Algebraic Cycles and Moduli Spaces

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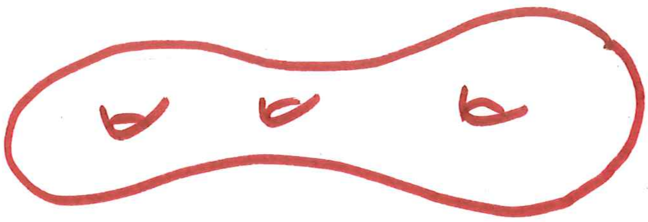
§. Curves

Let C be a complete, nonsingular curve $/\mathbb{C}$ of genus $g \geq 2$.



We are interested in cycles on $C^n = \underbrace{C \times C \times \cdots \times C}_{n}$

- $n = 2$

$$A^2(C^2) \ni K_1 \cdot K_2 - (2g-2) \cdot K \cdot \Delta$$

K_i canonical class on i^{th} factor

Δ diagonal class

Certainly, in cohomology,

$$K_1 K_2 - (2g-2)\cdot K \cdot \Delta = 0 \in H^4(C^2)$$

QUESTION (Faber - P. ~1999)

Does $K_1 K_2 - (2g-2) K \Delta$ vanish in Chow?

- If C is hyperelliptic $\Rightarrow$ YES
- If $g(C) = 3$ $\Rightarrow$ YES

Proof. Let $C \subset \mathbb{P}^2$ be a nonsingular quartic.

$$C \times C \xhookrightarrow{i} \mathbb{P}^2 \times \mathbb{P}^2$$

$$\Delta_{\mathbb{P}^2} = p \times \mathbb{P}^2 + \underset{\uparrow}{H \times H} + \mathbb{P}^2 \times p$$

in Chow

hyperplane

$$i^* : A^2(\mathbb{P}^2 \times \mathbb{P}^2) \to A^2(C \times C)$$

Excess intersection $i^*(\Delta_{\mathbb{P}^2}) = 4K\Delta$

$$i^*(p \times \mathbb{P}^2) = i^*(\mathbb{P}^2 \times p) = 0$$

$$i^*(H \times H) = K_1 K_2$$

So $K_1 K_2 - 4K \cdot \Delta = 0$

in Chow $\square$

- We could not prove the vanishing in Chow for $g \geq 4$.

Theorem (Green-Griffiths 2003)

$$K_1 K_2 - (2g-2) K\Delta \neq 0$$

in Chow generic C of genus ≥ 4.

[Another Proof: Yin 2013]

Bloch-Beilinson Conjecture

$$\Downarrow$$

$$K_1 K_2 - (2g-2) K\Delta = 0$$

in Chow for all C defined over $\overline{\mathbb{Q}}$.

- $n = 3$ Gross - Schoen cycle on C^3

$$(2g-2)^2 \cdot \Delta_{123} - (2g-2)\left(K_1 \Delta_{23} + K_2 \Delta_{13} + K_3 \Delta_{12} \right)$$
$$+ \left(K_1 K_2 + K_1 K_3 + K_2 K_3 \right) \in A^2(C^3)$$

Again, vanishes in cohomology but not in Chow for $g \geq 3$

[Ceresa 1983]

Define tautological rings

$$RH^*(C^n) \subset H^*(C^n)$$
$$R^*(C^n) \subset A^*(C^n)$$

generated by $K_1, \ldots, K_n$

Δ_{ij}

$RH^*(C^n)$ is easily calculated

in addition to $n = 2, 3$

relation we have discussed,

there is a relation

in degree $g+1$ for $n = 2g+2$

Question: What is $R^*(C^n)$?

Open, but there is

an explicit proposal:

all relations come from

Pixton's set.

2. K3 Surfaces

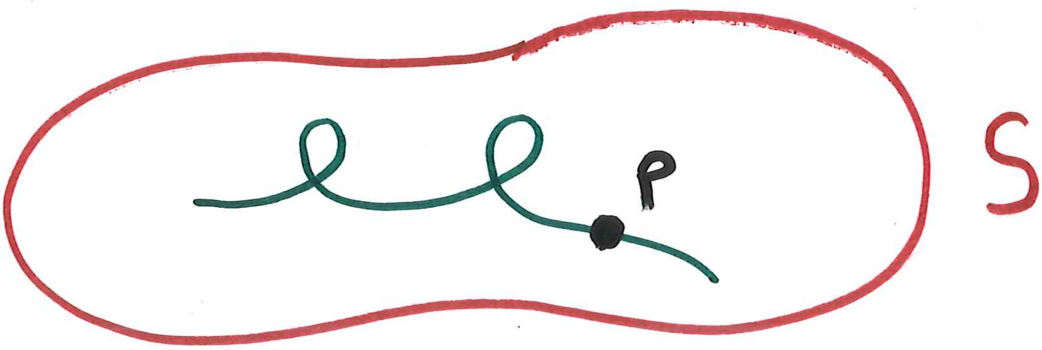
Let S be an algebraic K3 surface$/\mathbb{C}$.

While $A^2(S)$ is unruly there is a canonical

$$\mathbb{Z} \subset A^2(S)$$

Considered by Beauville and Voisin

$\mathbb{Z}$ generated by any point on any rational curve of S



• $n=3$ Special Cycle on S^3

$$\Delta_{123} - (p \times \Delta_{23} + \Delta_{12} \times p + \ldots)$$
$$+ (p \times p \times S + p \times S \times p + S \times p \times p)$$

in $A^4(S^3)$

Here p is any Beauville-Voisin point

Cycle easily shown to vanish in cohomology

Theorem (Beauville-Voisin 2004)

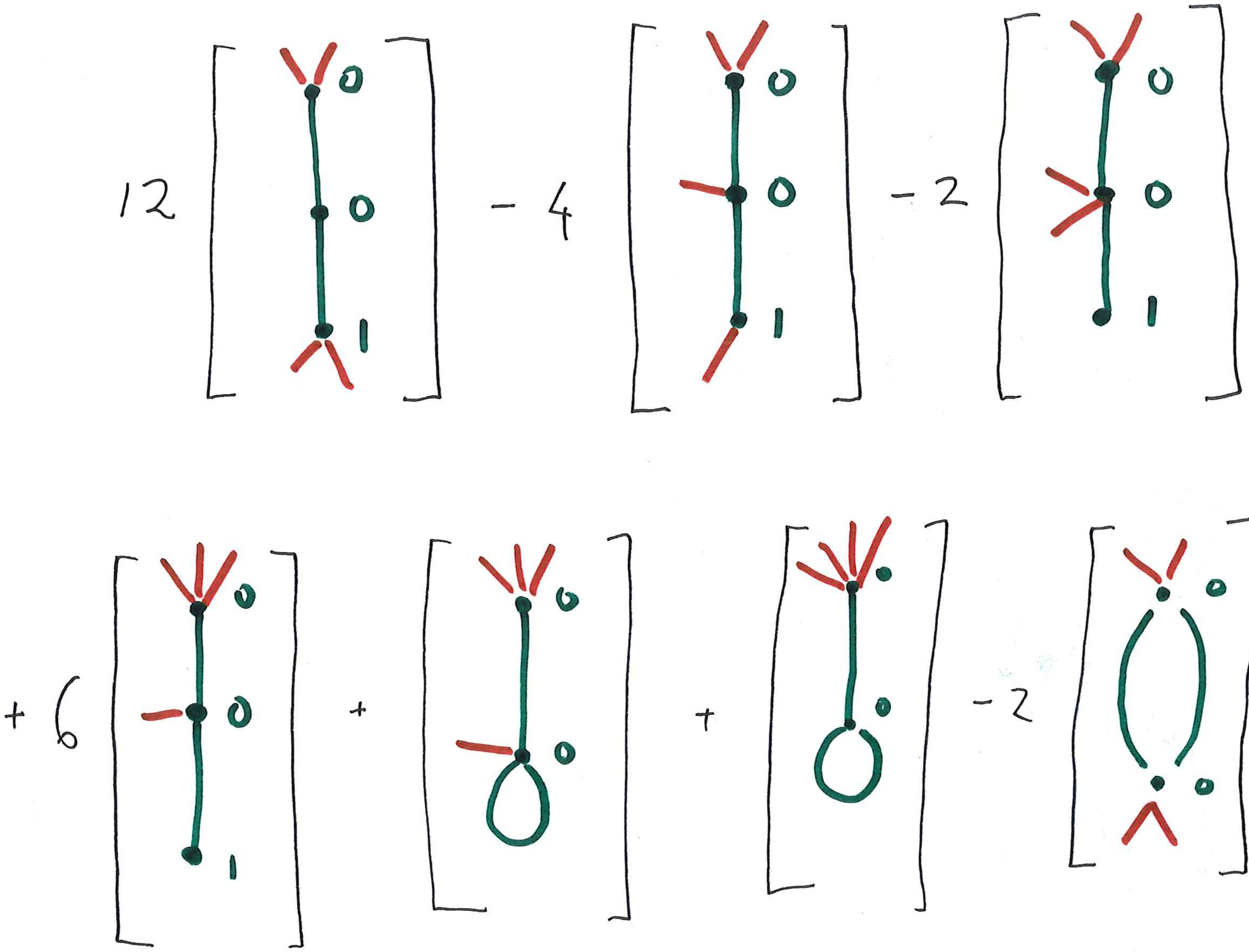
The cycle vanishes in Chow for every S.

K3 surface

A new proof of the Chow Vanishing

(a direction of work with Q. Yin)

Getzler's Relation in $A^2(\overline{M}_{1,4})$



$$= 0 \in A^2(\overline{M}_{1,4})$$

Let S be a K3 surface and let $\beta \in \mathrm{Pic}(S)$ be primitive, effective.

$$
\begin{tikzcd}
 & \left[\overline{M}_{1,4}(S,\beta)\right]^{\text{red vir}} \arrow[dl, "\pi"'] \arrow[dr, "\varepsilon_{123}"] & \\
\overline{M}_{1,4} & & S \times S \times S
\end{tikzcd}
$$

$$
\varepsilon_{123\,*}\left(\pi^*(\mathbf{GETZLER}) \cup ev_4^*(\beta) \right)
$$

$$
\text{Special Cycle (up to } \overset{\text{nonzero}}{\text{multiple}}\text{)} = \quad = 0
$$

in

$$
A^4(S^3)
$$

Tautological ring $R^*(S^n) \subset A^*(S^n)$

generated Δ_{ij} diagonals

and the pull-backs of $\mathrm{Pic}(S)$

on each factor

(contains Beauville-Voisin point)

$$\varepsilon_*\left[\overline{M}_{g,n}(S,\beta)\right]^{red\ vir} \in A_{g+n}(S^n)$$

$$\varepsilon : \overline{M}_{g,n}(S,\beta) \longrightarrow S \times S \times \cdots \times S$$

CONJECTURE (P., Q. Yin)

$$\varepsilon_*\left[\overline{M}_{g,n}(S,\beta)\right]^{red vir} \in R^*(S^n)$$

We expect much more to be true.

Let $\mathcal{M}_{K3}$ be the moduli space of quasi polarized K3 surfaces

$$[S, \beta] \in \mathcal{M}_{K3}$$

$$\begin{array}{c} \mathcal{S} \\ \downarrow \pi \\ \mathcal{M}_{K3} \end{array}$$

universal surface.

$$\begin{tikzcd} \overline{\mathcal{M}}_{g,n}(\pi, \beta) \arrow[r, "\varepsilon"] \arrow[dr] & \mathcal{S} \times \mathcal{S} \times \cdots \times \mathcal{S} \arrow[d] \\ & \mathcal{M}_{K3} \end{tikzcd}$$

Conjecture (P., Q. Yin)

$\varepsilon_* [\overline{M}_{g,n}(\pi, \beta)]^{\mathrm{red\,vir}}$ is tautological class on

$$\begin{array}{c} \mathcal{S}^n \\ \downarrow \\ \mathcal{M}_{K3} \end{array}$$

in the case of $\overline{M}_{1,4}(\pi, \beta)$

We can use **Getzler's relation** to calculate the Noether-Lef boundary terms of the universal special cycle on

$$\begin{array}{c} \mathcal{S}^3 \\ \downarrow \\ \mathcal{M}_{K3} \end{array}$$

Return to a fixed K3 surface S.

$$RH^*(S^n) \subset H^*(S^n)$$

$$R(S^n) \subset A^*(S^n)$$

Satisfied Poincaré duality

Is $R^*(S^n) \overset{?}{\cong} RH^*(S^n)$

Open question!

without $\mathrm{Pic}(S)$, true for $n \leq 45$

$(46 = 2 \cdot 22 + 2)$

For $n = 46$, there is a cohomological relation $H^{2\cdot 46}(S^{46})$

Does there exist a corresponding cycle relation in $A^{46}(S^{46})$?

Related to Kimura Finiteness

Cycles on $\overline{M}_g$

Let X be a nonsingular, projective variety / $\mathbb{C}$

$\beta \in H_2(X, \mathbb{Z})$

$$\overline{M}_g(X, \beta) \xrightarrow{\pi} \overline{M}_g$$

$$[\overline{M}_g(X, \beta)]^{vir} \rightsquigarrow \pi_*[\overline{M}_g(X, \beta)]^{vir}$$

$$\cap$$

$$A_*(\overline{M}_g)$$

Which cycles classes are obtained?

is $\pi_\star\left[\overline{M}_g(X,\beta)\right] \in R(\overline{M}_g)$

$$\begin{array}{c} R(\overline{M}_g) \\ \cap \\ A(\overline{M}_g) \end{array}$$

Tautological classes

Conjecture:

$$\pi_\star\left[\overline{M}_g(X,\beta)\right]^{vir} \in RH(\overline{M}_g)$$

Evidence:

Very little. Hasn't failed yet.

Wilder Statements needed to be tautological in Chow

Speculation: Let C be a nonsingular projective curve $/\overline{\mathbb{Q}}$.

Then

$$[C] \in R_0(\overline{M}_g) \subset A_0(\overline{M}_g).$$

Evidence: none at all

But perhaps 1^{st} step of proof is Belyi's Theorem.

3. 3-folds

Let X be a nonsingular, projective 3-fold $/ \mathbb{C}$

$$P_n(X, \beta)$$

Moduli space of stable pairs

$$(\mathcal{F}, s)$$

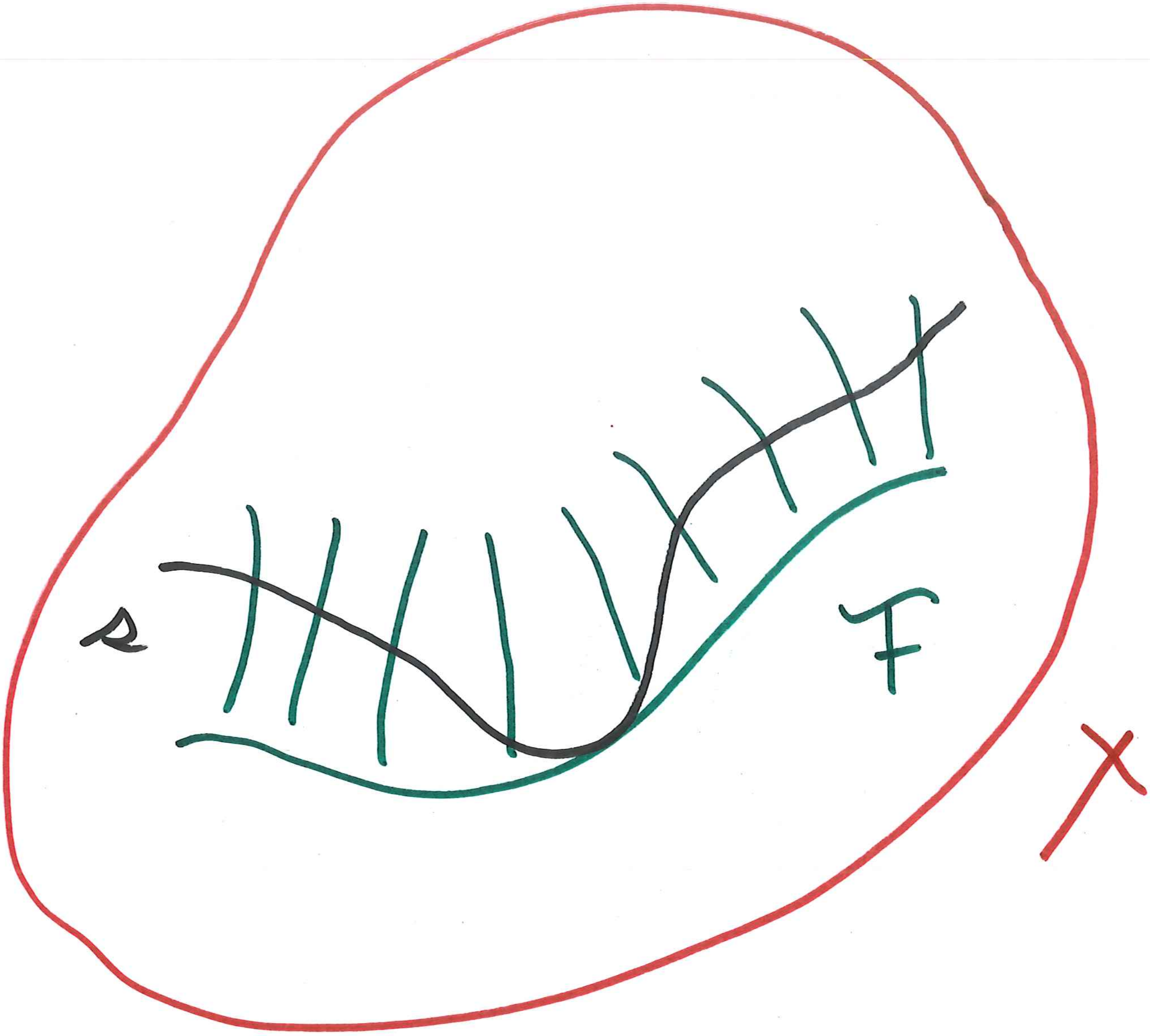
$\chi(F)$ $\quad$ $[F] \in H_2(X, \mathbb{Z})$.

- $\mathcal{F}$ is a pure sheaf of dim 1 on X
- $s \in H^0(X, \mathcal{F})$ is a section and

$$\mathcal{O}_X \xrightarrow{s} \mathcal{F}$$

has coker supported on dim 0

Studied with R. Thomas



$P_n(X, \beta)$ Carries perfect obstruction theory, virtual class.

GW/Pair Correspondence

$P_n(X, \beta)$ carries a Virtual Class

in **algebraic cobordism**

Theory developed by M. Levine, F. Morel

Let Y be a quasi proj Variety $/\mathbb{C}$

$\Omega_d(Y)$ generated by

$$\left[\begin{array}{c} Z \\ \downarrow \\ Y \end{array}\right]$$

- nonsingular
- projective over Y
- $\dim_{\mathbb{C}}$ d

modulo Cobordism relations.

Theorem (J. Shen 2014)

The obstruction theory canonically determines a virtual class in algebraic cobordism

$$[P_n(X,\beta)]^{vir\,\Omega} \in \Omega_d(P_n(X,\beta))$$

$$d = \int_\beta c_1(X)$$

whose image in Chow is the usual virtual fundamental class

$$[P_n(X,\beta)]^{vir\ Chow} \in A_d(P_n(X,\beta))$$

Consider push-forward to $\Omega(pt)$.

fix β, sum over n ← holomorphic Euler Char

$$Z^{\Omega}_{X,\beta} = \sum_{n \in \mathbb{Z}} q^n \left[P_n(X,\beta) \right]^{vir\ \Omega(pt)}$$

Conjecture (Shen 2014)

- $Z^{\Omega}_{X,\beta}$ is a rational function in q.

$$Z^{\Omega}_{X,\beta} \in \Omega_{(pt)}(q)$$

- functional equation

$$Z^{\Omega}_{X,\beta}(q^{-1}) = q^{-d} Z^{\Omega}_{X,\beta}(q)$$

Rationality of $Z^{\Omega}_{X,\beta}$

follows whenever we know

the rationality of all stable

pair descendent series.

Theorem (Shen 2014)

$Z^{\Omega}_{X,\beta}$ is rational in q for toric X.

(uses descendent results of P.–Pixton)

functional equation is
a mystery.

Only verified in local
Calabi-Yau toric cases MNOP.

Example (Shen):

$X = \mathbb{P}^3$, $\beta = [L]$, $d = 4$

$$Z^{\Omega, c_4}_{\mathbb{P}^3, L}(q)$$

$$\shortparallel$$

$$\frac{64q}{(1+q)^6}\Big(8 + 49q + 124q^2 + 3q^3 + 1040q^4 + 3q^5 + 124q^6 + 49q^7 + 8q^8\Big)$$

The End