(\imath u)^{-2}}{(k_1-1)!}\sum_{|\mu|=k_1-3}\frac{\mathfrak{a}_{\mu_1}\mathfrak{a}_{\mu_2}\mathfrak{a}_{\mu_3}(\gamma\cdot c_1^2)}{\mathrm{Aut}(\mu)},\end{aligned}$$

$$\begin{aligned}(14)\quad \mathfrak{C}^\circ\Big(\widetilde{\mathsf{ch}}_{k_1+2}(\gamma)\widetilde{\mathsf{ch}}_{k_2+2}(\gamma')\Big) &= -\frac{(\imath u)^{-1}}{k_1!k_2!}\mathfrak{a}_{k_1+k_2}(\gamma\gamma') - \frac{(\imath u)^{-2}}{k_1!k_2!}\mathfrak{a}_{k_1+k_2-1}(\gamma\gamma'\cdot c_1) \\ &- \frac{(\imath u)^{-2}}{k_1!k_2!}\sum_{|\mu|=k_1+k_2-2}\max(\max(k_1,k_2),\max(\mu_1+1,\mu_2+1))\frac{\mathfrak{a}_{\mu_1}\mathfrak{a}_{\mu_2}}{\mathrm{Aut}(\mu)}(\gamma\gamma'\cdot c_1),\end{aligned}$$

$$(15)\quad \mathfrak{C}^\circ\Big(\widetilde{\mathsf{ch}}_{k_1+2}(\gamma)\widetilde{\mathsf{ch}}_{k_2+2}(\gamma')\widetilde{\mathsf{ch}}_{k_3+2}(\gamma'')\Big) = \frac{(\imath u)^{-2}|k|}{k_1!k_2!k_3!}\mathfrak{a}_{|k|-1}(\gamma\gamma'\gamma''),\quad |k| = k_1+k_2+k_3.$$

The above sums are over *partitions* of μ of length 2 or 3. The parts of μ are *positive* integers, and we always write

$$\mu = (\mu_1,\mu_2) \quad \text{and} \quad \mu = (\mu_1,\mu_2,\mu_3)$$

with weakly decreasing parts. In equations (13)-(15), we have $k_i \geqslant 0$, and all occurrences of $\mathfrak{a}_0$ and $\mathfrak{a}_{-1}$ are set to 0.

The above formulas for the GW/PT descendent correspondence are proven here from the vertex operator formulas of [17] by a direct evaluation of the leading terms. In the toric case, we have the following explicit correspondence statement[10]

Theorem 6. *Let X be a nonsingular projective toric 3-fold. Let*

$$\prod_{i=1}^m \widetilde{\mathsf{ch}}_{k_i}(\gamma_i) \in \mathbb{D}_{\mathrm{PT}}^{X\bigstar}.$$

[10] A straightforward exercise using our new conventions is to show the abstract correspondence of Theorem 6 is a consequence of [25, Theorem 4]. The novelty of Theorem 6 is the closed formula for the transformation.

Let $\beta \in H_2(X,\mathbb{Z})$ with $d_\beta = \int_\beta c_1(X)$. Then, the GW/PT correspondence defined by formulas (13)-(15) holds:

$$(-q)^{-d_\beta/2}\left\langle \prod_{i=1}^{m} \widetilde{\mathrm{ch}}_{k_i}(\gamma_i)\right\rangle_\beta^{X,\mathrm{PT}} = (-\imath u)^{d_\beta}\left\langle \mathfrak{C}^\bullet\left(\prod_{i=1}^{m} \widetilde{\mathrm{ch}}_{k_i}(\gamma_i)\right)\right\rangle_\beta^{X,\mathrm{GW}},$$

after the change of variables $-q = e^{\imath u}$.

What is $\widetilde{\mathrm{ch}}_k(\gamma)$?

Definition: $\widetilde{\mathrm{ch}}_k(\gamma) = \mathrm{ch}_k(\gamma) + \frac{1}{24}\mathrm{ch}_{k-2}(\gamma \cdot c_2)$

2nd Chern class of T_X

These formulas (and their proof in the toric case) use a lot of previous work over the past 15 years.

Okounkov-P
MOOP
Pixton-P
OOP / MorOOP

GW/Hurwitz
GW/DT toric
Toric descendent GW/Pairs
Final formulas

$\mathrm{Hilb}^n(S)$ of a surface S

If the 3fold X is of the form

$$X = S \times \mathbb{P}^1$$

Simply connected nonsingular projective surface

and the curve class is $\beta = n[\mathbb{P}^1]$

then $P_n(S \times \mathbb{P}^1, n[\mathbb{P}^1]) = \mathrm{Hilb}^n(S)$.

Moreover $[P_n(S \times \mathbb{P}^1, n[\mathbb{P}^1])]^{vir}$ is

the usual fundamental class of $\mathrm{Hilb}^n(S)$.

The Virasoro constraints for stable pairs on $S \times \mathbb{P}^1$ specialize to Virasoro constraints for certain descendent integrals on $\mathrm{Hilb}^n(S)$.

Moreira's paper "Virasoro conjecture for stable pairs descendent theory of simply connected 3 folds"

What is a descendent for $\mathrm{Hilb}^n(S)$?

$$\Sigma_n \quad \text{universal subscheme}$$

$$
\begin{tikzcd}
 & \Sigma_n \arrow[d, phantom, "\cap"] & \\
 & \mathrm{Hilb}^n(S) \times S \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
\mathrm{Hilb}^n(S) & & S
\end{tikzcd}
$$

for $\gamma \in H^*(S)$

$$\mathrm{ch}_k(\gamma) = \pi_{1*}\left(\mathrm{ch}_k(\mathcal{O}_{\Sigma_n} - \mathcal{O}_{H\times S}) \cdot \pi_2^*(\gamma)\right)$$

Theorem [Moreira 2020]

S is a simply connected surface

$$\int_{\mathrm{Hilb}^n(S)} L_k \left(\mathrm{ch}_{k_1}(\gamma_1) \cdots \mathrm{ch}_{k_r}(\gamma_r) \right) = 0$$

where $L_k = T_k + R_k + S_k$

very similar to T_k, R_k for 3 folds, but now involve the Hodge grading

slightly different

To define S_k :

$$R_{-1}[\alpha]\,(ch_i(\gamma)) = ch_{i-1}(\alpha\cdot\gamma)$$

derivation on algebra $\mathbb{D}^S$ with generators $\{ch_i(\gamma)\}$

$$S_k = (k+1)! \sum_{p_i^L=0} R_{-1}[\gamma_i^L]\, ch_{k+1}(\gamma_i^R)$$

where the sum runs over the terms

$\gamma_i^L \otimes \gamma_i^R$ of the Künneth decomposition of the diagonal $\Delta \subset S \times S$ where

$$\gamma_i^L \in H^{0,q_i^L}(S).$$

Happy Birthday to Ezra from Leysin!

Workshop on the moduli space of curves

Bernese
Summit

The End