Descendents for stable pairs on 3-folds

R. Pandharipande

Dedicated to Simon Donaldson on the occasion of his 60^{th} birthday

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Abstract

We survey here the construction and the basic properties of descendent invariants in the theory of stable pairs on nonsingular projective 3-folds. The main topics covered are the rationality of the generating series, the functional equation, the Gromov-Witten/Pairs correspondence for descendents, the Virasoro constraints, and the connection to the virtual fundamental class of the stable pairs moduli space in algebraic cobordism. In all of these directions, the proven results constitute only a small part of the conjectural framework. A central goal of the article is to introduce the open questions as simply and directly as possible.

Contents

0 Introduction

0.1 Moduli space of stable pairs

Let X be a nonsingular projective 3-fold. The moduli of curves in X can be approached in several different ways.[1] For an algebraic geometer, perhaps the most straightforward is the Hilbert scheme of subcurves of X. The moduli of stable pairs is closely related to the Hilbert scheme, but is geometrically much more efficient. While the definition of a stable pair takes some time to understand, the advantages of the moduli theory more than justify the effort.

Definition 1. *A* stable pair (F, s) *on* X *is a coherent sheaf* F *on* X *and a section* $s \in H^0(X, F)$ *satisfying the following stability conditions:*

- F *is* pure *of dimension 1,*
- *the section* $s : \mathcal{O}_X \to F$ *has cokernel of dimensional 0.*

Let C be the scheme-theoretic support of F. By the purity condition, all the irreducible components of C are of dimension 1 (no 0-dimensional components are permitted). By [38, Lemma 1.6], the kernel of s is the ideal sheaf of C,

$$\mathcal{I}_C = \ker(s) \subset \mathcal{O}_X \,,$$

and C has no embedded points. A stable pair

$$\mathcal{O}_X \to F$$

therefore defines a Cohen-Macaulay subcurve $C \subset X$ via the kernel of s and a 0-dimensional subscheme[2] of C via the support of the cokernel of s.

To a stable pair, we associate the Euler characteristic and the class of the support C of F,

$$\chi(F) = n \in \mathbb{Z} \quad \text{and} \quad [C] = \beta \in H_2(X, \mathbb{Z})\,.$$

For fixed n and β, there is a projective moduli space of stable pairs $P_n(X, \beta)$. Unless β is an effective curve class, the moduli space $P_n(X, \beta)$ is empty.

[1] For a discussion of the different approaches, see [42].

[2] When C is Gorenstein (for instance if C lies in a nonsingular surface), stable pairs supported on C are in bijection with 0-dimensional subschemes of C. More precise scheme theoretic isomorphisms of moduli spaces are proved in [40, Appendix B].

A foundational treatment of the moduli space of stable pairs is presented in [38] via the results of Le Potier [16]. Just as the Hilbert scheme $I_n(X,\beta)$ of subcurves of X of Euler characteristic n and class β is a fine moduli space with a universal quotient sequence, $P_n(X,\beta)$ is a fine moduli space with a universal stable pair [38, Section 2.3]. While the Hilbert scheme $I_n(X,\beta)$ is a moduli space of curves with free and embedded points, the moduli space of stable pairs $P_n(X,\beta)$ should be viewed as a moduli space of curves with points *on the curve* determined by the cokernel of s. Though the additional points still play a role, $P_n(X,\beta)$ is much smaller than $I_n(X,\beta)$.

If $P_n(X,\beta)$ is non-empty, then $P_m(X,\beta)$ is non-empty for all $m > n$. Stable pairs with higher Euler characteristic can be obtained by suitably twisting stable pairs with lower Euler characteristic (in other words, by *adding points*). On the other hand, for a fixed class $\beta \in H_2(X,\mathbb{Z})$, the moduli space $P_n(X,\beta)$ is empty for all sufficiently negative n. The proof exactly parallels the same result for the Hilbert scheme of curves $I_n(X,\beta)$.

0.2 Action of the descendents

Denote the universal stable pair over $X \times P_n(X,\beta)$ by

$$\mathcal{O}_{X\times P_n(X,\beta)} \xrightarrow{s} \mathbb{F}.$$

For a stable pair $(F,s) \in P_n(X,\beta)$, the restriction of the universal stable pair to the fiber

$$X \times (F,s) \ \subset\ X \times P_n(X,\beta)$$

is canonically isomorphic to $\mathcal{O}_X \xrightarrow{s} F$. Let

$$\pi_X \colon X \times P_n(X,\beta) \to X,$$
$$\pi_P \colon X \times P_n(X,\beta) \to P_n(X,\beta)$$

be the projections onto the first and second factors. Since X is nonsingular and $\mathbb{F}$ is π_P-flat, $\mathbb{F}$ has a finite resolution by locally free sheaves.[3] Hence, the Chern character of the universal sheaf $\mathbb{F}$ on $X \times P_n(X,\beta)$ is well-defined.

Definition 2. *For each cohomology*[4] *class $\gamma \in H^*(X)$ and integer $i \in \mathbb{Z}_{\geq 0}$, the action of the* descendent $\tau_i(\gamma)$ *is defined by*

$$\tau_i(\gamma) = \pi_{P*}\big(\pi_X^*(\gamma)\cdot \mathrm{ch}_{2+i}(\mathbb{F}) \cap \pi_P^*(\ \cdot\ )\big)\,.$$

[3] Both X and $P_n(X,\beta)$ carry ample line bundles.

[4] All homology and cohomology groups will be taken with $\mathbb{Q}$-coefficients unless explicitly denoted otherwise.

The pull-back π_P^* is well-defined in homology since π_p is flat [8].

We may view the descendent action as defining a cohomology class

$$\tau_i(\gamma) \in H^*(P_n(X,\beta))$$

or as defining an endomorphism

$$\tau_i(\gamma) : H_*(P_n(X,\beta)) \to H_*(P_n(X,\beta))\,.$$

Definition 2 is the standard method of obtaining classes on moduli spaces of sheaves via universal structures. The construction has been used previously for the cohomology of the moduli space of bundles on a curve [28], for the cycle theory of the Hilbert schemes of points of a surface [10], and in Donaldson's famous μ map for gauge theory on 4-manifolds [6].

0.3 Tautological classes

Let $\mathbb{D}$ denote the polynomial $\mathbb{Q}$-algebra on the symbols

$$\{\, \tau_i(\gamma) \,|\, i \in \mathbb{Z}_{\geq 0} \text{ and } \gamma \in H^*(X) \,\}$$

subject to the basic linear relations

$$\begin{aligned} \tau_i(\lambda \cdot \gamma) &= \lambda \tau_i(\gamma)\,, \\ \tau_i(\gamma + \widehat{\gamma}) &= \tau_i(\gamma) + \tau_i(\widehat{\gamma})\,, \end{aligned}$$

for $\lambda \in \mathbb{Q}$ and $\gamma, \widehat{\gamma} \in H^*(X)$. The descendent action defines a $\mathbb{Q}$-algebra homomorphism

$$\alpha^X_{n,\beta} : \mathbb{D} \to H^*(P_n(X,\beta))\,.$$

The most basic questions about the descendent action are to determine

$$\mathrm{Ker}(\alpha^X_{n,\beta}) \subset \mathbb{D} \quad \text{and} \quad \mathrm{Im}(\alpha^X_{n,\beta}) \subset H^*(P_n(X,\beta))\,.$$

Both questions are rather difficult since the space $P_n(X,\beta)$ can be very complicated (with serious singularities and components of different dimensions). Few methods are available to study $H^*(P_n(X,\beta))$.

Following the study of the cohomology of the moduli of stable curves, we define, for the moduli space of stable pairs $P_n(X,\beta)$,

- $\mathrm{Im}(\alpha^X_{n,\beta}) \subset H^*(P_n(X,\beta))$ to be the algebra of *tautological classes*,

- $\mathrm{Ker}(\alpha^X_{n,\beta}) \subset \mathbb{D}$ to be the the ideal of *tautological relations* since

$$\frac{\mathbb{D}}{\mathrm{Ker}(\alpha^X_{n,\beta})} = \mathrm{Im}(\alpha^X_{n,\beta}) \,.$$

The basic expectation is that natural constructions yield tautological classes. For the moduli spaces of curves there is a long history of the study of tautological classes, geometric constructions, and relations, see [12, 32] for surveys.

As a simple example, consider the tautological classes in the case

$$X = \mathbb{P}^3 \,, \quad n = 1 \,, \quad \beta = \mathsf{L} \,,$$

where $\mathsf{L} \in H_2(\mathsf{P}^3, \mathbb{Z})$ is the class of a line. The moduli space $P_1(\mathbb{P}^3, \mathsf{L})$ is isomorphic to the Grassmannian $\mathbb{G}(2,4)$. The ring homomorphism

$$\alpha^{\mathbb{P}^3}_{1,\mathsf{L}} : \mathbb{D} \to H^*(P_1(\mathbb{P}^3, \mathsf{L}))$$

is surjective, so *all* classes are tautological. The tautological relations

$$\mathrm{Ker}(\alpha^{\mathbb{P}^3}_{1,\mathsf{L}}) \subset \mathbb{D}$$

can be determined by the Schubert calculus.

Our study of descendents here follows a different line which is more accessible than the full analysis of $\alpha^X_{n,\beta}$. The moduli space $P_n(X,\beta)$ carries a virtual fundamental class

$$[P_n(X,\beta)]^{vir} \in H_*(P_n(X,\beta))$$

obtained from the deformation theory of stable pairs. There is an associated integration map

$$\int_{[P_n(X,\beta)]^{vir}} : \ \mathbb{D} \to \mathbb{Q} \tag{1}$$

defined by

$$\int_{[P_n(X,\beta)]^{vir}} \mathsf{D} = \int_{P_n(X,\beta)} \alpha^X_{n,\beta}(\mathsf{D}) \cap [P_n(X,\beta)]^{vir}$$

for $\mathsf{D} \in \mathbb{D}$. Here,

$$\int_{P_n(X,\beta)} : \ H_*(P_n(X,\beta)) \to \mathbb{Q}$$

is the canonical point counting map factoring through $H_0(P_n(X,\beta))$. The standard theory of descendents is a study of the integration map (1).

0.4 Deformation theory

To define a virtual fundamental class [3, 21], a 2-term deformation/obstruction theory must be found on the moduli space of stable pairs $P_n(X,\beta)$. As in the case of the Hilbert scheme $I_n(X,\beta)$, the most immediate obstruction theory of $P_n(X,\beta)$ does *not* admit such a structure. For $I_n(X,\beta)$, a suitable obstruction theory is obtained by viewing $C \subset X$ *not* as a subscheme, but rather as an ideal sheaf $\mathcal{I}_C$ with trivial determinant [7, 44]. For $P_n(X,\beta)$, a suitable obstruction theory is obtained by viewing a stable pair *not* as sheaf with a section, but as an object

$$[\mathcal{O}_X \to F] \in D^b(X)$$

in the bounded derived category of coherent sheaves on X.

Denote the quasi-isomorphism equivalence class of the complex $[\mathcal{O}_X \to F]$ in $D^b(X)$ by $I^\bullet$. The quasi-isomorphism class $I^\bullet$ determines[5] the stable pair [38, Proposition 1.21], and the fixed-determinant deformations of $I^\bullet$ in $D^b(X)$ match those of the pair (F,s) to all orders [38, Theorem 2.7]. The latter property shows the scheme $P_n(X,\beta)$ may be viewed as a moduli space of objects in the derived category.[6] We can then use the obstruction theory of the complex $I^\bullet$ rather than the obstruction theory of sheaves with sections.

The deformation/obstruction theory for complexes at $[I^\bullet] \in P_n(X,\beta)$ is governed by

$$\mathrm{Ext}^1(I^\bullet, I^\bullet)_0 \quad \text{and} \quad \mathrm{Ext}^2(I^\bullet, I^\bullet)_0\,. \tag{2}$$

The obstruction theory (2) has all the formal properties of the Hilbert scheme case: 2 terms, a virtual class of (complex) dimension $d_\beta = \int_\beta c_1(X)$,

$$[P_n(X,\beta)]^{vir} \in H_{2d_\beta}\big(P_n(X,\beta), \mathbb{Z}\big)\,,$$

and a description via the χ^B-weighted Euler characteristics in the Calabi-Yau case [2].

0.5 Descendent invariants

Let X be a nonsingular projective 3-fold. For nonzero $\beta \in H_2(X,\mathbb{Z})$ and arbitrary $\gamma_i \in H^*(X)$, define the stable pairs invariant with descendent in-

[5] The claims require the dimension of X to be 3.

[6] The moduli of objects in the derived category usually yields Artin stacks. The space $P_n(X,\beta)$ is a rare example where the moduli of objects in the derived category has a component with coarse moduli space given by a scheme (uniformly for all 3-folds X).

sertions by

$$\left\langle \tau_{k_1}(\gamma_1)\dots\tau_{k_r}(\gamma_r)\right\rangle^X_{n,\beta} = \int_{[P_n(X,\beta)]^{vir}} \prod_{i=1}^r \tau_{k_i}(\gamma_i)\ . \tag{3}$$

The partition function is

$$\mathsf{Z}_{\mathsf{P}}\Big(X;q\ \Big|\prod_{i=1}^r \tau_{k_i}(\gamma_i)\Big)_\beta = \sum_{n\in\mathbb{Z}} \Big\langle \prod_{i=1}^r \tau_{k_i}(\gamma_i)\Big\rangle^X_{n,\beta} q^n. \tag{4}$$

Since $P_n(X,\beta)$ is empty for sufficiently negative n, the partition function is a Laurent series in q,

$$\mathsf{Z}_{\mathsf{P}}\Big(X;q\ \Big|\ \prod_{i=1}^r \tau_{k_i}(\gamma_i)\Big)_\beta \in \mathbb{Q}((q))\,.$$

The descendent invariants (3) and the associated partition functions (4) are the central topics of the paper. From the point of view of the complete tautological ring of descendent classes on $P_n(X,\beta)$, the descendent invariants (3) constitute only small part of the full data. However, among many advantages, the integrals (3) are deformation invariant as X varies in families. The same can not be said of the tautological ring nor of the full cohomology $H^*(P_n(X,\beta))$.

In addition to carrying data about the tautological classes on $P_n(X,\beta)$, the descendent series are related to the enumerative geometry of curves in X. The connection is clearest for the primary fields $\tau_0(\gamma)$ which correspond to incidence conditions for the support curve of the stable pair with a fixed cycle

$$V_\gamma \subset X$$

of class $\gamma \in H^*(X)$. But even for primary fields, the partition function

$$\mathsf{Z}_{\mathsf{P}}\Big(X;q\ \Big|\ \prod_{i=1}^r \tau_0(\gamma_i)\Big)_\beta$$

provides a virtual count and is rarely strictly enumerative.

Descendents $\tau_k(D)$, for $k \geq 0$ and $D \subset X$ a divisor, can be viewed as imposing tangency conditions of the support curve of the stable pair along the divisor D. The connection of $\tau_k(D)$ to tangency conditions is not as close as the enumerative interpretation of primary fields — the tangency condition is just the leading term in the understanding of $\tau_k(D)$. The topic will be discussed in Section 2.7.

0.6 Plan of the paper

The paper starts in Section 1 with a discussion of the rationality of the descendent partition function in absolute, equivariant, and relative geometries. While the general statement is conjectural, rationality in toric and hypersurface geometries has been proven in joint work with A. Pixton in [33, 35, 37]. Examples of exact calculations of descendents are given in Section 1.4. A precise conjecture for a functional equation related to the change of variable

$$q \mapsto \frac{1}{q}$$

is presented in Section 1.7, and a conjecture constraining the poles appears in Section 1.8.

The second topic, the Gromov-Witten/Pairs correspondence for descendents, is discussed in Section 2. The descendent theory of stable maps and stable pairs on a nonsingular projective 3-fold X are conjectured to be *equivalent* via a universal transformation. While the correspondence is proven in joint work with A. Pixton in toric [36] and hypersurface [37] cases and several formal properties are established, a closed formula for the transformation is not known.

The Gromov-Witten/Pairs correspondence has motivated much of the development of the descendent theory on the sheaf side. The first such conjectures for descendent series were made in joint work with D. Maulik, A. Okounkov, and N. Nekrasov [24, 25] in the context of the Gromov-Witten/Donaldson-Thomas correspondence[7] for the partition functions associated to the Hilbert schemes $I_n(X, \beta)$ of subcurves of X.

Given the Gromov-Witten/Pairs correspondence and the well-known Virasoro constraints for descendents in Gromov-Witten theory, there must be corresponding Virasoro constraints for the descendent theory of stable pairs. For the Hilbert schemes $I_n(X, \beta)$ of curves, descendent constraints were studied by A. Oblomkov, A. Okounkov, and myself in Princeton a decade ago [29]. In Section 3, conjectural descendent constraints for the stable pairs theory of $\mathbb{P}^3$ are presented (joint work with A. Oblomkov and A. Okounkov).

The moduli space of stable pairs $P_n(X, \beta)$ has a virtual fundamental class in homology $H_*(P_n(X, \beta))$. By construction, the class lifts to algebraic cycles

[7] A correspondence proposed in [38] between Hilbert scheme and stable pair counting (often termed DT/PT) has been well studied, especially in the Calabi-Yau case [4, 45], but is still conjectural for most 3-folds X.

$A_*(P_n(X,\beta))$. In a recent paper, Junliang Shen has lifted the virtual fundamental class further to algebraic cobordism $\Omega_*(P_n(X,\beta))$. Shen's results open a new area of exploration with beautiful structure. At the moment, the methods available to explore the virtual fundamental class in cobordism all use the theory of descendents (since the Chern classes of the virtual tangent bundle of $P_n(X,\beta)$ are *tautological*). Shen's work is presented in Section 4.

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The perspective of the paper is based in part on my talk *Why descendents?* at the Newton institute in Cambridge in the spring of 2011, though much of the progress discussed here has happened since then.

1 Rationality

1.1 Overview

Let X be a nonsingular projective 3-fold. Our goal here is to present the conjectures governing the *rationality* of the partition functions of descendent invariants for the stable pairs theory of X. The most straightforward statements are for the absolute theory, but we will present the rationality claims for the equivariant and relative stable pairs theories as well. The latter two appear naturally when studying the absolute theory: most results to date involve equivariant and relative techniques. In addition to rationality, we will also discuss the *functional equation* and the *pole constraints* for the descendent partition functions.

While rationality has been established in many cases, new ideas are required to prove the conjectures in full generality. The subject intertwines the Chern characters of the universal sheaves with the geometry of the virtual fundamental class. Perhaps, in the future, a point of view will emerge from which rationality is obvious. Hopefully, the functional equation will then also be clear. At present, the geometries for which the functional equation has been proven are rather few.

1.2 Absolute theory

Let X be a nonsingular projective 3-fold. The stable pairs theory for X as presented in the introduction is the *absolute* case. Let $\beta \in H_2(X,\mathbb{Z})$ be a nonzero class, and let $\gamma_i \in H^*(X)$. The following conjecture[8] was proposed[9] in [39].

Conjecture 1 (P.-Thomas, 2007)**.** *For X a nonsingular projective 3-fold, the descendent partition function*

$$\mathsf{Z}_{\mathsf{P}}\Big(X;q \mid \prod_{i=1}^{r} \tau_{k_i}(\gamma_i)\Big)_\beta$$

is the Laurent expansion in q of a rational function in $\mathbb{Q}(q)$.

In the absolute case, the descendent series satisfies a dimension constraint. For $\gamma_i \in H^{e_i}(X)$, the (complex) degree of the insertion $\tau_{k_i}(\gamma_i)$ is $\frac{e_i}{2}+k_i-1$. If the sum of the degrees of the descendent insertions does not equal the virtual dimension,

$$\dim_{\mathbb{C}} [P_n(X,\beta)]^{vir} = \int_\beta c_1(X)\,,$$

the partition function $\mathsf{Z}_{\mathsf{P}}\big(X;q \mid \prod_{i=1}^{r} \tau_{k_i}(\gamma_i)\big)_\beta$ vanishes.

In case X is a nonsingular projective Calabi-Yau 3-fold, the virtual dimension of $P_n(X,\beta)$ is always 0 (and no nontrivial insertions are allowed). The rationality of the basic partition function

$$\mathsf{Z}_{\mathsf{P}}\big(X;q \mid 1\big)_\beta$$

was proven[10] in [4, 45] by Serre duality, wall-crossing, and a weighted Euler characteristic approach to the virtual class [2]. At the moment, the proof for Calabi-Yau 3-folds does not appear to suggest an approach in the general case.

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[8] A weaker conjecture for descendent partition functions for the Hilbert scheme $I_n(X,\beta)$ was proposed earlier in [25].

[9] Theorems and Conjectures are dated in the text by the year of the arXiv posting. The published dates are later and can be found in the bibliography.

[10] See [40] for a similar rationality argument in a restricted (simpler) setting.

1.3 Equivariant theory

Let X be a nonsingular quasi-projective toric 3-fold equipped with an action of the 3-dimensional torus

$$\mathbf{T} = \mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^* .$$

The stable pairs descendent invariants can be lifted to equivariant cohomology (and defined by residues in the open case). For equivariant classes $\gamma_i \in H^*_{\mathbf{T}}(X)$, the descendent partition function is a Laurent series in q,

$$\mathsf{Z}_{\mathsf{P}}\Big(X; q \;\Big|\; \prod_{i=1}^{r} \tau_{k_i}(\gamma_i)\Big)^{\mathbf{T}}_{\beta} \in \mathbb{Q}(s_1, s_2, s_3)((q)) ,$$

with coefficients in the field of fractions of

$$H^*_{\mathbf{T}}(\bullet) = \mathbb{Q}[s_1, s_2, s_3] .$$

The stable pair theory for such toric X is the *equivariant* case. A central result of [33, 35] is the following rationality property.

Theorem 1 (P.-Pixton, 2012)**.** *For X a nonsingular quasi-projective toric 3-fold, the descendent partition function*

$$\mathsf{Z}_{P}\Big(X; q \;\Big|\; \prod_{i=1}^{r} \tau_{k_i}(\gamma_i)\Big)^{\mathbf{T}}_{\beta}$$

is the Laurent expansion in q of a rational function in $\mathbb{Q}(q, s_1, s_2, s_3)$.

The proof of Theorem 1 uses the virtual localization formula of [14], the capped vertex[11] perspective of [27], the quantum cohomology of the Hilbert scheme of points of resolutions of A_r-singularities [26, 31], and a delicate argument for pole cancellation at the vertex [33]. In the toric case, calculations can be made effectively, but the computational methods are not very efficient.

When X is a nonsingular projective toric 3-fold, Theorem 1 implies Conjecture 1 for X by taking the non-equivariant limit. However, Theorem 1

[11] A basic tool in the proof is the capped *descendent* vertex. The 1-leg capped descendent vertex is proven to be rational in [33]. The 2-leg and 3-leg capped descendent vertices are proven to be rational in [35].

is much stronger in the toric case than Conjecture 1 since the descendent insertions may exceed the virtual dimension in equivariant cohomology.

In addition to the Calabi-Yau and toric cases, Conjecture 1 has been proven in [37] for complete intersections in products of projective spaces (for descendents of cohomology classes γ_i restricted from the ambient space — the precise statement is presented in Section 1.9). Taken together, the evidence for Conjecture 1 is rather compelling.

1.4 First examples

Let X be a nonsingular projective Calabi-Yau 3-fold, and let

$$C \subset X$$

be a rigid nonsingular *rational* curve. Let

$$\mathsf{Z}_{\mathsf{P}}\Big(C \subset X; q \mid \mathsf{1}\Big)_{d[C]}$$

be the contribution to the partition function $\mathsf{Z}_{\mathsf{P}}\Big(X; q \mid \mathsf{1}\Big)_{d[C]}$ obtained from the moduli of stable pairs *supported on C*. A localization calculation which goes back to the Gromov-Witten evaluation of [11] yields

$$\mathsf{Z}_{\mathsf{P}}\Big(C \subset X; q \mid 1\Big)_{d[C]} = \sum_{\mu \vdash d} \frac{(-1)^{\ell(\mu)}}{\mathfrak{z}(\mu)} \prod_{i=1}^{\ell(\mu)} \frac{(-q)^{m_i}}{(1-(-q)^{m_i})^2} \,. \tag{5}$$

The sum here is over all (unordered) partitions of d,

$$\mu = (m_1, \ldots, m_{\ell(\mu)}) \,, \quad \sum_{i=1}^{\ell(\mu)} m_i = d \,,$$

and $\mathfrak{z}(\mu)$ is the standard combinatorial factor

$$\mathfrak{z}(\mu) = \prod_{i=1}^{\ell(\mu)} m_i \cdot |\mathrm{Aut}(\mu)| \,.$$

The evaluation (5) played an important role in the discovery of the Gromov-Witten/Donaldson-Thomas correspondence in [24].

In example (5), only the trivial descendent insertion $\mathsf{1}$ appears. For nontrivial insertions, consider the case where X is $\mathbb{P}^3$. Let

$$\mathsf{p}, \mathsf{L} \in H_*(\mathbb{P}^3)$$

be the point and line classes in $\mathbb{P}^3$ respectively. Geometrically, there is unique line through two points of $\mathbb{P}^3$. The corresponding partition function is also simple,

$$\mathsf{Z}_{\mathsf{P}}\Big(\mathbb{P}^3; q \mid \tau_0(\mathsf{p})\tau_0(\mathsf{p})\Big)_{\mathsf{L}} = q + 2q^2 + q^3 \,. \tag{6}$$

The resulting series is not only rational, but in fact polynomial. For curve class L, the descendent invariants in (6) vanish for Euler characteristic greater than 3.

In example (6), only primary fields (with descendent subscript 0) appear. An example with higher descendents is

$$\mathsf{Z}_{\mathsf{P}}\Big(\mathbb{P}^3; q \mid \tau_2(\mathsf{p})\Big)_{\mathsf{L}} = \frac{1}{12}q - \frac{5}{6}q^2 + \frac{1}{12}q^3 \,.$$

The fractions here come from the Chern character. Again, the result is a cubic polynomial. More interesting is the partition function

$$\mathsf{Z}_{\mathsf{P}}\Big(\mathbb{P}^3; q \mid \tau_5(\mathsf{1})\Big)_{\mathsf{L}} = \frac{-2q - q^2 + 31q^3 - 31q^4 + q^5 + 2q^6}{18(1+q)^3} \,. \tag{7}$$

The partition functions considered so far are all in the absolute case. For an equivariant descendent series, consider the $\mathbf{T}$-action on $\mathbb{P}^3$ defined by representation weights $\lambda_0, \lambda_1, \lambda_2, \lambda_3$ on the vector space $\mathbb{C}^4$. Let

$$\mathsf{p}_0 \in H^4_{\mathbf{T}}(\mathbb{P}^3)$$

be the class of the $\mathbf{T}$-fixed point corresponding to the weight λ_0 subspace of $\mathbb{C}^4$. Then,

$$\mathsf{Z}_{\mathsf{P}}\Big(\mathbb{P}^3; q \mid \tau_3(\mathsf{p}_0)\Big)_{\mathsf{L}} = \frac{\mathsf{A}q - \mathsf{B}q^2 + \mathsf{B}q^3 - \mathsf{A}q^4}{(1+q)}$$

where $A, B \in H^2_{\mathbf{T}}(\bullet)$ are given by

$$\begin{aligned} \mathsf{A} &= \frac{1}{8}\lambda_0 - \frac{1}{24}(\lambda_1 + \lambda_2 + \lambda_3)\,, \\ \mathsf{B} &= \frac{9}{8}\lambda_0 - \frac{3}{8}(\lambda_1 + \lambda_2 + \lambda_3)\,. \end{aligned}$$

The descendent insertion here has dimension 5 which exceeds the virtual dimension 4 of the moduli space of stable pair, so the invariants lie in $H^2_{\mathbf{T}}(\bullet)$. The obvious symmetry in all of these descendent series is explained by the conjectural function equation (discussed in Section 1.7).

All of the formulas discussed above are calculated by the virtual localization formula [14] for stable pairs. The $\mathbf{T}$-fixed points, virtual tangent weights, and virtual normal weights are described in detail in [39].

1.5 Example in degree 2

A further example in the absolute case is the degree 2 series $\mathsf{Z}_{\mathsf{P}}\big(\mathsf{P}^3; q \mid \tau_9(1)\big)_{\mathsf{2L}}$. While a rigorous answer could be obtained, the available computer calculation here outputs a conjecture,[12]

$$\mathsf{Z}_{\mathsf{P}}\big(\mathbb{P}^3; q \mid \tau_9(1)\big)_{\mathsf{2L}} =$$
$$-\frac{(73q^{12} - 825q^{11} - 124q^{10} + 5945q^9 + 779q^8 - 36020q^7 + 60224q^6 - 36020q^5 + 779q^4 + 5945q^3 - 124q^2 - 825q + 73)q}{60480(1+q)^3(-1+q)^3}\,.$$

The computer calculations of Section 1.4 all provide rigorous results and could be improved to handle higher degree curves, but the code has not yet been written.

1.6 Relative theory

Let X be a nonsingular projective 3-fold containing a nonsingular divisor

$$D \subset X\,.$$

The *relative* case concerns the geometry X/D.

While the definitions and constructions are more involved in the relative case, the basic idea is simple. The moduli space of stable pairs on X/D includes stable pairs on X which are *transverse* to D. The transversality condition here has two parts:

(i) the section s of the stable pair has cokernel supported away from D,

(ii) the equation of D is not permitted to be a zero divisor on the support of the stable pair.

Conditions (i) and (ii) are *not* closed conditions on stable pairs on X. In a family, the support of the cokernel of s may approach D. The solution is then to change the geometry of X by bubbling off D. In fact, by appropriately bubbling X, a compact moduli space of stable pairs $P_n(X/D,\beta)$ on X/D satisfying both (i) and (ii) can be obtained.

[12]The answer relies on an old program for the theory of ideal sheaves written by A. Okounkov and a newer DT/PT descendent correspondence [29].

The moduli space $P_n(X/D, \beta)$ parameterizes stable relative pairs

$$s : \mathcal{O}_{X[k]} \to F \tag{8}$$

on the k-step[13] degeneration $X[k]$.

• The algebraic variety $X[k]$ is constructed by attaching a chain of k copies of the 3-fold $\mathbb{P}(N_{X/D} \oplus \mathcal{O}_D)$ equipped with 0-sections and ∞-sections

$$D \xrightarrow{\iota_0} \mathbb{P}(N_{X/D} \oplus \mathcal{O}_D) \xleftarrow{\iota_\infty} D$$

defined by the summands $N_{X/D}$ and $\mathcal{O}_D$ respectively. The k-step degeneration $X[k]$ is a union

$$X \cup_D \mathbb{P}(N_{X/D} \oplus \mathcal{O}_D) \cup_D \mathbb{P}(N_{X/D} \oplus \mathcal{O}_D) \cup_D \cdots \cup_D \mathbb{P}(N_{X/D} \oplus \mathcal{O}_D)\,,$$

where the attachments are made along ∞-sections on the left and 0-sections on the right. The original divisor $D \subset X$ is considered an ∞-section for the attachment rules. The rightmost component of $X[k]$ carries the last ∞-section,

$$D_\infty \subset X[k],$$

called the *relative divisor*. The k-step degeneration also admits a canonical contraction map

$$X[k] \to X \tag{9}$$

collapsing all the attached components to $D \subset X$.

• The sheaf F on $X[k]$ is of Euler characteristic

$$\chi(F) = n$$

and has 1-dimensional support on $X[k]$ which pushes-down via the contraction (9) to the class

$$\beta \in H_2(X, \mathbb{Z}).$$

• The following stability conditions are required for stable relative pairs:

(i) F is pure with finite locally free resolution,

[13] We follow the terminology of [20, 22].

(ii) the higher derived functors of the restriction of F to the singular[14] loci of $X[k]$ vanish,

(iii) the section s has 0-dimensional cokernel supported away from the singular loci of $X[k]$.

(iv) the pair (8) has only finitely many automorphisms covering the automorphisms of $X[k]/X$.

The moduli space $P_n(X/D, \beta)$ of stable relative pairs is a complete Deligne-Mumford stack equipped with a map to the Hilbert scheme of points of D via the restriction of the pair to the relative divisor,

$$P_n(X/D, \beta) \to \mathrm{Hilb}(D, \int_\beta [D]) \ .$$

Cohomology classes on $\mathrm{Hilb}(D, \int_\beta [D])$ may thus be pulled-back to the moduli space $P_n(X/D, \beta)$.

We will use the *Nakajima basis* of $H^*(\mathrm{Hilb}(D, \int_\beta [D]))$ indexed by a partition μ of $\int_\beta [D]$ labeled by cohomology classes of D. For example, the class

$$\Big|\mu\Big\rangle \in H^*(\mathrm{Hilb}(D, \int_\beta [D]))\,,$$

with all cohomology labels equal to the identity, is $\prod \mu_i^{-1}$ times the Poincaré dual of the closure of the subvariety formed by unions of schemes of length

$$\mu_1, \ldots, \mu_{\ell(\mu)}$$

supported at $\ell(\mu)$ distinct points of D.

The stable pairs descendent invariants in the relative case are defined using the universal sheaf just as in the absolute case. The universal sheaf is defined here on the universal degeneration of X/D over $P_n(X/D, \beta)$. The cohomology classes $\gamma_i \in H^*(X)$ are pulled-back to the universal degeneration via the contraction map (9). The descendent partition function with boundary conditions μ is a Laurent series in q,

$$\mathsf{Z}_{\mathsf{P}}\Big(X/D; q \ \Big| \ \prod_{i=1}^{r} \tau_{k_i}(\gamma_i) \ \Big| \ \mu\Big)_\beta \in \mathbb{Q}((q))\,.$$

[14]The singular loci of $X[k]$, by convention, include also the relative divisor $D_\infty \subset X[k]$ even though $X[k]$ is nonsingular along D_∞ as a variety. The perspective of log geometry is more natural here.

The basic rationality statement here is parallel to the absolute and equivariant cases.

Conjecture 2. *For X/D a nonsingular projective relative 3-fold, the descendent partition function*

$$\mathsf{Z}_{\mathsf{P}}\Big(X/D;q \ \Big| \ \prod_{i=1}^{r} \tau_{k_i}(\gamma_i) \ \Big| \ \mu\Big)_\beta \in \mathbb{Q}((q))$$

is the Laurent expansion in q of a rational function in $\mathbb{Q}(q)$.

In case X is a nonsingular quasi-projective toric 3-fold and $D \subset X$ is a toric divisor, an *equivariant relative* stable pairs theory can be defined. The rationality conjecture then takes the form expected by combining the rationality statements in the equivariant and relative cases.

Conjecture 3. *For X/D a nonsingular quasi-projective relative toric 3-fold, the descendent partition function*

$$\mathsf{Z}_{\mathsf{P}}\Big(X/D;q \ \Big| \ \prod_{i=1}^{r} \tau_{k_i}(\gamma_i) \ \Big| \ \mu\Big)^{\mathbf{T}}_\beta \in \mathbb{Q}(s_1,s_2,s_3)(q)$$

is the Laurent expansion in q of a rational function in $\mathbb{Q}(q,s_1,s_2,s_3)$.

Of course, both $\gamma_i \in H^\bullet_{\mathbf{T}}(X)$ and the Nakajima basis element

$$\mu \in H^*_{\mathbf{T}}(\mathrm{Hilb}(D, \int_\beta [D]))$$

must be taken here in equivariant cohomology. While the full statement of Conjecture 3 remains open, a partial result follows from Theorem 1 and [33, Theorem 2] which addresses the non-equivariant limit in the projective relative toric case.

Theorem 2 (P.-Pixton, 2012)**.** *For X/D a nonsingular projective relative toric 3-fold, the descendent partition function*

$$\mathsf{Z}_{\mathsf{P}}\Big(X;q \ \Big| \ \prod_{i=1}^{r} \tau_{k_i}(\gamma_i) \ \Big| \ \mu\Big)_\beta$$

is the Laurent expansion in q of a rational function in $\mathbb{Q}(q)$.

As an example of a computation in closed form in the equivariant relative case, consider the geometry of the *cap*,

$$\mathbb{C}^2 \times \mathsf{P}^1 / \mathbb{C}^2_\infty \,,$$

where $\mathbb{C}^2_\infty \subset \mathbb{C}^2 \times \mathsf{P}^1$ is the fiber of

$$\mathbb{C}^2 \times \mathsf{P}^1 \to \mathsf{P}^1$$

over $\infty \in \mathsf{P}^1$. The first two factors of the 3-torus $\mathbf{T}$ act on the $\mathbb{C}^2$-factor of the cap with tangent weights $-s_1$ and $-s_2$. The third factor of $\mathbf{T}$ acts on $\mathbb{P}^1$ factor of the cap with tangent weights $-s_3$ and s_3 at $0 \in \mathbb{P}^1$ and $\infty \in \mathbb{P}^1$ respectively.

From several perspectives, the equivariant relative descendent partition function

$$\mathsf{Z}^{\mathsf{cap}}_{\mathsf{P}}\big(\,\tau_d(\mathsf{p})\,|\,(d)\big)^{\mathbf{T}}_d = \sum_n \Big\langle \tau_d(\mathsf{p}) \,\Big|\, (d) \Big\rangle^{\mathrm{Cap}}_{n,d} q^n \;, \qquad d > 0$$

is the most important series in the cap geometry [34]. Here,

$$\mathsf{p} \in H^2_{\mathbf{T}}(\mathbb{C}^2 \times \mathbb{P}^1)$$

is the class of the $\mathbf{T}$-fixed point of $\mathbb{C}^2 \times \mathbb{P}^1$ over $0 \in \mathbb{P}^1$, and the Nakaijima basis element (d) is weighted with the identity class in $H^*_{\mathbf{T}}(\mathrm{Hilb}(\mathbb{C}^2, d))$. A central result of [34] is the following calculation.[15]

Theorem 3 (P.-Pixton, 2011)**.** *We have*

$$\mathsf{Z}^{\mathsf{cap}}_{\mathsf{P}}\big(\,\tau_d(\mathsf{p})\,|\,(d)\big)^{\mathbf{T}}_d = \frac{q^d}{d!}\left(\frac{s_1+s_2}{2}\right)\sum_{i=1}^{d} \frac{1+(-q)^i}{1-(-q)^i} \;.$$

In the above formula, the coefficient of q^d,

$$\big\langle \tau_d(\mathsf{p}), (d) \big\rangle_{\mathrm{Hilb}(\mathbb{C}^2,d)} = \frac{s_1+s_2}{2\cdot(d-1)!} \,,$$

is the classical $(\mathbb{C}^*)^2$-equivariant pairing on the Hilbert scheme of points $\mathrm{Hilb}(\mathbb{C}^2, d)$. The proof of Theorem 3 is a rather delicate localization calculation (using several special properties such as the a priori divisibility of the answer by $s_1 + s_2$ from the holomorphic symplectic form on $\mathrm{Hilb}(\mathbb{C}^2, d)$).

[15] The formula here differs from [34] by a factor of $s_1 s_2$ since a different convention for the cohomology class p is taken.

The difficulty in Theorem 3 is obtaining a closed form evaluation for all d. Any particular descendent series can be calculated by the localization methods. A calculation, for example, *not* covered by Theorem 3 is

$$\mathsf{Z}^{\mathsf{cap}}_{\mathsf{P}}\big(\,\tau_2(\mathsf{p})\,|\,(1)\big)^{\mathbf{T}}_1 = \big(2s_1^2+3s_1s_2+2s_2^2\big)q\frac{(1+q^2)}{(1+q)^2}$$
$$+\big(6s_3(s_1+s_2)-2s_1^2-6s_1s_2-2s_2^2\big)\frac{q^2}{(1+q)^2}\,. \quad (10)$$

A simple closed formula for all descendents of the cap is unlikely to exist.

1.7 Functional equation

In case X is a nonsingular Calabi-Yau 3-fold, the descendent series viewed as a rational function in q satisfies the symmetry

$$\mathsf{Z}_{\mathsf{P}}\Big(X;\frac{1}{q}\,\Big|\,1\Big)_\beta = \mathsf{Z}_{\mathsf{P}}\Big(X;q\,\Big|\,1\Big)_\beta \quad (11)$$

as conjectured in [24, 38] and proven in [4, 45]. In fact, a functional equation for the descendent partition function is expected to hold in *all* cases (absolute, equivariant, and relative). For the relative case, the functional equation is given by the following formula[16] [33, 34].

Conjecture 4 (P.-Pixton, 2012)**.** *For X/D a nonsingular projective relative 3-fold, the descendent series viewed as a rational function in q satisfies the functional equation*

$$\mathsf{Z}_{\mathsf{P}}\Big(X;\frac{1}{q}\;\Big|\;\prod_{i=1}^r\tau_{k_i}(\gamma_i)\;\Big|\;\mu\Big)_\beta = (-1)^{|\mu|-\ell(\mu)+\sum_{i=1}^r k_i}q^{-d_\beta}\,\mathsf{Z}_{\mathsf{P}}\Big(X;q\;\Big|\;\prod_{i=1}^r\tau_{k_i}(\gamma_i)\;\Big|\;\mu\Big)_\beta$$

where the constants are

$$|\mu| = \int_\beta D\,, \quad \ell(\mu) = \text{length}(\mu)\,, \quad d_\beta = \int_\beta c_1(X)\,.$$

[16] The conjecture is stated in [33, 34] with a sign error: the factor of q^{d_β} on the right side of the functional equation [33, 34] should be $(-q)^{d_\beta}$. Then two factors of $(-1)^{d_\beta}$ multiply to 1 and yield Conjecture 4 as stated here.

The functional equation in the absolute case is obtained by specializing the divisor $D \subset X$ to the empty set in Conjecture 4:

$$\mathsf{Z}_{\mathsf{P}}\Big(X;\frac{1}{q}\,\Big|\,\prod_{i=1}^{r}\tau_{k_i}(\gamma_i)\Big)_\beta = (-1)^{\sum_{i=1}^r k_i} q^{-d_\beta}\,\mathsf{Z}_{\mathsf{P}}\Big(X;q\,\Big|\,\prod_{i=1}^{r}\tau_{k_i}(\gamma_i)\,\Big)_\beta\,.$$

The functional equation in the equivariant case is conjectured to be identical,

$$\mathsf{Z}_{\mathsf{P}}\Big(X;\frac{1}{q}\,\Big|\,\prod_{i=1}^{r}\tau_{k_i}(\gamma_i)\Big)^{\mathbf{T}}_\beta = (-1)^{\sum_{i=1}^r k_i} q^{-d_\beta}\,\mathsf{Z}_{\mathsf{P}}\Big(X;q\,\Big|\,\prod_{i=1}^{r}\tau_{k_i}(\gamma_i)\,\Big)^{\mathbf{T}}_\beta\,.$$

Finally, in the equivariant relative case, the functional equation is expected to be same as in Conjecture 4.

As an example, the descendent series for the cap evaluated in Theorem 3 satisfies the conjectured functional equation:

$$\begin{aligned}
\mathsf{Z}^{\mathsf{cap}}_{\mathsf{P}}\left(\frac{1}{q};\,\tau_d(\mathsf{p})\,\Big|\,(d)\right)^{\mathbf{T}}_d &= \frac{q^{-d}}{d!}\left(\frac{s_1+s_2}{s_1s_2}\right)\frac{1}{2}\sum_{i=1}^{d}\frac{1+(-q)^{-i}}{1-(-q)^{-i}}\\
&= \frac{1}{q^{2d}}\frac{q^d}{d!}\left(\frac{s_1+s_2}{s_1s_2}\right)\frac{1}{2}\sum_{i=1}^{d}\frac{(-q)^{i}+1}{(-q)^{i}-1}\\
&= \frac{(-1)^{d-1+d}}{q^{2d}}\mathsf{Z}^{\mathsf{cap}}_{\mathsf{P}}\big(q;\,\tau_d(\mathsf{p})\,|\,(d)\big)^{\mathbf{T}}_d\,.
\end{aligned}$$

Here, the constants for the exponent of (-1) in the functional equation are

$$|(d)| = d\,,\quad \ell(d)=1\,,\quad d_\beta = 2d\,.$$

It is straightforward to check the functional equation in all the examples of Section 1.4 - 1.5.

The evidence for the functional equation for descendent series is not as large as for the rationality. For the equivariant relative cap, the functional equation is proven in [34] for all descendents series

$$\mathsf{Z}^{\mathsf{cap}}_{\mathsf{P}}\left(\prod_{i=1}^{r}\tau_{k_i}(\mathsf{p})\,\Big|\,(\mu)\right)^{\mathbf{T}}_d$$

after the specialization $s_3=0$. The predicted functional equation for

$$\mathsf{Z}^{\mathsf{cap}}_{\mathsf{P}}(\,\tau_2(\mathsf{p})\,|\,(1))^{\mathbf{T}}_1$$

before the specialization $s_3=0$ can be easily checked from the formula (10). The functional equation is also known to hold for special classes of descendent insertions in the nonsingular projective toric case [36] as will be discussed in Section 2.8.

1.8 Pole constraints

Let X be a nonsingular projective 3-fold, and let $\beta \in H_2(X,\mathbb{Z})$ be a nonzero class. For β to be an effective curve class, the image of β in the lattice

$$H_2(X,\mathbb{Z})/\text{torsion} \tag{12}$$

must also be nonzero. Let $\text{div}(\beta) \in \mathbb{Z}_{>0}$ be the divisibility of the image of β in the lattice (12).

Conjecture 5. *For $d = \text{div}(\beta)$, the poles in q of the rational function*

$$\mathsf{Z}_P\Big(X;q \;\Big|\; \prod_{i=1}^{r} \tau_{k_i}(\gamma_i)\Big)_\beta$$

may occur only at $q = 0$ and the roots of the polynomials

$$\{\, 1-(-q)^m \,|\, 1 \leq m \leq d \,\}.$$

Of the above conjectures, the evidence for Conjecture 5 is the weakest. In the Calabi-Yau case with no insertions, the statment is consistent with the Gopakumar-Vafa conjectures concerning BPS state counts. The full prediction is based on a study of the stable pairs theory of local curves where the above pole restrictions are always found. For example, the evaluation of Theorem 3 is consistent with the pole statement (even though Theorem 3 concerns the equivariant relative case). A promotion of Conjecture 5 to cover all cases also appears reasonable.

1.9 Complete intersections

Rationality results for non-toric 3-folds are proven in [37] by degeneration methods for several geometries. The simplest to state concern nonsingular complete intersections of ample divisors

$$X \subset \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_m} \ .$$

Theorem 4 (P.-Pixton, 2012)**.** *Let X be a nonsingular Fano or Calabi-Yau complete intersection 3-fold in a product of projective spaces. For even classes $\gamma_i \in H^{2*}(X)$, the descendent partition function*

$$\mathsf{Z}_{\mathsf{P}}\Big(X;q \;\Big|\; \prod_{i=1}^{r} \tau_{k_i}(\gamma_i)\Big)_\beta$$

is the Laurent expansion of a rational function in $\mathbb{Q}(q)$.

By the Lefschetz hyperplane result, the even cohomology of such X is exactly the image of the restricted cohomology from the product of projective spaces. Theorem 4 does *not* cover the primitive cohomology in $H^3(X)$. Moreover, even for descendents of the even cohomology $H^{2*}(X)$ the functional equation and pole conjectures are open.

2 Gromov-Witten/Pairs correspondence

2.1 Overview

Let X be a nonsingular projective variety. Descendent classes on the moduli spaces of stable maps $\overline{M}_{g,r}(X,\beta)$ in Gromov-Witten theory, defined using cotangent lines at the marked points, have played a central role since the beginning of the subject in the early 90s. Topological recursion relations, J-functions, and Virasoro constraints all essentially concern descendents. The importance of descendents in Gromov-Witten theory was hardly a surprise: cotangent lines on the moduli spaces $\overline{M}_{g,r}$ of stable curves were basic to their geometric study before Gromov-Witten theory was developed.

In case X is a nonsingular projective *3-fold*, descendent invariants are defined for both Gromov-Witten theory and the theory of stable pairs. The geometric constructions are rather different, but a surprising correspondence conjecturally holds: the two descendent theories are related by a universal correspondence for *all* nonsingular projective 3-folds. In order words, the two descendent theories contain exactly the same data.

The origin of the Gromov-Witten/Pairs correspondence is found in the study of ideal sheaves in [24, 25]. Since the descendent theory of stable pairs is much better behaved, the results and conjectures take a better form for stable pairs [36, 37].

The rationality results and conjectures of Section 1 are needed for the statement of the Gromov-Witten/Pairs correspondence. Just as in Section 1, we present the absolute, equivariant, and relative cases. A more subtle discussion of diagonals is required for the relative case.

2.2 Descendents in Gromov-Witten theory

Let X be a nonsingular projective 3-fold. Gromov-Witten theory is defined via integration over the moduli space of stable maps. Let $\overline{M}_{g,r}(X,\beta)$ denote the moduli space of r-pointed stable maps from connected genus g curves to X representing the class $\beta \in H_2(X, \mathbb{Z})$. Let

$$\mathrm{ev}_i : \overline{M}_{g,r}(X, \beta) \to X \,,$$

$$\mathbb{L}_i \to \overline{M}_{g,r}(X, \beta)$$

denote the evaluation maps and the cotangent line bundles associated to the marked points. Let $\gamma_1, \ldots, \gamma_r \in H^*(X)$, and let

$$\psi_i = c_1(\mathbb{L}_i) \in H^2(\overline{M}_{g,n}(X, \beta)) \,.$$

The *descendent fields*, denoted by $\tau_k(\gamma)$, correspond to the classes $\psi_i^k \mathrm{ev}_i^*(\gamma)$ on the moduli space of stable maps. Let

$$\Big\langle \tau_{k_1}(\gamma_1) \cdots \tau_{k_r}(\gamma_r) \Big\rangle_{g,\beta} = \int_{[\overline{M}_{g,r}(X,\beta)]^{vir}} \prod_{i=1}^{r} \psi_i^{k_i} \mathrm{ev}_i^*(\gamma_i)$$

denote the descendent Gromov-Witten invariants. Foundational aspects of the theory are treated, for example, in [3, 21].

Let C be a possibly disconnected curve with at worst nodal singularities. The genus of C is defined by $1 - \chi(\mathcal{O}_C)$. Let $\overline{M}'_{g,r}(X, \beta)$ denote the moduli space of maps with possibly disconnected domain curves C of genus g with *no* collapsed connected components. The latter condition requires each connected component of C to represent a nonzero class in $H_2(X, \mathbb{Z})$. In particular, C must represent a nonzero class β.

We define the descendent invariants in the disconnected case by

$$\Big\langle \tau_{k_1}(\gamma_1) \cdots \tau_{k_r}(\gamma_r) \Big\rangle'_{g,\beta} = \int_{[\overline{M}'_{g,r}(X,\beta)]^{vir}} \prod_{i=1}^{r} \psi_i^{k_i} \mathrm{ev}_i^*(\gamma_i).$$

The associated partition function is defined by[17]

$$\mathsf{Z}'_{\mathsf{GW}}\Big(X; u \;\Big|\; \prod_{i=1}^{r} \tau_{k_i}(\gamma_i)\Big)_{\beta} = \sum_{g \in \mathbb{Z}} \Big\langle \prod_{i=1}^{r} \tau_{k_i}(\gamma_i) \Big\rangle'_{g,\beta} u^{2g-2}. \tag{13}$$

Since the domain components must map nontrivially, an elementary argument shows the genus g in the sum (13) is bounded from below.

[17] Our notation follows [25, 27] and emphasizes the role of the moduli space $\overline{M}'_{g,r}(X, \beta)$. The degree 0 collapsed contributions will not appear anywhere in the paper.

2.3 Dimension constraints

Descendents in Gromov-Witten and stable pairs theories are obtained via tautological structures over the moduli spaces

$$\overline{M}'_{g,r}(X,\beta)\,, \quad P_n(X,\beta)\times X$$

respectively. The descendents $\tau_k(\gamma)$ in both cases mix the characteristic classes of the tautological sheaves

$$\mathbb{L}_i \to \overline{M}'_{g,r}(X,\beta)\,, \quad \mathbb{F}\to P_n(X,\beta)\times X$$

with the pull-back of $\gamma \in H^*(X)$ via the evaluation/projective morphism.

In the absolute (nonequivariant) case, the Gromov-Witten and stable pairs descendent series

$$\mathsf{Z}'_{\mathsf{GW}}\Big(X;u \;\Big|\; \prod_{i=1}^r \tau_{k_i}(\gamma_i)\Big)_\beta\,, \quad \mathsf{Z}_{\mathsf{P}}\Big(X;q\;\Big|\; \prod_{i=1}^r \tau_{k_i}(\gamma_i)\Big)_\beta \tag{14}$$

both satisfy dimension constraints. For $\gamma_i\in H^{e_i}(X)$, the (real) dimension of the descendents Gromov-Witten and stable pairs theories are

$$\tau_{k_i}(\gamma_i)\in H^{e_i+2k_i}(\overline{M}'_{g,r}(X,\beta))\,, \quad \tau_{k_i}(\gamma_i)\in H^{e_i+2k_i-2}(P_n(X,\beta))\,.$$

Since the virtual dimensions are

$$\dim_{\mathbb{C}}\,[\overline{M}'_{g,r}(X,\beta)]^{vir} = \int_\beta c_1(T_X)+r\,, \quad \dim_{\mathbb{C}}\,[P_n(X,\beta)]^{vir} = \int_\beta c_1(T_X)$$

respectively, the dimension constraints

$$\sum_{i=1}^r \frac{e_i}{2}+k_i = \int_\beta c_1(T_X)+r\,, \quad \sum_{i=1}^r \frac{e_i}{2}+k_i-1 = \int_\beta c_1(T_X)$$

exactly match.

After the matching of the dimension constraints, we can further reasonably ask if there is a relationship between the Gromov-Witten and stable pairs descendent series (14). The question has two immediately puzzling features:

(i) The series involve different moduli spaces and universal structures.

(ii) The variables u and q of the two series are associated to different invariants (the genus and the Euler characteristic).

Though the worry (i) is correct, both moduli spaces are essentially based upon the geometry of curves in X, so there is hope for a connection. The *descendent correspondence* proposes a precise relationship between the Gromov-Witten and stable pairs descendent series, but only after a change of variables to address (ii).

2.4 Descendent notation

Let X be a nonsingular projective 3-fold. Let $\widehat{\alpha} = (\widehat{\alpha}_1, \ldots, \widehat{\alpha}_{\widehat{\ell}})$,

$$\widehat{\alpha}_1 \geq \ldots \geq \widehat{\alpha}_{\widehat{\ell}} > 0\,,$$

be a partition of size $|\widehat{\alpha}|$ and length $\widehat{\ell}$. Let

$$\iota_\Delta : \Delta \to X^{\widehat{\ell}}$$

be the inclusion of the small diagonal[18] in the product $X^{\widehat{\ell}}$. For $\gamma \in H^*(X)$, we write

$$\gamma \cdot \Delta = \iota_{\Delta *}(\gamma) \in H^*(X^{\widehat{\ell}})\,.$$

Using the Künneth decomposition, we have

$$\gamma \cdot \Delta = \sum_{j_1,\ldots,j_{\hat{\ell}}} c^{\gamma}_{j_1,\ldots,j_{\hat{\ell}}}\, \theta_{j_1} \otimes \ldots \otimes \theta_{j_{\hat{\ell}}}\,,$$

where $\{\theta_j\}$ is a $\mathbb{Q}$-basis of $H^*(X)$. We define the descendent insertion $\tau_{\widehat{\alpha}}(\gamma)$ by

$$\tau_{\widehat{\alpha}}(\gamma) = \sum_{j_1,\ldots,j_{\hat{\ell}}} c^{\gamma}_{j_1,\ldots,j_{\hat{\ell}}}\, \tau_{\widehat{\alpha}_1-1}(\theta_{j_1}) \cdots \tau_{\widehat{\alpha}_{\hat{\ell}}-1}(\theta_{j_{\hat{\ell}}})\,. \tag{15}$$

Three basic examples are:

- If $\widehat{\alpha} = (\widehat{a}_1)$, then
$$\tau_{(\widehat{a}_1)}(\gamma) = \tau_{\widehat{a}_1-1}(\gamma)\,.$$
The convention of shifting the descendent by 1 allows us to index descendent insertions by standard partitions $\widehat{\alpha}$ and follows the notation of [36].

[18] The small diagonal Δ is the set of points of $X^{\widehat{\ell}}$ for which the coordinates $(x_1, \ldots, x_{\hat{\ell}})$ are all equal $x_i = x_j$.

- If $\widehat{\alpha} = (\widehat{a}_1, \widehat{a}_2)$ and $\gamma = 1$ is the identity class, then

$$\tau_{(\widehat{a}_1, \widehat{a}_2)}(1) = \sum_{j_1, j_2} c^1_{j_1,j_2} \tau_{\widehat{a}_1 - 1}(\theta_{j_1})\, \tau_{\widehat{a}_2 - 1}(\theta_{j_2})\,,$$

  where $\Delta = \sum_{j_1,j_2} c^1_{j_1,j_2}\, \theta_{j_1} \otimes \theta_{j_2}$ is the standard Künneth decomposition of the diagonal in X^2.

- If γ is the class of a point, then

$$\tau_{\widehat{\alpha}}(\mathsf{p}) = \tau_{\widehat{\alpha}_1 - 1}(\mathsf{p}) \cdots \tau_{\widehat{\alpha}_{\hat{\ell}} - 1}(\mathsf{p}).$$

By the multilinearity of descendent insertions, formula (15) does not depend upon the basis choice $\{\theta_j\}$.

While definition (15) provides an explicit formula for the descendent insertion $\tau_{\widehat{\alpha}}(\gamma)$, the action of the descendent on the moduli space of stable maps $\overline{M}'_{g,\hat{\ell}}(X, \beta)$ is expressed geometrically by

$$\tau_{\widehat{\alpha}}(\gamma) = \psi_1^{\widehat{\alpha}_1 - 1} \cdots \psi_{\hat{\ell}}^{\widehat{\alpha}_{\hat{\ell}} - 1} \cdot \mathrm{ev}^*_{1,\ldots,\hat{\ell}}(\gamma \cdot \Delta)\ ,$$

where the evaluation map is

$$\mathrm{ev}_{1,\ldots,\hat{\ell}} : \overline{M}'_{g,\hat{\ell}}(X, \beta) \to X^{\hat{\ell}}\,.$$

The diagonals play a crucial role in the Gromov-Witten/Pairs correspondence for descendents – the two moduli spaces treat the diagonals differently.

2.5 Correspondence matrix

A central result of [36] is the construction of a universal correspondence matrix $\widetilde{\mathsf{K}}$ indexed by partitions α and $\widehat{\alpha}$ of positive size with[19]

$$\widetilde{\mathsf{K}}_{\alpha,\widehat{\alpha}} \in \mathbb{Q}[i, c_1, c_2, c_3]((u))\,.$$

The elements of $\widetilde{\mathsf{K}}$ are constructed from the capped descendent vertex [36] and satisfy two basic properties:

(i) The vanishing $\widetilde{\mathsf{K}}_{\alpha,\widehat{\alpha}} = 0$ holds unless $|\alpha| \geq |\widehat{\alpha}|$.

[19] Here, $i^2 = -1$.

(ii) The u coefficients of $\widetilde{\mathsf{K}}_{\alpha,\widehat{\alpha}} \in \mathbb{Q}[i,c_1,c_2,c_3]((u))$ are homogeneous[20] in the variables c_i of degree

$$|\alpha| + \ell(\alpha) - |\widehat{\alpha}| - \ell(\widehat{\alpha}) - 3(\ell(\alpha)-1).$$

Via the substitution

$$c_i = c_i(T_X), \tag{16}$$

the matrix elements of $\widetilde{\mathsf{K}}$ act by cup product on the cohomology of X with $\mathbb{Q}[i]((u))$-coefficients.

The matrix $\widetilde{\mathsf{K}}$ is used to define a correspondence rule

$$\tau_{\alpha_1-1}(\gamma_1)\cdots\tau_{\alpha_\ell-1}(\gamma_\ell) \quad \mapsto \quad \overline{\tau_{\alpha_1-1}(\gamma_1)\cdots\tau_{\alpha_\ell-1}(\gamma_\ell)}\ . \tag{17}$$

The definition of the right side of (17) requires a sum over all set partitions P of $\{1,\ldots,\ell\}$. For such a set partition P, each element $S \in P$ is a subset of $\{1,\ldots,\ell\}$. Let α_S be the associated subpartition of α, and let

$$\gamma_S = \prod_{i\in S} \gamma_i.$$

In case all cohomology classes γ_j are even, we define the right side of the correspondence rule (17) by

$$\overline{\tau_{\alpha_1-1}(\gamma_1)\cdots\tau_{\alpha_\ell-1}(\gamma_\ell)} = \sum_{P \text{ set partition of } \{1,\ldots,\ell\}} \ \prod_{S\in P} \ \sum_{\widehat{\alpha}} \tau_{\widehat{\alpha}}(\widetilde{\mathsf{K}}_{\alpha_S,\widehat{\alpha}}\cdot\gamma_S)\ . \tag{18}$$

The second sum in (18) is over all partitions $\widehat{\alpha}$ of positive size. However, by the vanishing of property (i),

$$\widetilde{\mathsf{K}}_{\alpha_S,\widehat{\alpha}} = 0 \ \text{ unless } \ |\alpha_S| \geq |\widehat{\alpha}|\,,$$

the summation index may be restricted to partitions $\widehat{\alpha}$ of positive size bounded by $|\alpha_S|$.

Suppose $|\alpha_S| = |\widehat{\alpha}|$ in the second sum in (18). The homogeneity property (ii) then places a strong constraint. The u coefficients of

$$\widetilde{\mathsf{K}}_{\alpha_S,\widehat{\alpha}} \in \mathbb{Q}[i,c_1,c_2,c_3]((u))$$

[20]The variable c_i has degree i for the homogeneity.

are homogeneous of degree

$$3 - 2\ell(\alpha_S) - \ell(\widehat{\alpha})\,. \tag{19}$$

For the matrix element $\widetilde{\mathsf{K}}_{\alpha_S,\widehat{\alpha}}$ to be nonzero, the degree (19) must be nonnegative. Since the lengths of α_S and $\widehat{\alpha}$ are at least 1, nonnegativity of (19) is only possible if

$$\ell(\alpha_S) = \ell(\widehat{\alpha}) = 1\,.$$

Then, we also have $\alpha_S = \widehat{\alpha}$ since the sizes match.

The above argument shows that the descendents on the right side of (18) all correspond to partitions of size *less* than $|\alpha|$ except for the *leading term* obtained from the the maximal set partition

$$\{1\} \cup \{2\} \cup \ldots \cup \{\ell\} = \{1, 2, \ldots, \ell\}$$

in ℓ parts. The leading term of the descendent correspondence, calculated in [36], is a third basic property of $\widetilde{\mathsf{K}}$:

(iii) $\overline{\tau_{\alpha_1-1}(\gamma_1)\cdots\tau_{\alpha_\ell-1}(\gamma_\ell)} = (iu)^{\ell(\alpha)-|\alpha|}\,\tau_{\alpha_1-1}(\gamma_1)\cdots\tau_{\alpha_\ell-1}(\gamma_\ell) + \ldots.$

In case $\alpha = 1^\ell$ has all parts equal to 1, then α_S also has all parts equal to 1 for every $S \in P$. By property (ii), the u coefficients of $\widetilde{\mathsf{K}}_{\alpha_S,\widehat{\alpha}}$ are homogeneous of degree

$$3 - \ell(\alpha_S) - |\widehat{\alpha}| - \ell(\widehat{\alpha}),$$

and hence vanish unless

$$\alpha_S = \widehat{\alpha} = (1)\ .$$

Therefore, if α has all parts equal to 1, the leading term is therefore the entire formula. We obtain a fourth property of the matrix $\widetilde{\mathsf{K}}$:

(iv) $\overline{\tau_0(\gamma_1)\cdots\tau_0(\gamma_\ell)} = \tau_0(\gamma_1)\cdots\tau_0(\gamma_\ell)\,.$

In the presence of odd cohomology, a natural sign must be included in formula (18). We may write set partitions P of $\{1, \ldots, \ell\}$ indexing the sum on the right side of (18) as

$$S_1 \cup \ldots \cup S_{|P|} = \{1, \ldots, \ell\}.$$

The parts S_i of P are unordered, but we choose an ordering for each P. We then obtain a permutation of $\{1, \ldots, \ell\}$ by moving the elements to the ordered parts S_i (and respecting the original order in each group). The permutation, in turn, determines a sign $\sigma(P)$ determined by the anti-commutation of the associated odd classes. We then write

$$\overline{\tau_{\alpha_1-1}(\gamma_1)\cdots\tau_{\alpha_\ell-1}(\gamma_\ell)} = \sum_{P \text{ set partition of } \{1,\ldots,\ell\}} (-1)^{\sigma(P)} \prod_{S_i \in P} \sum_{\widehat{\alpha}} \tau_{\widehat{\alpha}}(\widetilde{\mathsf{K}}_{\alpha_{S_i},\widehat{\alpha}} \cdot \gamma_{S_i}) \, .$$

The descendent $\overline{\tau_{\alpha_1-1}(\gamma_1)\cdots\tau_{\alpha_\ell-1}(\gamma_\ell)}$ is easily seen to have the same commutation rules with respect to odd cohomology as $\tau_{\alpha_1-1}(\gamma_1)\cdots\tau_{\alpha_\ell-1}(\gamma_\ell)$.

The geometric construction of $\widetilde{\mathsf{K}}$ in [36] expresses the coefficients explicitly in terms of the 1-legged capped descendent vertex for stable pairs and stable maps. These vertices can be computed (as a rational function in the stable pairs case and term by term in the genus parameter for stable maps). Hence, the coefficient

$$\widetilde{\mathsf{K}}_{\alpha,\widehat{\alpha}} \in \mathbb{Q}[i, c_1, c_2, c_3]((u))$$

can, in principle, be calculated term by term in u. The calculations in practice are quite difficult, and complete closed formulas are not known for all of the coefficients.

2.6 Absolute case

To state the descendent correspondence proposed in [36] for all nonsingular projective 3-folds X, the basic degree

$$d_\beta = \int_\beta c_1(X) \ \in \mathbb{Z}$$

associated to the class $\beta \in H_2(X,\mathbb{Z})$ is required.

Conjecture 6 (P.-Pixton (2011))**.** *Let X be a nonsingular projective 3-fold. For $\gamma_i \in H^*(X)$, we have*

$$(-q)^{-d_\beta/2}\mathsf{Z}_{\mathsf{P}}\Big(X;q \ \Big|\tau_{\alpha_1-1}(\gamma_1)\cdots\tau_{\alpha_\ell-1}(\gamma_\ell)\Big)_\beta$$
$$= (-iu)^{d_\beta}\mathsf{Z}'_{\mathsf{GW}}\Big(X;u \ \Big| \ \overline{\tau_{\alpha_1-1}(\gamma_1)\cdots\tau_{\alpha_\ell-1}(\gamma_\ell)} \ \Big)_\beta$$

under the variable change $-q = e^{iu}$.

Since the stable pairs side of the correspondence

$$\mathsf{Z}_{\mathsf{P}}\Big(X;q\ \Big|\tau_{\alpha_1-1}(\gamma_1)\cdots\tau_{\alpha_\ell-1}(\gamma_\ell)\Big)_\beta \in \mathbb{Q}((q))$$

is defined as a series in q, the change of variable $-q = e^{iu}$ is *not* a priori well-defined. However, the stable pairs descendent series is predicted by Conjecture 1 to be a rational function in q. The change of variable $-q = e^{iu}$ is well-defined for a rational function in q by substitution. The well-posedness of Conjecture 6 therefore depends upon Conjecture 1.

2.7 Geometry of descendents

Let X be a nonsingular projective 3-fold, and let $D \subset X$ be a nonsingular divisor. The Gromov-Witten descendent insertion $\tau_1(D)$ has a simple geometric leading term. Let

$$[f:(C,p)\to X]\in \overline{M}_{g,1}(X,\beta)$$

be a stable map. Let

$$\mathrm{ev}_1:\overline{M}_{g,1}(X,\beta)\to X$$

be the evaluation map at the marking. The cycle

$$\mathrm{ev}_1^{-1}(D)\subset \overline{M}_{g,1}(X,\beta)$$

corresponds to stable maps with $f(p)\in D$. On the locus $\mathrm{ev}_1^{-1}(D)$, there is a differential

$$df: T_{C,p}\to N_{X/D,f(p)} \tag{20}$$

from the tangent space of C at p to the normal space of $D\subset X$ at $f(p)\in D$. The differential (20) on $\mathrm{ev}_1^{-1}(D)$ vanishes on the locus where $f(C)$ is *tangent* to D at p. In other words,

$$\tau_1(D)+\tau_0(D^2)=\mathrm{ev}_1^{-1}(D)\left(-c_1(T_{C,p})+\mathrm{ev}_1^*(N_{X/D})\right)$$

has the tangency cycle as a leading term. There are correction terms from the loci where p lies on a component of C contracted by f to a point of D.

A parallel relationship can be pursued for $\tau_k(D)$ for for higher k in terms of the locus of stable maps with higher tangency along D at $f(p)$. A full correction calculus in case X has dimension 1 (instead of 3) was found in [30].

The method has also been successfully applied to calculate the characteristic numbers of curves in $\mathbb{P}^2$ for genus at most 2 in [13].[21]

By the Gromov-Witten/Pairs correspondence of Conjecture 6, the stable pairs descendent $\tau_k(D)$ has leading term on the Gromov-Witten side

$$\overline{\tau_k(D)} = (iu)^{-k}\tau_k(D) + \dots .$$

Hence, the descendents $\tau_k(D)$ on the stable pairs side should be viewed as essentially connected to the tangency loci associated to the divisor $D \subset X$.

2.8 Equivariant case

If X is a nonsingular quasi-projective toric 3-fold, all terms of the descendent correspondence have $\mathbf{T}$-equivariant interpretations. We take the equivariant Künneth decomposition in (15), and the equivariant Chern classes $c_i(T_X)$ with respect to the canonical $\mathbf{T}$-action on T_X in (16). The toric case is proven in [36].

Theorem 5 (P.-Pixton, 2011)**.** *Let X be a nonsingular quasi-projective toric 3-fold. For $\gamma_i \in H^*_{\mathbf{T}}(X)$, we have*

$$(-q)^{-d_\beta/2}\mathsf{Z}_{\mathsf{P}}\Big(X;q\ \Big|\tau_{\alpha_1-1}(\gamma_1)\cdots\tau_{\alpha_\ell-1}(\gamma_\ell)\Big)^{\mathbf{T}}_\beta$$
$$= (-iu)^{d_\beta}\mathsf{Z}'_{\mathsf{GW}}\Big(X;u\ \Big|\ \overline{\tau_{\alpha_1-1}(\gamma_1)\cdots\tau_{\alpha_\ell-1}(\gamma_\ell)}\ \Big)^{\mathbf{T}}_\beta$$

under the variable change $-q = e^{iu}$.

Since the stable pairs side of the correspondence

$$\mathsf{Z}_{\mathsf{P}}\Big(X;q\ \Big|\tau_{\alpha_1-1}(\gamma_1)\cdots\tau_{\alpha_\ell-1}(\gamma_\ell)\Big)^{\mathbf{T}}_\beta \in \mathbb{Q}(s_1,s_2,s_3)((q))$$

is a rational function in q by Theorem 1, the change of variable $-q = e^{iu}$ is well-defined by substitution.

When X is a nonsingular projective toric 3-fold, Theorem 5 implies Conjecture 6 for X by taking the non-equivariant limit. However, Theorem 5 is much stronger in the toric case than Conjecture 6 since the descendent insertions may exceed the virtual dimension in equivariant cohomology.

[21] In higher genus, the correction calculus in $\mathbb{P}^2$ was too complicated to easily control.

In case $\alpha = (1)^\ell$ has all parts equal to 1, Theorem 5 specializes by property (iv) of Section 2.5 to the simpler statement

$$(-q)^{-d_\beta/2}\mathsf{Z}_{\mathsf{P}}\Big(X;q\ \Big|\ \tau_0(\gamma_1)\cdots\tau_0(\gamma_\ell)\Big)^{\mathbf{T}}_\beta \\ = (-iu)^{d_\beta}\mathsf{Z}'_{\mathsf{GW}}\Big(X;u\ \Big|\ \tau_0(\gamma_1)\cdots\tau_0(\gamma_\ell)\Big)^{\mathbf{T}}_\beta \quad (21)$$

which was first proven in the context of ideal sheaves in [27]. Viewing both sides of (21) as series in u, we can complex conjugate the coefficients. Imaginary numbers only occur in

$$-q = e^{iu} \quad \text{and} \quad (-iu)^{d_\beta}\,.$$

After complex conjugation, we find

$$(-q)^{d_\beta/2}\mathsf{Z}_{\mathsf{P}}\Big(X;\frac{1}{q}\ \Big|\ \tau_0(\gamma_1)\cdots\tau_0(\gamma_\ell)\Big)^{\mathbf{T}}_\beta \\ = (iu)^{d_\beta}\mathsf{Z}'_{\mathsf{GW}}\Big(X;u\ \Big|\ \tau_0(\gamma_1)\cdots\tau_0(\gamma_\ell)\Big)^{\mathbf{T}}_\beta$$

and thus obtain the functional equation

$$\mathsf{Z}_{\mathsf{P}}\Big(X;\frac{1}{q}\ \Big|\ \tau_0(\gamma_1)\cdots\tau_0(\gamma_\ell)\Big)^{\mathbf{T}}_\beta = q^{-d_\beta}\mathsf{Z}_{\mathsf{P}}\Big(X;q\ \Big|\ \tau_0(\gamma_1)\cdots\tau_0(\gamma_\ell)\Big)^{\mathbf{T}}_\beta$$

as predicted by Conjecture 4.

2.9 Relative case

2.9.1 Relative Gromov-Witten theory

Let X be a nonsingular projective 3-fold with a nonsingular divisor

$$D \subset X\,.$$

The relative theory of stable pairs was discussed in Section 1.6. A parallel relative Gromov-Witten theory of stable maps with specified tangency along the divisor D can also be defined.

In Gromov-Witten theory, relative conditions are represented by a partition μ of the integer $\int_\beta [D]$, each part μ_i of which is marked by a cohomology class $\delta_i \in H^*(D, \mathbb{Z})$,

$$\mu = ((\mu_1, \delta_1), \ldots, (\mu_\ell, \delta_\ell)) \,. \tag{22}$$

The numbers μ_i record the multiplicities of intersection with D while the cohomology labels δ_i record where the tangency occurs. More precisely, let $\overline{M}'_{g,r}(X/D, \beta)_\mu$ be the moduli space of stable relative maps with tangency conditions μ along D. To impose the full boundary condition, we pull-back the classes δ_i via the evaluation maps

$$\overline{M}'_{g,r}(X/D, \beta)_\mu \to D \tag{23}$$

at the points of tangency. Also, the tangency points are considered to be unordered.[22]

Relative Gromov-Witten theory was defined before the study of stable pairs. For the foundations, including the definition of the moduli space of stable relative maps and the construction of the virtual class

$$[\overline{M}'_{g,r}(X/D, \beta)_\mu] \in H_*(\overline{M}'_{g,r}(X/D, \beta)_\mu) \,,$$

we refer the reader to [19, 20].

2.9.2 Diagonal classes

Definition (18) of the Gromov-Witten/Pairs correspondence in the absolute case involves the diagonal

$$\iota_\Delta : \Delta \to X^s$$

via (15). For the correspondence in the relative case, the diagonal has a more subtle definition.

For the absolute geometry X, the product X^s naturally parameterizes s ordered (possibly coincident) points on X. For the relative geometry X/D, the parallel object is the moduli space $(X/D)^s$ of s ordered (possibly coincident) points

$$(p_1, \ldots, p_s) \in X/D \,.$$

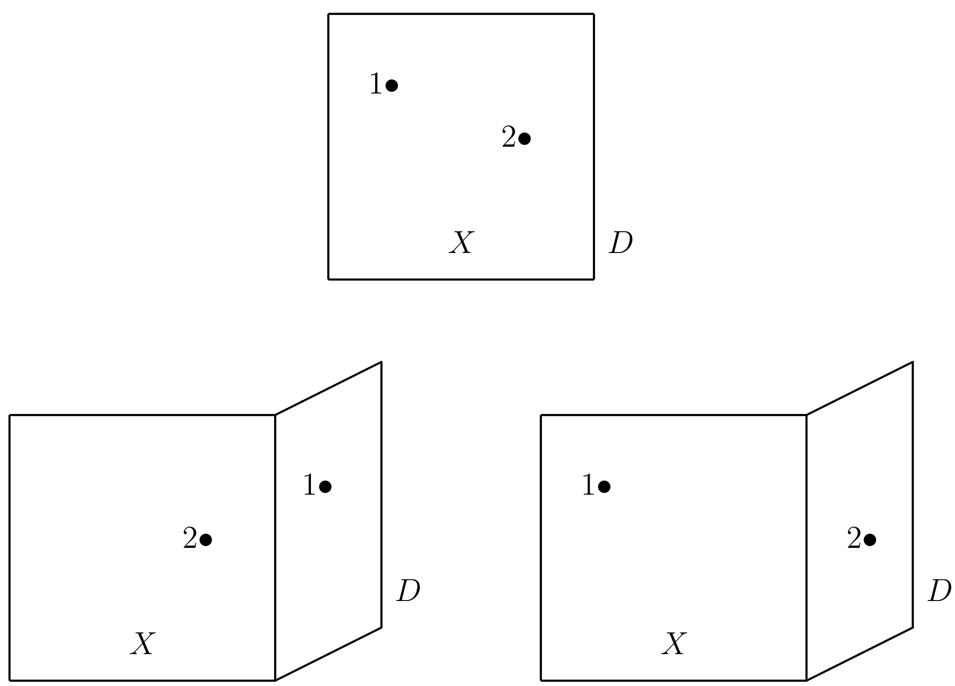
[22]The evaluation maps are well-defined only after ordering the points. We define the theory first with ordered tangency points. The unordered theory is then defined by dividing by the automorphisms of the cohomology weighted partition μ.

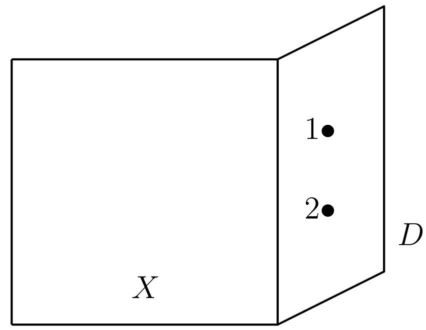
The points parameterized by $(X/D)^s$ are not allowed to lie on the relative divisor D. When the points approach D, the target X degenerates. The resulting moduli space $(X/D)^s$ is a nonsingular variety. Let

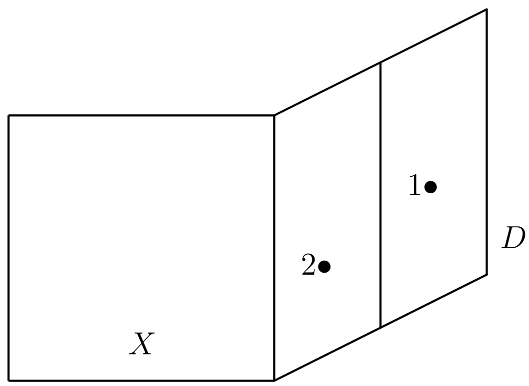
$$\Delta_{\mathsf{rel}} \subset (X/D)^s$$

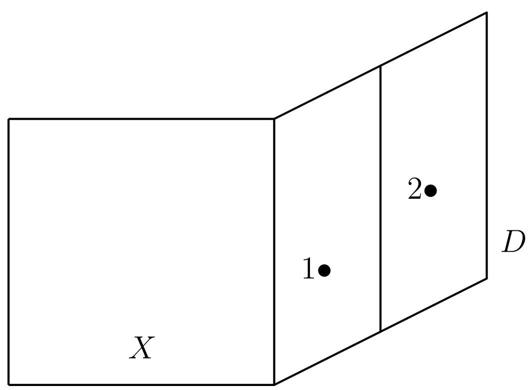
be the small diagonal where all the points p_i are coincident. As a variety, Δ_{rel} is isomorphic to X.

The space $(X/D)^s$ is a special case of well-known constructions in relative geometry. For example, $(X/D)^2$ consists of 6 strata:









As a variety, $(X/D)^2$ is the blow-up of X^2 along D^2. And, $\Delta_{\mathsf{rel}} \subset (X/D)^2$ is the strict transform of the standard diagonal.

Select a subset S of cardinality s from the r markings of the moduli space of maps. Just as $\overline{M}'_{g,r}(X,\beta)$ admits a canonical evaluation to X^s via the selected markings, the moduli space $\overline{M}'_{g,r}(X/D,\beta)_\mu$ admits a canonical evaluation

$$\mathrm{ev}_S : \overline{M}'_{g,r}(X/D,\beta)_\mu \to (X/D)^s,$$

well-defined by the definition of a relative stable map (the markings never map to the relative divisor). The class

$$\mathrm{ev}_S^*(\Delta_{\mathsf{rel}}) \in H^*(\overline{M}'_{g,r}(X/D,\beta)_\mu)$$

plays a crucial role in the relative descendent correspondence.

By forgetting the relative structure, we obtain a projection

$$\pi : (X/D)^s \to X^s \ .$$

The product contains the standard diagonal $\Delta \subset X^s$. However,

$$\pi^*(\Delta) \neq \Delta_{\mathsf{rel}} \ .$$

The former has more components in the relative boundary if $D \neq \emptyset$.

2.9.3 Relative descendent correspondence

Let $\widehat{\alpha}$ be a partition of length $\widehat{\ell}$. Let Δ_{rel} be the cohomology class of the small diagonal in $(X/D)^{\widehat{\ell}}$. For a cohomology class γ of X, let

$$\gamma \cdot \Delta_{\mathsf{rel}} \in H^*\big((X/D)^{\widehat{\ell}}\big),$$

where Δ_{ref} is the small diagonal of Section 2.9.2. Define the relative descendent insertion $\tau_{\widehat{\alpha}}(\gamma)$ by

$$\tau_{\widehat{\alpha}}(\gamma) = \psi_1^{\widehat{\alpha}_1-1} \cdots \psi_{\widehat{\ell}}^{\widehat{\alpha}_{\widehat{\ell}}-1} \cdot \mathrm{ev}^*_{1,\ldots,\hat{\ell}}(\gamma \cdot \Delta_{\mathsf{rel}}) \ . \tag{24}$$

In case, $D = \emptyset$, definition (24) specializes to (15).

Let $\Omega_X[D]$ denote the locally free sheaf of differentials with logarithmic poles along D. Let

$$T_X[-D] = \Omega_X[D]^{\ \vee}$$

denote the dual sheaf of tangent fields with logarithmic zeros.

For the relative geometry X/D, the coefficients of the correspondence matrix $\widetilde{\mathsf{K}}$ act on the cohomology of X via the substitution

$$c_i = c_i(T_X[-D])$$

instead of the substitution $c_i = c_i(T_X)$ used in the absolute case. Then, we define

$$\overline{\tau_{\alpha_1-1}(\gamma_1)\cdots\tau_{\alpha_\ell-1}(\gamma_\ell)} = \sum_{P \text{ set partition of } \{1,\ldots,l\}} \prod_{S\in P} \sum_{\widehat{\alpha}} \tau_{\widehat{\alpha}}(\widetilde{\mathsf{K}}_{\alpha_S,\widehat{\alpha}} \cdot \gamma_S) \qquad (25)$$

as before via (24) instead of (15). Definition (25) is for even classes γ_i. In the presence of odd γ_i, a sign has to be included exactly as in the absolute case.

Conjecture 7. *For $\gamma_i \in H^*(X)$, we have*

$$(-q)^{-d_\beta/2}\mathsf{Z}_{\mathsf{P}}\Big(X/D; q \,\Big|\tau_{\alpha_1-1}(\gamma_1)\cdots\tau_{\alpha_\ell-1}(\gamma_\ell)\,\Big|\, \mu\Big)_\beta$$
$$= (-iu)^{d_\beta+\ell(\mu)-|\mu|}\mathsf{Z}'_{\mathsf{GW}}\Big(X/D; u \,\Big|\, \overline{\tau_{a_1-1}(\gamma_1)\cdots\tau_{\alpha_\ell-1}(\gamma_\ell)} \,\Big|\, \mu\Big)_\beta$$

under the variable change $-q = e^{iu}$.

The change of variables is well-defined by the rationality of Conjecture 2. A case in which Conjecture 7 is proven is when X is a nonsingular projective toric 3-fold and $D \subset X$ is a toric divisor. The rationality of the stable pairs series is given by Theorem 2. The following result can be obtained by the methods of [37].

Theorem 6. *For X/D a nonsingular projective relative toric 3-fold, the descendent partition function For $\gamma_i \in H^*(X)$, we have*

$$(-q)^{-d_\beta/2}\mathsf{Z}_{\mathsf{P}}\Big(X/D; q \,\Big|\tau_{\alpha_1-1}(\gamma_1)\cdots\tau_{\alpha_\ell-1}(\gamma_\ell)\,\Big|\, \mu\Big)_\beta$$
$$= (-iu)^{d_\beta+\ell(\mu)-|\mu|}\mathsf{Z}'_{\mathsf{GW}}\Big(X/D; u \,\Big|\, \overline{\tau_{a_1-1}(\gamma_1)\cdots\tau_{\alpha_\ell-1}(\gamma_\ell)} \,\Big|\, \mu\Big)_\beta$$

under the variable change $-q = e^{iu}$.

Conjecture 7 can be lifted in a canonical way to the equivariant relative case (as in the the rationality of Conjecture 3). Some equivariant relative results are proven in [37].

2.10 Complete intersections

Let X be a Fano or Calabi-Yau complete intersection of ample divisors in a product of projective spaces,

$$X \subset \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_m} .$$

A central result of [37] is the proof of the descendent correspondence for even classes.

Theorem 7 (P.-Pixton, 2012)**.** *Let X be a nonsingular Fano or Calabi-Yau complete intersection 3-fold in a product of projective spaces. For even classes $\gamma_i \in H^{2*}(X)$, we have*

$$(-q)^{-d_\beta/2} \mathsf{Z}_{\mathsf{P}}\Big(X;q \;\Big|\tau_{\alpha_1-1}(\gamma_1)\cdots\tau_{\alpha_\ell-1}(\gamma_\ell)\Big)_\beta$$
$$= (-iu)^{d_\beta} \mathsf{Z}'_{\mathsf{GW}}\Big(X;u \;\Big|\; \overline{\tau_{\alpha_1-1}(\gamma_1)\cdots\tau_{\alpha_\ell-1}(\gamma_\ell)}\;\Big)_\beta$$

under the variable change $-q = e^{iu}$.

Theorem 7 relies on the rationality of the stable pairs series of Theorem 4. For $\gamma_i \in H^{2*}(X)$ even classes of *positive* degree, we obtain from Theorem 7 (under the same complete intersection hypothesis for X) the following result where only the leading term of the correspondence contributes:

$$(-q)^{-d_\beta/2}\ \mathsf{Z}_{\mathsf{P}}\left(X;q \;\middle|\; \prod_{i=1}^{r}\tau_0(\gamma_i)\prod_{j=1}^{s}\tau_{k_j}(\mathsf{p})\right)_\beta =$$
$$(-iu)^{d_\beta}(iu)^{-\sum k_j}\ \mathsf{Z}'_{\mathsf{GW}}\left(X;u \;\middle|\; \prod_{i=1}^{r}\tau_0(\gamma_i)\prod_{j=1}^{s}\tau_{k_j}(\mathsf{p})\right)_\beta$$

under the variable change $-q = e^{iu}$. Just as in the analysis of (21), the above correspondence proves the functional equation of Conjecture 4 in the case at hand.

If we specialize Theorem 7 further to the case where there are no descendent insertions, we obtain

$$\mathsf{Z}_{\mathsf{P}}\Big(X;q\Big)_\beta = \mathsf{Z}'_{\mathsf{GW}}\Big(X;u\Big)_\beta$$

under the variable change $-q = e^{iu}$ for Calabi-Yau complete intersections in a product of projective spaces. In particular, the Gromov-Witten/Pairs correspondence hold for the famous quintic Calabi-Yau 3-fold

$$X_5 \subset \mathbb{P}^4 .$$

2.11 $K3$ fibrations

Let Y be a nonsingular projective toric 3-fold for which the anticanonical class K_Y^* is base point free and the generic anticanonical divisor is a nonsingular projective $K3$ surface S. Let

$$X \subset Y \times \mathbb{P}^1 \tag{26}$$

be a nonsingular hypersurface in the class $K_Y^* \otimes K_{\mathbb{P}^1}^*$. Using the degeneration

$$X \rightsquigarrow Y \cup S \times \mathbb{P}^1 \cup Y$$

obtained by factoring a divisor of $K_Y^* \otimes K_{\mathbb{P}^1}^*$, the results of [37] yield the Gromov-Witten/Pairs correspondence for the Calabi-Yau 3-fold X.[23]

The hypersurface X defined by (26) is a $K3$-fibered Calabi-Yau 3-fold. A very natural question to ask is whether the Gromov-Witten/Pairs correspondence can be proven for all $K3$-fibered 3-folds. While the general case is open, results for the correspondence in fiber classes can be found in [42].[24]

3 Virasoro constraints

3.1 Overview

Descendent partition functions in Gromov-Witten theory are conjectured to satisfy Virasoro constraints [9] for every target variety X. Via the Gromov-Witten/Pairs descendent correspondence, we expect parallel constraints for the descendent theory of stable pairs. An ideal path to finding the constraints for stable pairs would be to start with the explicit Virasoro constraints in Gromov-Witten theory and then apply the correspondence. However, our

[23]The strategy here is simpler than presented in Appendix B of [42] for a particular toric 4-fold Y.

[24]Parallel questions can be pursued for other surfaces. For surfaces of general type (involving the stable pairs theory of descendents), see [15].

knowledge of the correspondence matrix is not yet sufficient for such an application.

Another method is to look experimentally for relations which are of the expected shape. In a search conducted almost 10 years ago with A. Oblomkov and A. Okounkov, we found a set of such relations for the theory of ideal sheaves [29] for every nonsingular projective 3-fold X. As an example, the equations for $\mathbb{P}^3$ are presented here for stable pairs.[25]

3.2 First equations

Let X be a nonsingular projective 3-fold. The descendent insertions

$$\tau_0(1)\,, \quad \tau_0(D) \text{ for } D \in H^2(X), \quad \tau_1(1)$$

all satisfy simple equations (parallel to the string, divisor, and dilation equations in Gromov-Witten theory):

(i) $\mathsf{Z}_P\Big(X;q \;\Big|\; \tau_0(1)\cdot \prod_{i=1}^r \tau_{k_i}(\gamma_i)\Big)_\beta = 0,$

(ii) $\mathsf{Z}_P\Big(X;q \;\Big|\; \tau_0(D)\cdot \prod_{i=1}^r \tau_{k_i}(\gamma_i)\Big)_\beta = \left(\int_\beta D\right)\, \mathsf{Z}_P\Big(X;q \;\Big|\; \prod_{i=1}^r \tau_{k_i}(\gamma_i)\Big)_\beta\,,$

(iii) $\mathsf{Z}_P\Big(X;q \;\Big|\; \tau_1(1)\cdot \prod_{i=1}^r \tau_{k_i}(\gamma_i)\Big)_\beta = \left(q\frac{d}{dq} - \frac{d_\beta}{2}\right)\, \mathsf{Z}_P\Big(X;q \;\Big|\; \prod_{i=1}^r \tau_{k_i}(\gamma_i)\Big)_\beta\,.$

All three are obtained directly from the definition of the descendent action given in Section 0.2. To prove (iii), the Hirzebruch-Riemann-Roch equation

$$\mathrm{ch}_3(F) = n - \frac{d_\beta}{2}$$

is used for a stable pair

$$[F,s] \in P_n(X,\beta)\,, \quad d_\beta = \int_\beta c_1(X)\,.$$

The compatibility of (i) and (ii) with the functional equation of Conjecture 4 is trivial. While not as obvious, the differential operator

$$q\frac{d}{dq} - \frac{d_\beta}{2}$$

[25] Since [29] is written for ideal sheaves, a DT/PT correspondence for descendents is needed to move the relations to the theory of stable pairs. Such a correspondence is also studied in [29]. I am very grateful to A. Oblomkov for his help with the formulas here.

is also beautifully consistent with Conjecture 4. We can easily prove using (iii) that Conjecture 4 holds for

$$\mathsf{Z}_P\Big(X;q\ \Big|\ \tau_1(1)\cdot\prod_{i=1}^r \tau_{k_i}(\gamma_i)\Big)_\beta$$

if and only if Conjecture 4 holds for

$$\mathsf{Z}_P\Big(X;q\ \Big|\ \prod_{i=1}^r \tau_{k_i}(\gamma_i)\Big)_\beta .$$

For example, equation (iii) yields

$$\mathsf{Z}_{\mathsf{P}}\big(\mathbb{P}^3;q\ |\ \tau_1(1)\tau_5(1)\big)_{\mathsf{L}} = \frac{q+4q^2+17q^3-62q^4+17q^5+4q^6+q^7}{9(1+q)^4}$$

when applied to (7).

3.3 Operators and constraints

A basis of the cohomology $H^*(\mathbb{P}^3)$ is given by

$$\mathsf{1}\,,\ \mathsf{H}\,,\ \mathsf{L}=\mathsf{H}^2\,,\ \mathsf{p}=\mathsf{H}^3$$

where H is the hyperplane class. The divisor and dilaton equations here are

$$\begin{aligned}\mathsf{Z}_P\Big(\mathbb{P}^3;q\ \Big|\ \tau_0(\mathsf{H})\cdot\mathsf{D})\Big)_{d\mathsf{L}} &= d\mathsf{Z}_P\Big(\mathbb{P}^3;q\ \Big|\ \mathsf{D}\Big)_{d\mathsf{L}}\,,\\ \mathsf{Z}_P\Big(\mathbb{P}^3;q\ \Big|\ \tau_1(1)\cdot\mathsf{D}\Big)_{d\mathsf{L}} &= \Big(q\frac{d}{dq}-2d\Big)\ \mathsf{Z}_P\Big(\mathbb{P}^3;q\ \Big|\ \mathsf{D}\Big)_{d\mathsf{L}}\,,\end{aligned}$$

where $\mathsf{D}=\prod_{i=1}^r \tau_{k_i}(\gamma_i)$ is an arbitrary descendent insertion.

Before presenting the formulas, we introduce two conventions which simplify the notation. The first concerns descendents with negative subscripts. We define the descendent action in two negative cases:

$$\tau_{-2}(\mathsf{H}^j) = -\delta_{j,3}\,,\quad \tau_{-1}(\mathsf{H}^j)=0\,. \tag{27}$$

In particular, these all vanish except for $\tau_{-2}(\mathsf{p})=-1$. Convention (27) is consistent with Definition 2 via the replacement

$$\mathrm{ch}_{2+i}(\mathbb{F})\mapsto \mathrm{ch}_{2+i}(\mathbb{I}[1]^\bullet)\ ,$$

where $\mathbb{I}^\bullet$ is the universal stable pair on $X \times P_n(X,\beta)$.

For the Virasoro constraints, the formulas are more naturally stated in terms of the Chern character subscripts (instead of including the shift by 2 in Definition 2). As a second convention, we define the insertions $\mathsf{ch}_i(\gamma)$ by

$$\mathsf{ch}_i(\gamma) = \tau_{i-2}(\gamma) \tag{28}$$

for all $i \geq 0$. In particular, $\mathsf{ch}_0(\mathsf{p})$ acts as -1 and $\mathsf{ch}_1(\mathsf{H}^j)$ acts as 0.

Let $\mathbb{D}^+$ be the free $\mathbb{Q}$-polynomial ring with generators

$$\left\{ \mathsf{ch}_i(\mathsf{H}^j) \,\middle|\, i \geq 0\,, \quad j = 0,1,2,3 \right\}.$$

Via equation (28), we view $\mathbb{D}^+$ as an extension

$$\mathbb{D} \subset \mathbb{D}^+$$

of the algebra of descendents defined in Section 0.3. We define

$$\mathsf{ch}_a\mathsf{ch}_b(\mathsf{H}^j) \in \mathbb{D}^+$$

in terms of the generators by

$$\mathsf{ch}_a\mathsf{ch}_b(\mathsf{H}^j) = \sum_r \mathsf{ch}_a(\gamma_r^L)\mathsf{ch}_b(\gamma_r^R)$$

where the sum is indexed by the Künneth decomposition

$$\mathsf{H}^j \cdot \Delta = \sum_r \gamma_r^L \otimes \gamma_r^R \in H^*(\mathbb{P}^3 \times \mathbb{P}^3)$$

and $\Delta \subset \mathbb{P}^3 \times \mathbb{P}^3$ is the diagonal. Both $\mathsf{ch}_i(\mathsf{H}^j)$ and $\mathsf{ch}_a\mathsf{ch}_b(\mathsf{H}^j)$ define operators on $\mathbb{D}^+$ by multiplication.

To write the Virasoro relations, we will define derivations

$$\mathrm{R}_k : \mathbb{D}^+ \to \mathbb{D}^+$$

for $k \geq -1$ by the following action on the generators of $\mathbb{D}^+$,

$$\mathrm{R}_k\left(\mathsf{ch}_i(\mathsf{H}^j)\right) = \left(\prod_{n=0}^{k}(i+j-3+n)\right)\mathsf{ch}_{k+i}(\mathsf{H}^j)\,.$$

In case $k=-1$, the product on the right is empty and

$$\mathrm{R}_{-1}\left(\mathsf{ch}_i(\mathsf{H}^j)\right) = \mathsf{ch}_{i-1}(\mathsf{H}^j)\,.$$

Definition 3. *Let* $\mathcal{L}_k : \mathbb{D}^+ \to \mathbb{D}^+$ *for* $k \geq -1$ *be the operator*

$$\begin{aligned}\mathcal{L}_k &= -2 \sum_{a+b=k+2} (-1)^{d^L d^R} (a+d^L-3)!(b+d^R-3)!\, \mathsf{ch}_a\mathsf{ch}_b(\mathsf{H}) \\ &\quad + \sum_{a+b=k} a!b!\, \mathsf{ch}_a\mathsf{ch}_b(\mathsf{p}) \\ &\quad + \mathrm{R}_k + (k+1)!\, \mathrm{R}_{-1}\mathsf{ch}_{k+1}(\mathsf{p})\,.\end{aligned}$$

The first term in the formula for $\mathcal{L}_k$ requires explanation. By definition,

$$\mathsf{ch}_a\mathsf{ch}_b(\mathsf{H}) = \mathsf{ch}_a(\mathsf{p})\mathsf{ch}_b(\mathsf{H}) + \mathsf{ch}_a(\mathsf{L})\mathsf{ch}_b(\mathsf{L}) + \mathsf{ch}_a(\mathsf{H})\mathsf{ch}_b(\mathsf{p}) \tag{29}$$

via the three terms of the Künneth decomposition of $\mathsf{H} \cdot \Delta$. The notation

$$(-1)^{d^L d^R} (a+d^L-3)!(b+d^R-3)!\, \mathsf{ch}_a\mathsf{ch}_b(\mathsf{H})$$

is shorthand for the sum

$$\begin{aligned} & (-1)^{3\cdot 1}(a+3-3)!(b+1-3)!\, \mathsf{ch}_a(\mathsf{p})\mathsf{ch}_b(\mathsf{H}) \\ + \quad & (-1)^{2\cdot 2}(a+2-3)!(b+2-3)!\, \mathsf{ch}_a(\mathsf{L})\mathsf{ch}_b(\mathsf{L}) \\ + \quad & (-1)^{1\cdot 3}(a+1-3)!(b+3-3)!\, \mathsf{ch}_a(\mathsf{H})\mathsf{ch}_b(\mathsf{p})\,. \end{aligned}$$

The three summands of (29) are each weighted by the factor

$$(-1)^{d^L d^R} (a+d^L-3)!(b+d^R-3)!$$

where d^L is the (complex) degree of γ^L and d^R is the (complex) degree of γ^R with respect to the Künneth summand $\gamma^L \otimes \gamma^R$.

In the second term of the formula, $a!b!\, \mathsf{ch}_a\mathsf{ch}_b(\mathsf{p})$ can be expanded as

$$a!b!\, \mathsf{ch}_a\mathsf{ch}_b(\mathsf{p}) = a!b!\, \mathsf{ch}_a(\mathsf{p})\mathsf{ch}_b(\mathsf{p})\,.$$

The summations over a and b in the first two terms in the formula for $\mathcal{L}_k$ require $a \geq 0$ and $b \geq 0$. All factorials with negative arguments vanish.

For example, the formula for the first operator $\mathcal{L}_{-1}$ is

$$\mathcal{L}_{-1} = \mathsf{R}_{-1} + 0!\, \mathsf{R}_{-1}\mathsf{ch}_0(\mathsf{p})\,.$$

For $\mathcal{L}_0$, we have

$$\begin{aligned}\mathcal{L}_0 &= -2 \cdot (-1)^{3\cdot 1}(0+3-3)!(2+1-3)!\, \mathsf{ch}_0(\mathsf{p})\mathsf{ch}_2(\mathsf{H}) \\ &\quad -2 \cdot (-1)^{2\cdot 2}(1+2-3)!(1+2-3)!\, \mathsf{ch}_1(\mathsf{L})\mathsf{ch}_1(\mathsf{L}) \\ &\quad -2 \cdot (-1)^{1\cdot 3}(2+1-3)!(0+3-3)!\, \mathsf{ch}_2(\mathsf{H})\mathsf{ch}_0(\mathsf{p}) \\ &\quad +\mathsf{ch}_0(\mathsf{p})\mathsf{ch}_0(\mathsf{p}) \\ &\quad +\mathsf{R}_0 + \mathsf{R}_{-1}\mathsf{ch}_1(\mathsf{p})\,.\end{aligned}$$

After simplification, we obtain

$$\mathcal{L}_0 = 4\mathsf{ch}_0(\mathsf{p})\mathsf{ch}_2(\mathsf{H}) - 2\mathsf{ch}_1(\mathsf{L})\mathsf{ch}_1(\mathsf{L}) + \mathsf{ch}_0(\mathsf{p})\mathsf{ch}_0(\mathsf{p}) + \mathsf{R}_0 + \mathsf{R}_{-1}\mathsf{ch}_1(\mathsf{p})\,.$$

The operators $\mathcal{L}_k$ on $\mathbb{D}^+$ are conjectured to be the analogs for stable pairs of the Virasoro constraints for the Gromov-Witten theory of $\mathbb{P}^3$.

Conjecture 8 (Oblomkov-Okounkov-P.)**.** *We have*

$$\mathsf{Z}_{\mathsf{P}}(\mathbb{P}^3; q \mid \mathcal{L}_k\, \mathsf{D})_{d\mathsf{L}} = 0$$

for all $k \geq -1$, for all $\mathsf{D} \in \mathbb{D}^+$, and for all curve classes $d\mathsf{L}$.

For example, for $k = -1$, Conjecture 8 states

$$\mathsf{Z}_{\mathsf{P}}(\mathbb{P}^3; q \mid \mathcal{L}_{-1}\mathsf{D})_{d\mathsf{L}} = 0\,.$$

By the above calculation of $\mathcal{L}_{-1}$,

$$\begin{aligned}\mathsf{Z}_{\mathsf{P}}(\mathbb{P}^3; q \mid \mathcal{L}_{-1}\mathsf{D})_{d\mathsf{L}} &= \mathsf{Z}_{\mathsf{P}}\Big(\mathbb{P}^3; q \,\Big|\, (\mathsf{R}_{-1} + 0!\,\mathsf{R}_{-1}\mathsf{ch}_0(\mathsf{p}))\,\mathsf{D}\Big)_{d\mathsf{L}} \\ &= \mathsf{Z}_{\mathsf{P}}\Big(\mathbb{P}^3; q \,\Big|\, (\mathsf{R}_{-1} - \mathsf{R}_{-1})\,\mathsf{D}\Big)_{d\mathsf{L}} \\ &= 0\,,\end{aligned}$$

where we have also used the descendent action $\mathsf{ch}_0(\mathsf{p}) = -1$. The claim

$$\mathsf{Z}_{\mathsf{P}}(\mathbb{P}^3; q \mid \mathcal{L}_0\mathsf{D})_{d\mathsf{L}} = 0\,.$$

is easily reduced to the divisor equation (ii) of Section 3.2 and is also true.

The first nontrivial assertion of Conjecture 8 occurs for $k = 1$,

$$\mathsf{Z}_{\mathsf{P}}(\mathbb{P}^3; q \mid \mathcal{L}_1\mathsf{D})_{d\mathsf{L}} = \mathsf{Z}_{\mathsf{P}}\Big(\mathbb{P}^3; q \,\Big|\, \big(-4\mathsf{ch}_3(\mathsf{H}) + \mathsf{R}_1 + 2\mathsf{ch}_2(\mathsf{p})\mathsf{R}_{-1}\big)\,\mathsf{D}\Big)_{d\mathsf{L}} = 0\,,$$

which is at the moment unproven. For example, let $\mathsf{D} = \mathsf{ch}_3(\mathsf{p})$ and $d = 1$. We obtain a prediction for descendent series for $\mathbb{P}^3$,

$$-4\mathsf{Z}_{\mathsf{P}}(\mathsf{ch}_3(\mathsf{H})\mathsf{ch}_3(\mathsf{p}))_{\mathsf{L}} + 12\mathsf{Z}_{\mathsf{P}}(\mathsf{ch}_4(\mathsf{p}))_{\mathsf{L}} + 2\mathsf{Z}_{\mathsf{P}}(\mathsf{ch}_2(\mathsf{p})\mathsf{ch}_2(\mathsf{p}))_{\mathsf{L}} = 0\,,$$

which can be checked using the evaluations

$$\begin{aligned}\mathsf{Z}_{\mathsf{P}}(\mathsf{ch}_3(\mathsf{H})\mathsf{ch}_3(\mathsf{p}))_{\mathsf{L}} &= \mathsf{Z}_{\mathsf{P}}(\tau_1(\mathsf{H})\tau_1(\mathsf{p}))_{\mathsf{L}} = \frac{3}{4}q - \frac{3}{2}q^2 + \frac{3}{4}q^3\,, \\ \mathsf{Z}_{\mathsf{P}}(\mathsf{ch}_4(\mathsf{p}))_{\mathsf{L}} &= \mathsf{Z}_{\mathsf{P}}(\tau_2(\mathsf{p}))_{\mathsf{L}} = \frac{1}{12}q - \frac{5}{6}q^2 + \frac{1}{12}q^3\,, \\ \mathsf{Z}_{\mathsf{P}}(\mathsf{ch}_2(\mathsf{p})\mathsf{ch}_2(\mathsf{p}))_{\mathsf{L}} &= \mathsf{Z}_{\mathsf{P}}(\tau_0(\mathsf{p})\tau_0(\mathsf{p}))_{\mathsf{L}} = q + 2q^2 + q^3\,.\end{aligned}$$

3.4 The bracket

To find the Virasoro bracket, we introduce the operators

$$\begin{aligned} L_k &= -2 \sum_{a+b=k+2} (-1)^{d^L d^R} (a+d^L-3)!(b+d^R-3)!\mathsf{ch}_a\mathsf{ch}_b(\mathsf{H}) \\ &\quad + \sum_{a+b=k} a!b!\mathsf{ch}_a\mathsf{ch}_b(\mathsf{p}) \\ &\quad + \mathsf{R}_k \,. \end{aligned}$$

We then obtain the Virasoro relations and the bracket with $\mathsf{ch}_k(\mathsf{p})$,

$$[L_k, L_m] = (m-k)L_{k+m}, \quad [L_n, k!\mathsf{ch}_k(\mathsf{p})] = k \cdot (k+n)!\mathsf{ch}_{n+k}(\mathsf{p}).$$

The operators $\mathcal{L}_k$ are expressed in terms of L_k by:

$$\mathcal{L}_k = L_k + (k+1)!L_{-1}\mathsf{ch}_{k+1}(\mathsf{p}).$$

4 Virtual class in algebraic cobordism

4.1 Overview

Let X be nonsingular projective 3-fold. From the work of J. Shen [43], the virtual fundamental class of the moduli space of stable pairs

$$[P_n(X,\beta)]^{vir} \in A_{d_\beta}(P_n(X,\beta))$$

admits a canonical lift to the theory of algebraic cobordism[26]

$$[P_n(X,\beta)]^{vir} \in \Omega_{d_\beta}(P_n(X,\beta)) \tag{30}$$

where $d_\beta = \int_\beta c_1(X)$. Shen's construction depends only upon the 2-term perfect obstruction theory of $P_n(X,\beta)$ and is closely related to earlier work of Ciocan-Fontantine and Kapranov [5] and Lowrey-Schürg [23].

[26] We do not review the foundations of the theory of algebraic cobordism here. The reader can find discussions in [17, 18]. As for cohomology, we always take $\mathbb{Q}$-coefficients. Shen constructs a canonical lift to algebraic cobordism $[M]^{vir} \in \Omega_*(M)$ of the virtual class in Chow $[M]^{vir} \in A_*(M)$ obtained from a 2-term perfect obstruction theory on a quasi-projective scheme M.

The lift (30) leads to several natural questions. The simplest is *how does the virtual class in algebraic cobordism vary with n?* Let

$$\pi : P_n(X,\beta) \to \bullet$$

be the structure map to the point $\bullet$. Then, for fixed β, we define

$$\mathsf{Z}^{\Omega}_{\mathsf{P}}(X;q)_\beta = \sum_{n\in\mathbb{Z}} q^n \, \pi_*[P_n(X,\beta)]^{vir} \ \in \Omega_{d_\beta}(\bullet) \otimes_{\mathbb{Q}} \mathbb{Q}((q))\,.$$

Is there an analogue for $\mathsf{Z}^{\Omega}_{\mathsf{P}}(X;q)_\beta$ of the rationality and functional equation in the descendent theory of the standard virtual class?

4.2 Chern numbers

While the full data of the cobordism class (30) is difficult to analyze, the push-forward

$$\pi_*[P_n(X,\beta)]^{vir} \in \Omega_{d_\beta}(\bullet)$$

is characterized by the virtual Chern numbers of $P_n(X,\beta)$.

Since $P_n(X,\beta)$ has a 2-term perfect obstruction theory, there is a virtual tangent complex $\mathsf{T}^{vir} \in D^b(P_n(X,\beta))$ with Chern classes

$$c_i(\mathsf{T}^{vir}) \in H^{2i}(P_n(X,\beta))\,.$$

For every partition of the virtual dimension d_β,

$$\sigma = (s_1,\ldots,s_\ell)\,, \qquad d_\beta = \sum_{i=1}^{\ell} s_i\,,$$

we define an associated Chern number

$$c^\sigma_{n,\beta} = \int_{[P_n(X,\beta)]^{vir}} \prod_{i=1}^{\ell} c_{s_i}(\mathsf{T}^{vir}) \ \in \mathbb{Z}$$

by integration against the standard virtual class

$$[P_n(X,\beta)]^{vir} \in H_{2d_\beta}(P_n(X,\beta))\,.$$

The complete collection of Chern numbers

$$\left\{\, c^\sigma_n \,\middle|\, \sigma \in \text{Partitions}(d_\beta) \,\right\}$$

uniquely determines the algebraic cobordism class

$$\pi_*[P_n(X,\beta)]^{vir} \in \Omega_{d_\beta}(\bullet)\,.$$

4.3 Rationality and the functional equation

The rationality of the partition function $\mathsf{Z}_{\mathsf{P}}^{\Omega}(X;q)_\beta$ is equivalent to the rationality of *all* the functions

$$\mathsf{Z}_{\mathsf{P}}^{\sigma}(X;q)_\beta = \sum_{n\in\mathbb{Z}} c^{\sigma}_{n,\beta} q^n$$

for $\sigma \in \text{Partitions}(d_\beta)$.

Theorem 8 (Shen 2014)**.** *The Chern class $c_i(\mathsf{T}^{vir}) \in H^{2i}(P_n(X,\beta))$ can be written as a $\mathbb{Q}$-linear combination of products of descendent classes*

$$\left\{ \prod_{i=1}^{r} \tau_{k_i}(\gamma_i) \;\middle|\; \sum_{i=1}^{r} k_i \equiv 0 \bmod 2\,,\ \gamma_i \in H^*(X) \right\}$$

by a formula which is independent of n and β.

Shen's proof is geometric and constructive. Following the notation of Section 0.2, let

$$\pi_P : X \times P_n(X,\beta) \to P_n(X,\beta)$$

be the projection and let $\mathbb{I}^\bullet \in D^b(X \times P_n(X,\beta))$ be the universal stable pair. The class of the virtual tangent complex in $K^0(P_n(X,\beta))$ is

$$\begin{aligned} [-\mathsf{T}^{vir}] &= [R\pi_{P*} R\mathcal{H}om(\mathbb{I}^\bullet, \mathbb{I}^\bullet)_0] \\ &= [R\pi_{P*}(\mathbb{I}^\bullet \otimes^L (\mathbb{I}^\bullet))^\vee] - [R\pi_{P*}\mathcal{O}_{X\times P_n(X,\beta)}]\,. \end{aligned}$$

The Chern character of $-\mathsf{T}^{vir}$ is then computed by the Grothendieck-Riemman-Roch formula,

$$\text{ch}[-\mathsf{T}^{vir}] = \pi_{P*}\Big(\text{ch}(\mathbb{I}^\bullet) \cdot \text{ch}((\mathbb{I}^\bullet)^\vee) \cdot \text{Td}(X)\Big) - \pi_{P*}\Big(\text{Td(X)}\Big)\,. \qquad (31)$$

The second term of (31) is just $\int_X \text{Td}_3(X)$ times the identity $1 \in H^0(P_n(X,\beta))$.

More interesting is the first term of (31) which can be written as

$$\epsilon_*\Big(\text{ch}(\mathbb{I}^\bullet) \cdot \text{ch}((\widetilde{\mathbb{I}}^\bullet)^\vee) \cdot \Delta \cdot \text{Td}(X)\Big) \qquad (32)$$

where ϵ is the projection

$$\epsilon : X \times X \times P_n(X,\beta) \to P_n(X,\beta)\,,$$

$\mathbb{I}^\bullet$ and $\widetilde{\mathbb{I}}^\bullet$ are the universal stable pairs pulled-back via the first and second projections

$$X \times P_n(X,\beta) \leftarrow X \times X \times P_n(X,\beta) \rightarrow X \times P_n(X,\beta)$$

respectively, and Δ is the pull-back of the diagonal in $X \times X$. Using the Künneth decomposition of Δ, Shen easily writes (32) as a quadratic expression in the descendent classes — see [43, Section 3.1]. The answer is a universal formula independent of n and β.

Though not explicitly remarked (nor needed) in [43], Shen's universal formula for $\mathrm{ch}[-\mathsf{T}^{vir}]$ is a $\mathbb{Q}$-linear combination of classes

$$\left\{ \tau_{k_1}(\gamma_1)\tau_{k_2}(\gamma_2) \,\middle|\, k_1 + k_2 \equiv 0 \mod 2\,,\ \gamma_1, \gamma_2 \in H^*(X) \right\}$$

since each quadratic term appears in (32) in a form proportional to

$$((-1)^{k_1} + (-1)^{k_2}) \cdot \tau_{k_1}(\gamma_1)\tau_{k_2}(\gamma_2)$$

because of the universal stable pair $\mathrm{ch}(\mathbb{I}^\bullet)$ appears together with the dual $\mathrm{ch}((\widetilde{\mathbb{I}}^\bullet)^\vee)$.

There are two immediate consequences of Theorem 8. If the rationality of descendent series of Conjecture 1 holds for X, then

$\mathsf{Z}^{\Omega}_{\mathsf{P}}(X;q)_\beta$ *is the Laurent expansion of a rational function in* $\Omega_{d_\beta}(\bullet) \otimes_{\mathbb{Q}} \mathbb{Q}(q)\,.$

In particular, Shen's results yield the rationality of the partition functions in algebraic cobordism in case X is a nonsingular projective toric variety (where rationality of the descendent series is proven).

The second consequence concerns the functional equation. The descendents which arise in Theorem 8 have *even* subscript sum. Hence, if the functional equation of Conjecture 4 holds for X, then

$$\mathsf{Z}^{\Omega}_{\mathsf{P}}\left(X; \frac{1}{q}\right)_\beta = q^{-d_\beta}\mathsf{Z}^{\Omega}_{\mathsf{P}}(X;q)_\beta\,. \tag{33}$$

The functional equation (33) should be regarded as the correct generalization to all X of the symmetry

$$\mathsf{Z}_{\mathsf{P}}\left(Y; \frac{1}{q}\right)_\beta = \mathsf{Z}_{\mathsf{P}}(Y;q)_\beta$$

of stable pairs invariants for *Calabi-Yau* 3-folds Y.

4.4 An example

A geometric basis of $\Omega_*(\bullet)$ is given by the classes of products of projective spaces. As an example, we write the series

$$\mathsf{Z}^{\Omega}_{\mathsf{P}}(\mathbb{P}^3; q)_{\mathsf{L}} \in \Omega_4(\bullet) \otimes_{\mathbb{Q}} \mathbb{Q}(q)$$

in terms of products of projective spaces:

$$\begin{aligned}\mathsf{Z}^{\Omega}_{\mathsf{P}}(\mathbb{P}^3; q)_{\mathsf{L}} \;=\; & [\mathbb{P}^4] \cdot f_4(q) \\ & +[\mathbb{P}^3 \times \mathbb{P}^1] \cdot f_{31}(q) \\ & +[\mathbb{P}^2 \times \mathbb{P}^2] \cdot f_{22}(q) \\ & +[\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1] \cdot f_{211}(q) \\ & +[\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1] \cdot f_{1111}(q)\,,\end{aligned}$$

where the rational functions[27] are given by

$$\begin{aligned}
f_4(q) &= -4q - 40q^2 - 4q^3\,, \\
f_{31}(q) &= \frac{q}{(1+q)^4}\left(\frac{21}{2} + 139q + \frac{823}{2}q^2 + 446q^3 + \frac{823}{2}q^4 + 139q^5 + \frac{21}{2}q^6\right)\,, \\
f_{22}(q) &= 6q + 60q^2 + 6q^3\,, \\
f_{211}(q) &= \frac{q}{(1+q)^4}\left(-18 - 264q - 774q^2 - 816q^3 - 774q^4 - 264q^5 - 18q^6\right)\,, \\
f_{1111}(q) &= \frac{q}{(1+q)^6}\left(\frac{13}{2} + 115q + 490q^2 + 889q^3 + 1215q^4\right. \\
&\qquad \left. +889q^5 + 490q^6 + 115q^7 + \frac{13}{2}q^8\right)\,.
\end{aligned}$$

4.5 Further directions

The study of the virtual class in algebraic cobordism of the moduli space of stable pairs $P_n(X, \beta)$ is intimately connected with the study of descendents invariants. The basic reason is because the Chern classes of the virtual tangent complex are *tautological classes* of $P_n(X, \beta)$ in the sense of Section 0.3. If another approach to the virtual class in algebraic cobordism class could be found, perhaps the implications could be reversed and results about descendent series could be proven.

[27] I am very grateful to J. Shen for providing these formulas.

References

[1] M. Aganagic, A. Klemm, M. Marino, and C. Vafa, *The topological vertex*, Comm. Math. Phys. **254**, 425–478.

[2] K. Behrend, *Donaldson-Thomas type invariants via microlocal geometry*, Ann. of Math. **170** (2009), 1307–1338.

[3] K. Behrend and B. Fantechi, *The intrinsic normal cone,* Invent. Math. **128** (1997), 45–88.

[4] T. Bridgeland, *Hall algebras and curve-counting invariants*, JAMS **24** (2011), 969–998, 2011.

[5] I. Ciocan-Fontanine and M. Kapranov, *Virtual fundamental classes via dg-manifolds*, Geom. Top. **13** (2009), 1779–1804.

[6] S.K. Donaldson and P. B. Kronheimer, *The geometry of four-manifolds*, Oxford Mathematical Monographs, Oxford University Press: New York, 1990.

[7] S. K. Donaldson and R. P. Thomas, *Gauge theory in higher dimensions* in *The geometric universe (Oxford, 1996)*, 31–47, Oxford Univ. Press: Oxford, 1998.

[8] A. Douady and J.-L. Verdier, *Séminaire de Géométrie Analytique de l'Ecole Normale Supérieure 1974/1975*, Astérisque **36-37** (1976), Exposés VI-IX.

[9] T. Eguchi, K. Hori, and C.-S. Xiong, *Quantum cohomology and Virasoro algebra*, Phys. Lett. **B 402** (1997), 71 – 80.

[10] G. Ellingsrud and S. A. Strömme, *Towards the Chow ring of the Hilbert scheme of* $\mathbb{P}^2$, J. Reine Angew. Math. **441** (1993), 33–44.

[11] C. Faber and R. Pandharipande, *Hodge integrals and Gromov-Witten theory*, Invent. Math. **139** (2000), 173–199.

[12] C. Faber and R. Pandharipande, *Tautological and non-tautological cohomology of the modul space of curves* in *Handbook of moduli, Vol. I*, 293–330, Advanced Lectures in Mathematics 24, International Press: Beijing, 2013.

[13] T. Graber, J. Kock, and R. Pandharipande, *Descendent invariants and characteristic numbers*, Amer. J. Math. **124** (2002), 611–647.

[14] T. Graber and R. Pandharipande, *Localization of virtual classes*, Invent. Math., **135**, 487–518, 1999.

[15] M. Kool and R. Thomas, *Stable pairs with descendents on local surfaces I: the vertical component*, arXiv:1605.02576.

[16] J. Le Potier, *Faisceaux semi-stable et systèmes cohérents*, in *Vector bundles in algebraic geometry (Durham, 1993)*, LMS Lecture Note Ser. **208**, 179–239, Cambridge Univ. Press: Cambridge, 1995.

[17] M. Levine and F. Morel, *Algebraic cobordism*, Springer Monographs in Mathematics, Springer: Berlin, 2007.

[18] M. Levine and R. Pandharipande, *Algebraic cobordism revisited*, Invent. Math. **176** (2009), 63–130.

[19] A.-M. Li and Y. Ruan, *Symplectic surgery and Gromov-Witten invariants of Calabi-Yau 3-folds I*, Invent. Math. **145** (2001), 151–218.

[20] J. Li, *A degeneration formula of GW-invariants*, JDG **60** (2002), 199–293.

[21] J. Li and G. Tian, *Virtual moduli cycles and Gromov-Witten invariants of algebraic varieties,* JAMS **11**, 119–174, 1998.

[22] J. Li and B. Wu, *Good degeneration of Quot schemes and coherent systems*, Comm. Ana. Geom. **23** (2015), 841–921.

[23] P. Lowrey and T. Schrürg, *Derived algebraic bordism*, J. Inst. Math. Jussieu **15** (2016), 407–443.

[24] D. Maulik, N. Nekrasov, A. Okounkov, and R. Pandharipande, *Gromov-Witten theory and Donaldson-Thomas theory. I*, Compos. Math. **142** (2006), 1263–1285.

[25] D. Maulik, N. Nekrasov, A. Okounkov, and R. Pandharipande, *Gromov-Witten theory and Donaldson-Thomas theory. II*, Compos. Math. **142** (2006), 1286–1304.

[26] D. Maulik and A. Oblomkov, *The quantum cohomology of the Hilbert scheme of points of A_n-resolutions*, JAMS **22** (2009), 1055–1091.

[27] D. Maulik, A. Oblomkov, A. Okounkov, and R. Pandharipande, *The Gromov-Witten/Donaldson-Thomas correspondence for toric 3-folds*, Invent. Math. **186** (2011), 435–479.

[28] P. E. Newstead, *Characteristic classes of stable bundles of rank 2 over an algebraic curve*, Trans. Amer. Math. Soc. **169** (1972), 337–345.

[29] A. Oblomkov, A. Okounkov, and R. Pandharipande, *in preparation.*

[30] A. Okounkov and R. Pandharipande, *Gromov-Witten theory, Hurwitz numbers, and completed cycles*, Ann. of Math. **163** (2006), 517–560.

[31] A. Okounkov and R. Pandharipande, *The quantum cohomology of the Hilbert scheme of points of the plane*, Invent. Math. **179** (2010), 523–557.

[32] R. Pandharipande, *A calculus for the moduli space of curves*, Proc. of Algebraic geometry - Salt Lake City 2015, Proc. Sympos. Pure Math. (to appear), arXiv:1603.05151.

[33] R. Pandharipande and A. Pixton, *Descendents on local curves: Rationality*, Comp. Math. **149** (2013), 81–124.

[34] R. Pandharipande and A. Pixton, *Descendents on local curves: Stationary theory* in *Geometry and arithmetic*, 283–307, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2012.

[35] R. Pandharipande and A. Pixton, *Descendent theory for stable pairs on toric 3-folds*, Jour. Math. Soc. Japan. **65** (2013), 1337–1372.

[36] R. Pandharipande and A. Pixton, *Gromov-Witten/Pairs descendent correspondence for toric 3-folds*, Geom. Topol. **18** (2014), 2747–2821.

[37] R. Pandharipande and A. Pixton, *Gromov-Witten/Pairs correspondence for the quintic*, JAMS **30** (2017), 389–449.

[38] R. Pandharipande and R. P. Thomas, *Curve counting via stable pairs in the derived category*, Invent Math. **178** (2009), 407–447.

[39] R. Pandharipande and R. P. Thomas, *The 3-fold vertex via stable pairs*, Geom. Topol. **13** (2009), 1835–1876.

[40] R. Pandharipande and R. P. Thomas, *Stable pairs and BPS invariants*, JAMS **23** (2010), 267–297.

[41] R. Pandharipande and R. P. Thomas, *The Katz-Klemm-Vafa conjecture for $K3$ surfaces*, Forum of Mathematics Pi **4** (2016).

[42] R. Pandharipande and R. P. Thomas, *13/2 ways of counting curves*, in Moduli spaces, 282–333, LMS Lecture Note Ser. **411**, Cambridge Univ. Press, Cambridge, 2014.

[43] J. Shen, *Cobordism invariants of the moduli space of stable pairs*, J. Lond. Math. Soc. **94** (2016), 427–446.

[44] R. P. Thomas, *A holomorphic Casson invariant for Calabi-Yau 3-folds, and bundles on $K3$ fibrations,* J. Differential Geom. **54** (2000), 367–438.

[45] Y. Toda, *Curve counting theories via stable objects I: DT/PT correspondence*, JAMS **23** (2010), 1119-1157.

Departement Mathematik
ETH Zürich
rahul@math.ethz.ch