ENUMERATIVITY OF VIRTUAL TEVELEV DEGREES

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Abstract. Tevelev degrees in Gromov-Witten theory are defined whenever there are virtually a finite number of genus g maps of fixed complex structure in a given curve class β through n general points of a target variety X. These virtual Tevelev degrees often have much simpler structure than general Gromov-Witten invariants. We explore here the question of the enumerativity of such counts in the asymptotic range for large curve class β. A simple speculation is that for all Fano X, the virtual Tevelev degrees are enumerative for sufficiently large β. We prove the claim for all homogeneous varieties and all hypersurfaces of sufficiently low degree (compared to dimension). As an application, we prove a new result on the existence of very free curves of low degree on hypersurfaces in positive characteristic.

Contents

1. Tevelev degrees

1.1. **Definitions.** Let X be a nonsingular, projective, complex algebraic variety of dimension r, and let $\beta \in H_2(X,\mathbb{Z})$ be a class satisfying

$$\int_\beta c_1(T_X) > 0\,. \tag{1}$$

If X is Fano, the positivity (1) is always satisfied for classes of curves. Fano varieties will be our main interest here.

Let $g \geq 0$ and $n \geq 0$ be in the stable range $2g-2+n>0$ so that the moduli space of stable curves $\overline{\mathcal{M}}_{g,n}$ is well-defined. The moduli space of stable maps $\overline{\mathcal{M}}_{g,n}(X,\beta)$ has virtual dimension equal to the dimension of $\overline{\mathcal{M}}_{g,n} \times X^n$ if and only if

$$\int_\beta c_1(T_X) = r(n+g-1)\,. \tag{2}$$

If the dimension constraint (2) holds, we expect to find a finite number of maps from a fixed curve $(C,p_1,\ldots,p_n)$ of genus g to X of curve class β where the p_i are incident to general

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points of X. Tevelev degrees in Gromov-Witten theory are defined to be the corresponding virtual count.

Definition 1. *Let $g \geq 0$, $n \geq 0$, and $\beta \in H_2(X, \mathbb{Z})$ satisfy $2g - 2 + n > 0$ and the dimension constraint* (2). *Let*

$$\tau : \overline{\mathcal{M}}_{g,n}(X, \beta) \to \overline{\mathcal{M}}_{g,n} \times X^n$$

*be the canonical morphism. The **virtual Tevelev degree** $\mathsf{vTev}^X_{g,\beta,n} \in \mathbb{Q}$ is defined by*

$$\tau_*([\overline{\mathcal{M}}_{g,n}(X, \beta)]^{vir}) = \mathsf{vTev}^X_{g,\beta,n} \cdot [\overline{\mathcal{M}}_{g,n} \times X^n] \in A^0(\overline{\mathcal{M}}_{g,n} \times X^n)\,.$$

Here, $[\,]^{vir}$ and $[\,]$ denote the virtual and usual fundamental classes, respectively. If the dimension constraint (2) fails, we define $\mathsf{vTev}^X_{g,\beta,n}$ to vanish.

We say that the virtual Tevelev count for g and β is **well-posed** if

$$n(g, \beta) = 1 - g + \frac{1}{r} \int_\beta c_1(T_X)$$

is a non-negative integer satisfying $2g - 2 + n(g, \beta) > 0$. We will use the notation

$$\mathsf{vTev}^X_{g,\beta} = \mathsf{vTev}^X_{g,\beta,n(g,\beta)}$$

in the well-posed case.

Let $(C, p_1, \ldots, p_n)$ be a fixed *general* nonsingular curve of genus g with n distinct points. Let $x_1, \ldots, x_n \in X$ be n general points.

Definition 2. *If the virtual Tevelev count for g and β is well-posed and the actual count of maps*

$$f : (C, p_1, \ldots, p_n) \to X$$

*in class β satisfying $f(p_i) = x_i$ is finite and transverse, we define the **geometric Tevelev degree** $\mathsf{Tev}^X_{g,\beta,n} \in \mathbb{Z}$ to be the set-theoretic count.*

Equivalently, $\mathsf{Tev}^X_{g,\beta,n}$ is defined by the set-theoretic degree of the morphism

$$\tau_{\mathcal{M}} : \mathcal{M}_{g,n}(X, \beta) \to \mathcal{M}_{g,n} \times X^n$$

computed along a general fibre (which is required to be everywhere transverse). Transversality here is the condition that $d\tau_{\mathcal{M}}$ is an isomorphism on Zariski tangent spaces. The definition ignores all components of $\mathcal{M}_{g,n}(X, \beta)$ which fail to dominate $\mathcal{M}_{g,n} \times X^n$.

The virtual Tevelev degree is **enumerative** if the geometric Tevelev degree is well-defined and

$$\mathsf{vTev}^X_{g,\beta,n} = \mathsf{Tev}^X_{g,\beta,n}\,.$$

Remark 3. The virtual Tevelev count is never well-posed for constant maps: if $\int_\beta c_1(T_X) = 0$ and $n(g, 0) \geq 0$, then g must be 0 or 1 with

$$2g - 2 + n(g, 0) < 0\,.$$

The geometric Tevelev degree therefore requires a nonzero class β.

Remark 4. Stable maps with automorphisms *never* occur in a general transverse fiber of $\tau_{\mathcal{M}}$ when the geometric Tevelev degree is well-defined. Such automorphisms could only occur in the cases

$$(g, n) = (1, 1) \text{ or } (2, 0)\,. \tag{3}$$

In both cases (3), a stable map to X in a general fiber of $\tau_{\mathcal{M}}$ with a nontrivial automorphism must factor through a map to $\mathbb{P}^1$. Using the infinite automorphism group of $\mathbb{P}^1$, we see the finiteness condition for the geometric Tevelev count is violated.

1.2. **Calculations of virtual Tevelev degrees.** While Gromov-Witten invariants are in general rarely enumerative (especially in higher genus) and complicated to compute, the situation is much better for virtual Tevelev degrees.

• The projective space case $X = \mathbb{P}^r$ has a particularly simple answer:

$$\mathsf{vTev}^{\mathbb{P}^r}_{g,d} = (r+1)^g \tag{4}$$

whenever the virtual Tevelev count is well-posed [4, 5, 24].

• For low degree hypersurfaces, a similar result is true.

Theorem 5. [6] *Let $X_e \subset \mathbb{P}^{r+1}$ be a nonsingular hypersurface of degree $e \geq 3$ and dimension $r \geq 2e-3$. We index curve classes of X_e by their associated degree d in $\mathbb{P}^{r+1}$. Then,*

$$\mathsf{vTev}^{X_e}_{g,d} = ((e-1)!)^{n(g,d)} \cdot (r+2-e)^g \cdot e^{(d-n(g,d))e-g+1}$$

whenever the virtual Tevelev count is well-posed.

• The $e = 2$ case of quadric hypersurfaces $Q^r \subset \mathbb{P}^{r+1}$ takes a special form. Let

$$\delta = \begin{cases} 1 & \text{if } r \text{ is odd,} \\ 2 & \text{if } r \text{ is even.} \end{cases}$$

Theorem 6. [6] *For quadrics of dimension $r \geq 3$,*

$$\mathsf{vTev}^{Q^r}_{g,d} = \frac{(2r)^g + (-1)^d (2\delta)^g}{2}$$

whenever the virtual Tevelev count is well-posed.

The method of [6] expresses the virtual Tevelev degrees explicitly in terms of the small quantum cohomology ring of X. When $QH^*(X)$ is sufficiently well known, exact calculations are possible. For further recent progress on the Gromov-Witten of hypersurfaces, see [1, 15].

1.3. **Enumerativity.** Our main topic here is the enumerativity of virtual Tevelev degrees. When β is sufficiently large, enumerativity is much more likely as evidenced by the following example, where $X = \mathbb{P}^1$.

Theorem 7. [9, 12] *Suppose the virtual Tevelev count is well-posed. Then*

$$\begin{aligned} \mathsf{Tev}^{\mathbb{P}^1}_{g,d} &= 2^g - 2\sum_{i=0}^{-\ell-2} \binom{g}{i} + (-\ell-2)\binom{g}{-\ell-1} + \ell\binom{g}{-\ell} \\ &= \int_{Gr(2,d+1)} \sigma_1^g \cdot \left[\sum_{a+b=2d-2-g} \sigma_a \sigma_b \right], \end{aligned}$$

where $\ell = d-g-1$ in the first formula.

When $\ell \leq 0$, all terms except 2^g in the first formula are interpreted to be zero. In particular, virtual Tevelev degrees for $\mathbb{P}^1$ are enumerative if $d \geq g+1$, but not in general. For further results related to moduli spaces of Hurwitz covers, see [8].

More generally, virtual Tevelev degrees for $\mathbb{P}^r$ are enumerative whenever $d \geq rg + r$ but not when $d = r + \frac{gr}{r+1}$ is as small as possible [12]. The geometric Tevelev degree in the case $d = r + \frac{gr}{r+1}$ recovers Castelnuovo's count of linear series of minimal degree [7]. The general computation of geometric Tevelev degrees for $\mathbb{P}^r$ for intermediate d is in completed in [20].

These current developments on Tevelev degrees for $\mathbb{P}^1$ and higher dimensional projective spaces may be viewed (in part) as developing a theory intiated by Castelnuovo in the 19^{th} century. The question of connecting classical counting to virtual counting can be formulated as follows.

Question 8. *For which X does the following property hold: $\mathsf{vTev}^X_{g,\beta}$ is enumerative whenever well-posed and $\int_\beta c_1(T_X)$ is sufficiently large (depending on g)?*

Remark 9. The 2^g formula for $\mathbb{P}^1$ is connected to many directions in geometry and physics, see Tevelev's article [26]. In the Gromov-Witten theory of $\mathbb{P}^1$, the 2^g formula appeared in Janda's work [16]. The geometric Tevelev degrees for $\mathbb{P}^r$ for large curve classes were studied earlier by Bertram, Daskalopoulos, and Wentworth [4, 5] using the classical geometry of the Quot scheme before the development of the virtual fundamental class. To connect the Quot scheme fully (for all curve classes) to the Gromov-Witten calculation (4), a straightforward path is to consider the virtual fundamental class of the Quot scheme [21] and then apply the comparison result [22, Theorem 3]. Alternatively, formula (4) is a direct consequence of the Vafa-Intriligator formula [24].

1.4. **Main results.** We have positive answers to Question 8 for homogeneous spaces and hypersurfaces. Together with the above calculations of $\mathsf{vTev}^X_{g,d}$ from [6], we obtain calculations of geometric Tevelev degrees in many new cases.

Theorem 10. *Let $X = G/P$ be a homogeneous space for a linear algebraic group. Then, for fixed g, the virtual Tevelev degree $\mathsf{vTev}^X_{g,\beta}$ is enumerative whenever well-posed and $\int_\beta c_1(T_X)$ is sufficiently large.*

In case $g = 0$, a stronger result holds for $X = G/P$: the virtual Tevelev degrees $\mathsf{vTev}^X_{0,\beta}$ are enumerative for *all* positive curve classes β. The stronger genus 0 claims follows easily from the unobstructedness of genus 0 stable maps to G/P. In case X is a Grassmannian, the enumerativity claim of Theorem 10 is known from the results of [4, 5] and the comparison results of [22].

Theorem 11. *Let $X \subset \mathbb{P}^{r+1}$ be a nonsingular hypersurface of degree $e \geq 3$ and dimension*

$$r > (e+1)(e-2)\,.$$

Then, for fixed g, the virtual Tevelev degree $\mathsf{vTev}^X_{g,\beta}$ is enumerative whenever well-posed and $\int_\beta c_1(T_X)$ is sufficiently large.

In case $g = 0$ and $r > (e+1)(e-2)$, we again have a stronger result: the virtual Tevelev degrees $\mathsf{vTev}^X_{0,\beta}$ are enumerative for *all* positive curve classes β, see Corollary 33. If X is a *very general* hypersurface, the stronger claim follows from the fact that $\overline{M}_{0,n}(X,\beta)$ is irreducible of the expected dimension for $r > e$ and all positive curve classes β [2, 14, 23]. Our proof works for *all* X in the more restrictive range for e.

It is natural to hope for the following result which we formulate as a speculation.

Speculation 12. *Let X be a nonsingular projective Fano variety. For fixed g, the virtual Tevelev degree $\mathsf{vTev}^X_{g,\beta}$ is enumerative whenever well-posed and $\int_\beta c_1(T_X)$ is sufficiently large.*

In case $g = 0$, $\mathsf{vTev}^X_{0,\beta}$ is enumerative whenever well-posed for all positive curve classes β in all Fano examples that we know the speculation to be true.[1]

As we will see, there are two main difficulties in proving a general enumerativity statement for Fano varieties X:

• The first concerns controlling the excess dimensions of families of *general* curves of positive genus in X. When $X = \mathbb{P}^r$, Brill-Noether theory provides optimal statements (see Remark 15), but we do not have such results in general.

• The second concerns controlling the excess dimensions of families of rational curves. Similar issues for hypersurfaces are studied in [2, 14, 23] in characteristic 0 and in [25] in positive characteristic.

Using our study of the enumerativity of virtual Tevelev degrees of hypersurfaces, we can can prove a new result on the existence of very free rational curves in characterstic p (where p does not divide the virtual Tevelev degree). The positive characteristic results are presented in Section 5.

1.5. **Acknowledgments.** Our project started during a visit to HU Berlin by R.P. in July 2021 and continued at the *Helvetic Algebraic Geometry Seminar* in Geneva and at the *Forschungsinsitut für Mathematik* at ETH Zürich in August 2021. We thank A. Buch, A. Cela, G. Farkas, D. Ranganathan, E. Riedl, and J. Schmitt for many discussions about Tevelev degrees in various contexts. The application to very free rational curves on hypersurfaces in characteristic p was suggested to us by J. Starr.

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2. Analysis of the moduli space of stable maps

2.1. **Overview.** Let X be a nonsingular projective variety of dimension r. Let $g \geq 0$ be the genus, and let $\beta \in H_2(X, \mathbb{Z})$ be an effective curve class.

We study here the enumerativity of virtual Tevelev degrees when β (or, equivalently, n) is sufficiently large. There are two main aspects of the analysis:

[1] Speculation 12 was proposed in 2021. In 2023, counterexamples to Speculation 12 have been constructed by Beheshti-Lehmann-Riedl-Starr-Tanimoto [3] when X is a special Fano hypersurface of large degree. We thank Eric Riedl for communicating these examples to us. A revised speculation should perhaps include a condition on the Fano index.

(i) We must control the fibers of τ when restricted to the open locus

$$\mathcal{M}_{g,n}(X,\beta) \subset \overline{\mathcal{M}}_{g,n}(X,\beta)$$

of stable maps with nonsingular domains in order to verify that the geometric Tevelev degrees are well-defined.

(ii) We must show that a general fiber of τ contains no stable maps at the boundary.

2.2. **Stable maps with nonsingular domains.** We start with a criterion for unobstructedness of maps of nonsingular curves C to a nonsingular projective variety X. We do not require X to be Fano in Section 2.2.

Proposition 13. *Suppose* $[f : (C, p_1, \ldots, p_n) \to X] \in \mathcal{M}_{g,n}(X,\beta)$ *lies over a point*

$$(C, p_1, \ldots, p_n, x_1, \ldots, x_n) \in \mathcal{M}_{g,n} \times X^n$$

and the evaluation map

$$\mathrm{ev} : \mathcal{M}_{g,n}(X,\beta) \to X^n$$

is surjective on tangent spaces at $[f]$*. Assume further that* $(C, p_1, \ldots, p_n) \in \mathcal{M}_{g,n}$ *is general. If* $n \geq g+1$*, or equivalently* $\int_\beta c_1(T_X) \geq 2gr$*, then* $H^1(C, f^*T_X) = 0$*.*

Proof. Let v be a non-zero tangent vector of X at x_1. By assumption, there exists a tangent vector of $\mathcal{M}_{g,n}(X,\beta)$ at $[f]$ mapping to $(v, 0, \ldots, 0) \in T_{(x_1,\ldots,x_n)}X^n$. This is equivalent to the data of a section $\phi_v : \mathcal{O}_C(\sum_{i=2}^n p_i) \to f^*T_X$ evaluating to $v \in T_{x_1}X$ at p_1.

Varying over a basis of $T_{x_1}X$, we obtain a map of vector bundles on C,

$$\phi : \mathcal{O}_C\left(\sum_{i=2}^n p_i\right)^{\oplus r} \to f^*T_X\,,$$

that is surjective at p_1 and therefore generically surjective. Thus, the induced map on H^1 is surjective.

On the other hand, we claim that $H^1(\mathcal{O}_C(\sum_{i=2}^n p_i)) = 0$, which implies the needed conclusion. The H^1-vanishing is an open condition, so it suffices to degenerate to the situation in which the p_i become equal to a single general point p. If $h^1(\mathcal{O}_C((n-1)p)) > 0$, then C is a general curve possessing a linear series V of degree $d = n-1 \geq g$ and rank $s \geq d-g+1$, ramified to order $d-1$ at p. The pointed Brill-Noether number of V is

$$\widehat{\rho} = g - (s+1)(g-d+s) - (d-1) = -s(s-(d-g)+1) + 1 < 0,$$

as $d \geq g$ and $s \geq 1$, contradicting the pointed Brill-Noether theorem [11, Proposition 1.2]. □

If $n \geq 2g$, we also obtain the conclusion for *arbitrary* pointed curves $(C, p_1, \ldots, p_n)$, as $H^1(\mathcal{O}_C(\sum_{i=2}^n p_i)) = 0$ for degree reasons. When $g = 0$, we must assume further that $n \geq 1$ in order to choose the point x_1.

Suppose $\mathsf{vTev}^X_{g,\beta}$ is well posed, and let $n = n(g,\beta)$. Let

$$(C, p_1, \ldots, p_n, x_1, \ldots, x_n) \in \mathcal{M}_{g,n} \times X^n$$

be a general point.

Proposition 14. *If* $n \geq g+1$*, then there are finitely many maps*

$$[f : (C, p_1, \ldots, p_n) \to X] \in \mathcal{M}_{g,n}(X,\beta)$$

lying over $(C, p_1, \ldots, p_n, x_1, \ldots, x_n) \in \overline{\mathcal{M}}_{g,n} \times X^n$*, and all such maps are transverse.*

Proof. Suppose $[f : (C, p_1, \ldots, p_n) \to X] \in \mathcal{M}_{g,n}(X, \beta)$ is such a map. Then, $[f]$ must lie on a component of $Z \subset \mathcal{M}_{g,n}(X, \beta)$ dominating $\mathcal{M}_{g,n} \times X^n$. In particular, the evaluation map

$$\text{ev} : \mathcal{M}_{g,n}(X, \beta) \to X^n$$

is dominant on Z, so the map ev is surjective on tangent spaces at $[f]$ (as the x_i are general).

By Proposition 13, $\mathcal{M}_{g,n}(X, \beta)$ is nonsingular of the expected dimension at $[f]$. Therefore, Z is étale over $\mathcal{M}_{g,n} \times X^n$ at $[f]$. Finiteness and transversality then follow. □

By Proposition 14, in the well-posed case with $n \geq 2g$, the geometric Tevelev degree is well-defined and equal to the degree of

$$\tau : \mathcal{M}_{g,n}(X, \beta) \to \mathcal{M}_{g,n} \times X^n .$$

Remark 15. The bound $n \geq g + 1$ is not sharp. For example, if $X = \mathbb{P}^r$, then by the Brill-Noether Theorem, geometric Tevelev degrees are also well-defined whenever $n \geq r + 1$, in which case $f : C \to \mathbb{P}^r$ is necessarily non-degenerate.

2.3. **Stable maps with singular domains.** We now assume X is a projective Fano variety of dimension r.

Lemma 16. *For maps from $\mathbb{P}^1$ to X, we have the following basic results:*

(a) If $\text{ev}_1 : \mathcal{M}_{0,2}(X, \beta) \to X$ is dominant (on every component of the source), then $\text{ev}_2 : \mathcal{M}_{0,2}(X, \beta) \to X$ is also dominant, and $\mathcal{M}_{0,2}(X, \beta)$ is generically nonsingular of the expected dimension.

(b) If f^*T_X is globally generated for every $f : \mathbb{P}^1 \to X$, then every boundary stratum of $\overline{M}_{0,n}(X, \beta)$ is nonsingular of the expected dimension.

Proof. First, consider claim (a). At a generic point $[f : \mathbb{P}^1 \to X]$ of any irreducible component of $\mathcal{M}_{0,2}(X, \beta)$ on which ev_1 is dominant, all summands of $T_X|_{\mathbb{P}^1}$ are non-negative, so ev_2 is also dominant on that component. Moreover, $H^1(\mathbb{P}^1, T_X) = 0$ at $[f]$, so $\mathcal{M}_{0,2}(X, \beta)$ is generically nonsingular of the expected dimension.

Claim (b) follows from the same argument as (a), by induction on the number of components of the domain curve in the stratum in question. See, for example, [13]. □

Definition 17. *Let $s(X) > 0$ be the smallest positive integer for which there exists an effective curve class $\beta \in H_2(X, \mathbb{Z})$ such that*

$$s(X) = \int_\beta c_1(T_X)$$

and $ev_1 : \overline{M}_{0,1}(X, \beta) \to X$ is surjective.

Let $t(X) > 0$ be the smallest positive integer for which there exists an effective curve class $\beta \in H_2(X, \mathbb{Z})$ such that

$$t(X) = \int_\beta c_1(T_X) .$$

Definition 18. *We say X has property $(\star)_g$ if, for every curve class β and for every $[f : C \to X] \in \mathcal{M}_g(X, \beta)$, we have*

$$h^1(C, f^*T_X) \leq K_{g,X},$$

for some constant $K_{g,X}$ depending only on g and X (so not *on C, β, and f).*

We say that X has property $(\star\star)_g$ if

$$\liminf \frac{\int_\beta c_1(T_X)}{h^1(C, f^*T_X)} > r - s(X)\,.$$

where we range over all curve classes β and $[f : C \to X] \in \mathcal{M}_g(X, \beta)$ and order by $\int_X \beta \cdot c_1(T_X)$.

Property $(\star\star)_g$ is automatically satisfied if $(\star)_g$ is, or if $s(X) > r$.

Example 19. If $X = G/P$, then T_X is globally generated, so

$$h^1(C, f^*T_X) \le gr,$$

In particular, X satisfies $(\star)_g$ for any g.

Example 20. *We will see in later in Proposition 25 and the proof of Theorem 11 that hypersurfaces $X_e \subset \mathbb{P}^{r+1}$ of sufficiently low degree e satisfy $(\star\star)_g$.*

We are now ready to show that, under certain hypotheses, stable maps $[f : C \to X]$ at the boundary of the moduli space cannot contribute to the virtual Tevelev degree if n is sufficiently large. We first consider the case in which C is the union of a nonsingular component and disjoint nonsingular rational tails, each containing a marked point p_i.

Proposition 21. *Suppose X satisfies property $(\star\star)_g$ and n is sufficiently large (depending on X and g). Let*

$$\mathcal{M}_\Gamma \subset \overline{\mathcal{M}}_{g,n}(X, \beta)$$

be a locally closed boundary stratum consisting of stable maps $f : C \to X$ such that C is the union of a nonsingular genus g curve C_0 and disjoint nonsingular rational tails $R_1, \ldots, R_{n-m}$, such that $x_i \in R_i$ for $i = 1, 2, \ldots, n - m$ and $x_i \in C_0$ for $i = n - m + 1, \ldots, n$. (See Figure 1.)

Suppose further that the virtual dimension of $\overline{\mathcal{M}}_{g,n}(X, \beta)$ is at most the dimension of $\overline{\mathcal{M}}_{g,n} \times X^n$,

$$r(1-g) + \int_\beta c_1(T_X) \le rn\,, \tag{5}$$

and furthermore that, if equality holds, then $n - m > 0$.

Then, $\dim(\mathcal{M}_\Gamma) < \dim(\overline{\mathcal{M}}_{g,n} \times X^n)$. In particular, $\mathcal{M}_\Gamma$ fails to dominate $\overline{\mathcal{M}}_{g,n} \times X^n$.

Proof. If $n - m = 0$ and $n \ge g + 1$, then $\mathcal{M}_\Gamma = \mathcal{M}_{g,n}(X, \beta)$ has expected dimension by Proposition 13, so we obtain the claim.

If $m \ge g+1$ and $n-m > 0$, then $\mathcal{M}_{g,n}(X, f_*[C_0])$ has expected dimension by the argument of Proposition 13. Moreover, $\mathcal{M}_{0,2}(X, f_*[R_i])$ has expected dimension by Lemma 16(a), and any incidence condition on the nodal point of R_i imposes the expected number of conditions. Thus, the boundary stratum $\mathcal{M}_\Gamma$ has expected dimension, which is strictly less than that of $\mathcal{M}_{g,n}(X, \beta)$, so we are again done.

Assume now that $m < g + 1$, so in particular m is bounded above by a constant. Applying Lemma 16(a) again, we find

$$\dim(\mathcal{M}_\Gamma) \le \dim(\overline{\mathcal{M}}_{g,n} \times X^n) + h^1(C_0, T_X) - n + O(1)\,.$$

where the term $O(1)$ denotes a constant upper bound depending only on g and X.

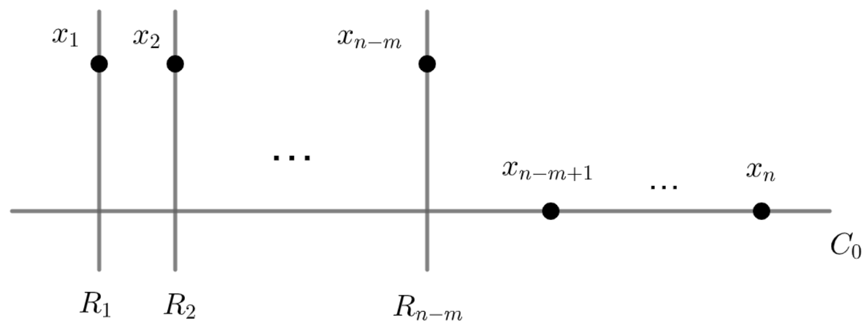

FIGURE 1. Domain of a stable map given by a nonsingular C_0 attached to disjoint nonsingular rational tails

Furthermore, we have

$$\int_{[C_0]} c_1(T_X) \leq \int_\beta c_1(T_X) - n \cdot s(X)$$
$$\leq (r - s(X))n + O(1)$$

by (9), where again the term $O(1)$ depends only on g and X and not on C, f, β. Since X satisfies property $(\star\star)_g$, we conclude $\dim(\mathcal{M}_\Gamma) < \dim(\overline{\mathcal{M}}_{g,n} \times X^n)$ for n sufficiently large, as desired. ☐

Under stronger assumptions as in Proposition 21, we now rule out stable maps at the boundary in a general fiber of τ with arbitrary topology.

Proposition 22. *Suppose X satisfies property $(\star\star)_g$ and n is sufficiently large (depending on X and g). Let*

$$\mathcal{M}_\Gamma \subset \overline{\mathcal{M}}_{g,n}(X, \beta)$$

be a locally closed boundary stratum consisting of stable maps $f : C \to X$ such that C is the union of a nonsingular genus g curve C_0, disjoint trees of nonsingular rational curves $T_1, \ldots, T_{n-m}, T'_1, T'_2, \ldots, T'_k$, such that $x_i \in T_i$ for $i = 1, 2, \ldots, n-m$ and $x_i \in C_0$ for $i = n-m+1, \ldots, n$. (See Figure 2.)

Suppose the virtual dimension of $\overline{\mathcal{M}}_{g,n}(X, \beta)$ is equal to the dimension of $\overline{\mathcal{M}}_{g,n} \times X^n$,

$$r(1-g) + \int_\beta c_1(T_X) = rn\,.$$

Assume further that at least one of the following two conditions hold:

(i) f^*T_X is globally generated for every $f : \mathbb{P}^1 \to X$,
(ii) $s(X) + t(X) \geq r + 1$.

Then, $\mathcal{M}_\Gamma$ fails to dominate $\overline{\mathcal{M}}_{g,n} \times X^n$.

Proof. For (i), we may apply the proof of Proposition 21 using the stronger Lemma 16(b) instead of Lemma 16(a) to conclude.

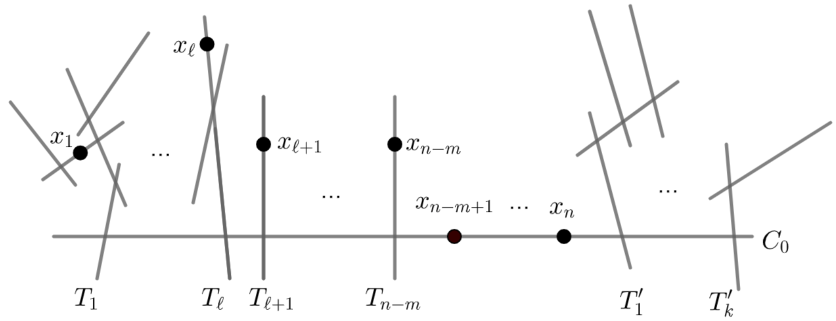

FIGURE 2. Arbitrary domain of a stable map whose stable contraction is nonsingular

Consider now (ii). Without loss of generality, let ℓ be such that among the trees $T_1, \ldots, T_{n-m}$, those containing more than one rational curve are $T_1, \ldots, T_\ell$. We have

$$\ell \leq \frac{1}{(s(X)+t(X))} \int_\beta c_1(T_X) \leq \frac{rn}{r+1} + O(1)\,. \tag{6}$$

where the constant term $O(1)$ depends only on g and X.

We now apply Proposition 21 to the stable maps $\widehat{f} : \widehat{C} \to X$ obtained by deleting $T_1, \ldots, T_\ell$, $x_1, \ldots, x_\ell$, and $T'_1, \ldots, T'_k$ from C. Since X is Fano, $\int_{T'_i} c_1(T_X) \geq 0$. We have

$$\begin{aligned}\int_{[\widehat{C}]} c_1(T_X) &\leq \int_\beta c_1(T_X) - \ell(s(X)+t(X)) \\ &= r(n+g-1) - \ell(s(X)+t(X)) \\ &< r(n-\ell+g-1)\,,\end{aligned}$$

so Proposition 21 indeed applies. We find that when $n-\ell$ is sufficiently large (which occurs whenever n is sufficiently large, by (6)), the space of stable maps $\widehat{C} \to X$ does not dominate $\overline{\mathcal{M}}_{g,n-\ell} \times X^{n-\ell}$. In particular, $\mathcal{M}_\Gamma$ fails to dominate $\overline{\mathcal{M}}_{g,n} \times X^n$. □

3. Asymptotic enumerativity

We can now state our most general result concerning the enumerativity of virtual Tevelev degrees.

Theorem 23. *Suppose that X satisfies property $(\star\star)_g$ and additionally that one of the following two conditions hold:*

(i) f^*T_X is globally generated for every $f : \mathbb{P}^1 \to X$,

(ii) $s(X)+t(X) \geq r+1$.

Then, $\mathsf{vTev}^X_{g,\beta}$ is enumerative whenever $\int_\beta c_1(T_X)$ is sufficiently large (depending only on X and g).

Proof. The requirement that $\int_\beta c_1(T_X)$ be sufficiently large is equivalent to the requirement that n be sufficiently large.

Proposition 22 shows that the general fiber of

$$\tau : \overline{\mathcal{M}}_{g,n}(X,\beta) \to \overline{\mathcal{M}}_{g,n} \times X^n$$

consists only of stable maps with nonsingular domains. Proposition 14 then shows that the general fiber consists of finitely many reduced points, the number of which is equal to $\mathsf{Tev}^X_{g,\beta}$. □

Proof of Theorem 10. Every $X = G/P$ satisfies property $(\star)_g$ and hence property $(\star\star)_g$, and also satisfies property (i) above. The result is then immediate from Theorem 23. □

Example 24. Every X satisfying $s(X) \geq r+1$ satisfies $(\star\star)_g$ and (ii), so for any such X, virtual Tevelev degrees are enumerative for β sufficiently large.

In order to prove Theorem 11, we show that if X is a complete intersection of low degree in a Fano variety Y satisfying $(\star)_g$, then X satisfies $(\star\star)_g$. For simplicity, we assume the Picard rank of Y is 1.

Proposition 25. *Let Y be a nonsingular projective Fano variety of Picard rank 1 with positive generator $\mathcal{O}(1) \in \mathrm{Pic}(Y)$. Suppose further that Y satisfies property $(\star)_g$.*

Let $X \subset Y$ is a nonsingular complete intersection of dimension r and degree $(e_1, \ldots, e_k)$ (with the degrees computed with respect to $\mathcal{O}(1)$). Suppose, for all curve classes $\beta \in H_2(X,\mathbb{Z})$, that

$$\frac{\int_\beta c_1(T_X)}{\int_\beta c_1(\mathcal{O}(1))} > (r - s(X)) \sum_{i=1}^{k} e_i \,. \tag{7}$$

Then, X satisfies property $(\star\star)_g$.

Remark 26. *If $\dim(X) \geq 3$, the Lefschetz hyperplane theorem guarantees that the left hand side of (7) only needs to be computed for one non-zero effective class β.*

Proof of Proposition 25. Let C be nonsingular, and let $f : C \to X$ be a stable map. Let $i : X \to Y$ be the inclusion. From the exact sequence

$$H^0(C, f^*N_{X/Y}) \to H^1(C, f^*T_X) \to H^1(C, f^*i^*T_Y)\,,$$

we have

$$\begin{aligned} h^1(C, f^*T_X) &\leq h^0(C, f^*N_{X/Y}) + h^1(C, f^*i^*T_Y) \\ &= h^0(C, f^*(\mathcal{O}(e_1) \oplus \cdots \mathcal{O}(e_k))) + O(1) \\ &= \left(\int_\beta c_1(\mathcal{O}(1))\right) \sum_{i=1}^{k} e_i + O(1)\,, \end{aligned}$$

where the constant upper bound $O(1)$ depends only on g and Y. This implies the claim. □

Proof of Theorem 11. In Proposition 25, we take $r \geq 3$, $Y = \mathbb{P}^{r+1}$, and X to be a hypersurface of degree $e = e_1 \leq r$. Then,

$$s(X) \geq t(X) = \int_{[L]} c_1(T_X) = r + 2 - e\,,$$

where $[L]$ is the class of a line. If $r > (e+1)(e-2)$, then Proposition 25 applies and X satisfies property $(\star\star)_g$. Moreover, we have

$$s(X) + t(X) \geq 2(r + 2 - e) \geq r + 1\,,$$

so we are done by Theorem 23. □

Hypersurfaces $X_e \subset \mathbb{P}^{r+1}$ are homogeneous spaces for $e = 1$ and 2. By Theorem 10, the virtual Tevelev degrees are enumerative for all curve classes of sufficiently high degree (depending upon the genus). While a direct approach to the geometric Tevelev degrees is explained in the $e = 1$ case in [12], how to directly calculate the geometric Tevelev degrees for quadrics in the asympototic range (and to match the the quadric formula of Theorem 6) is an interesting question in projective geometry.

For cubic hypersurfaces $X_3 \subset \mathbb{P}^{r+1}$, the virtual Tevelev degrees are calculated in [6] for all $r \geq 3$ by the simple formula of Theorem 5. By Theorem 11, the virtual Tevelev degrees are enumerative for all curves classes of sufficiently high degree (depending upon the genus) for all $r \geq 5$.

Question 27. *Find a direct calculation of the geometric Tevelev degrees in the asymptotic range for hypersurfaces via the projective geometry of curves.*

In fact, a geometric derivation of the formula of Theorem 5 has been recently given in [19], but the case of quadrics remains open.

4. Refined results in the genus 0 case

Our arguments yield stronger results for the enumerativity of virtual Tevelev degrees in the genus 0 case: for certain X, *all* well-posed virtual Tevelev degrees are enumerative, with no assumption on the positivity of β. The improvements will be needed for the application to curves on hypersurfaces in positive characteristic in Section 1.

Let X be a projective Fano variety of dimension r. We first introduce the following more precise version of the property $(\star\star)_0$.

Definition 28. *We say that X has property $(\star\star)_0'$ if*

$$(r - s(X)) \cdot h^1(\mathbb{P}^1, f^*T_X) < r + \int_\beta c_1(T_X) \tag{8}$$

for every curve class β and for every $[f : \mathbb{P}^1 \to X] \in M_0(X, \beta)$.

Remark 29. *Property $(\star\star)_0'$ is immediate when $s(X) \geq r$ or when $h^1(\mathbb{P}^1, f^*T_X) = 0$ for all maps $f : C \to X$ in class β.*

We now have the following refined version of Proposition 21.

Proposition 30. *Suppose that $g = 0$, and that X satisfies $(\star\star)_0'$. As in Proposition 21, let*

$$M_\Gamma \subset \overline{M}_{0,n}(X, \beta)$$

be a locally closed boundary stratum with topology as in Figure 1, where we require the spine C_0 now to be rational.

Suppose further that the virtual dimension of $\overline{M}_{0,n}(X,\beta)$ is at most the dimension of $\overline{M}_{0,n} \times X^n$,

$$r + \int_\beta c_1(T_X) \leq rn\,, \tag{9}$$

and furthermore that, if equality holds, then $n - m > 0$.

Then, $\dim(M_\Gamma) < \dim(\overline{M}_{0,n} \times X^n)$. In particular, M_Γ fails to dominate $\overline{M}_{0,n} \times X^n$.

Proof. We employ the same proof as in Proposition 21. We immediately reduce to the case $m = 0$. By Lemma 16(a), we have

$$\dim(M_\Gamma) \leq \dim(\overline{M}_{0,n} \times X^n) + h^1(C_0, T_X) - n.$$

Furthermore, we have

$$\begin{aligned} \int_{[C_0]} c_1(T_X) &\leq \int_\beta c_1(T_X) - n \cdot s(X) \\ &\leq (r - s(X))n - r \end{aligned}$$

by (9). Since X satisfies property $(\star\star)'_0$, we conclude $\dim(M_\Gamma) < \dim(\overline{M}_{0,n} \times X^n)$ for *all* n, as desired. □

Next, we have the following analog of Proposition 22:

Proposition 31. *Suppose that $g = 0$, and that X satisfies property $(\star\star)'_0$. Let*

$$M_\Gamma \subset \overline{M}_{0,n}(X,\beta)$$

be any locally closed boundary stratum as in Proposition 22.

Suppose the virtual dimension of $\overline{M}_{0,n}(X,\beta)$ is equal to the dimension of $\overline{M}_{0,n} \times X^n$,

$$r + \int_\beta c_1(T_X) = rn\,.$$

Assume further that at least one the conditions (i) or (ii) of Proposition 22 holds. Then, M_Γ fails to dominate $\overline{M}_{0,n} \times X^n$.

Proof. The proof of Proposition 22 goes through immediately; note that no inequality on ℓ is needed because Proposition 30 holds for n arbitrary. □

Corollary 32. *Suppose that $g = 0$, and that X satisfies property $(\star\star)'_0$, If condition (i) or (ii) of Theorem 23 holds, then, $\mathsf{vTev}^X_{0,\beta}$ is enumerative.*

Proof. Immediate from Propositions 14 and 31. □

Corollary 33. *Suppose $X_e \subset \mathbb{P}^{r+1}$ is a nonsingular hypersurface of degree e and dimension*

$$r > (e+1)(e-2).$$

Then, all genus 0 virtual Tevelev degrees $\mathsf{vTev}^{X_e}_{0,\beta}$ are enumerative.

Proof. We follow the proofs of Proposition 25 and Theorem 11. If $g = 0$ and $Y = \mathbb{P}^{r+1}$ (more generally, if $Y = G/P$), then the $O(1)$ term in the upper bound on $h^1(C, f^*T_X)$ goes away:

$$h^1(C, f^*T_X) \leq \left(\int_\beta c_1(\mathcal{O}(1))\right) e + 1\,.$$

We find that if $r > (e+1)(e-2)$, then X_e satisfies $(\star\star)'_0$, so we conclude by Corollary 32. □

5. Very free rational curves in characteristic p

Let k be an algebraically closed field of arbitrary characteristic. Let X be a nonsingular projective variety over k. A morphism

$$f : \mathbb{P}^1 \to X$$

is a **very free rational curve** on X if f^*T_X is ample. The variety X is **separably rationally connected** if it has a very free rational curve. The condition is equivalent by [18, Theorem 3.7] to the existence of a curve class β on X such that the evaluation map

$$\mathrm{ev} : M_{0,2}(X, \beta) \to X \times X$$

is dominant and separable on at least one component of the space $M_{0,2}(X, \beta)$. In characteristic zero, the separability of ev is immediate. It is known that Fano varieties are (separably) rationally connected in characteristic zero [17], and are conjectured to be so in arbitrary characteristic.

Our results on the enumerativity of Tevelev degrees in characteristic zero imply the existence of very free curves on certain Fano hypersurfaces in characteristic p.

Theorem 34. *Let k be an algebraically closed field of characteristic $p > 0$, and let $X_e \subset \mathbb{P}^{r+1}_k$ be a nonsingular hypersurface of degree $e \geq 3$. Fix integers $d, n \geq 3$ satisfying*

$$d = (n-1) \cdot \frac{r}{r+2-e}.$$

Assume that:

(i) $r > (e+1)(e-2)$, and
(ii) $p > e$.

Then, X_e contains a very free rational curve of degree at most d (where the degree is measured in the ambient projective space). In particular, X_e is separably rationally connected.

On a *general* hypersurface $X_e \subset \mathbb{P}^{r+1}_k$, very free curves of degree $r+1$ were constructed by Zhu [27] without assuming (i) or (ii). Very free curves on general complete intersections were similarly constructed by Chen-Zhu in [10]. When $\gcd(r+2-e, r) > 1$, our result gives very free curves of lower degree for *arbitrary* hypersurfaces satisfying assumptions (i) and (ii).

The separable rational connectivity for *arbitrary* Fano hypersurfaces X_e (and more generally, for arbitrary Fano complete intersections) was proven only assuming (ii) when $e < r+1$, and with an additional divisibility condition when $e = r+1$, by Starr-Tian-Zong in [25], but there the very free curves constructed are of unspecified degree.

Proof of Theorem 34. Let R be a discrete valuation ring with fraction field K of characteristic 0 and residue field isomorphic to k. Let $\mathfrak{X} \subset \mathbb{P}^{r+1}_R$ be a smooth hypersurface of degree e over $\mathrm{Spec}(R)$ with special fiber isomorphic to X_e.

Let $\overline{M}_{0,n}(\mathfrak{X}, d[L])_R$ be the relative moduli space (over R) of stable maps of degree d (as computed against the hyperplane class) in $\mathfrak{X}$, and let

$$\pi_R : \mathcal{C}_{0,n}(\mathfrak{X}, d[L])_R \to \overline{M}_{0,n}(\mathfrak{X}, d[L])_R$$

be the universal family. Let

$$h_R = \tau_R : \overline{M}_{0,n}(\mathfrak{X}, d[L])_R \to (\overline{M}_{0,n})_R \times \mathfrak{X}^n$$

be the forgetful map.

Combining Theorem 5 and Corollary 33, the degree of the map (in characteristic 0)

$$h_{\overline{K}} : \overline{M}_{0,n}(X_{\overline{K}}, d[L]) \to \overline{M}_{0,n} \times X_{\overline{K}}^n$$

is equal to

$$\mathsf{vTev}_{0,d}^{X_{\overline{K}}} = ((e-1)!)^n \cdot e^{(d-n)e+1},$$

which in particular is not divisible by p.

Applying [25, Corollary 3.3], we conclude that the special fiber $\overline{M}_{0,n} \times X^n$ has a free curve $\mathbb{P}^1_k \to \overline{M}_{0,n} \times X^n$, and upon projection we obtain a very free curve of degree at most d on X. □

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