THE CHOW RING OF THE MODULI SPACE OF DEGREE 2 QUASI-POLARIZED K3 SURFACES

SAMIR CANNING, DRAGOS OPREA, AND RAHUL PANDHARIPANDE

Abstract. We study the Chow ring with rational coefficients of the moduli space $\mathcal{F}_2$ of quasi-polarized $K3$ surfaces of degree 2. We find generators, relations, and calculate the Chow Betti numbers. The highest nonvanishing Chow group is $\mathsf{A}^{17}(\mathcal{F}_2) \cong \mathbb{Q}$. We prove that the Chow ring consists of tautological classes and is isomorphic to the even cohomology. The Chow ring is not generated by divisors and does not satisfy duality with respect to the pairing into $\mathsf{A}^{17}(\mathcal{F}_2)$. In the appendix, we revisit Kirwan-Lee's calculation of the Poincaré polynomial of $\mathcal{F}_2$.

Contents

1. Introduction

1.1. **Main results.** Let $\mathcal{F}_{2\ell}$ denote the moduli space of quasi-polarized $K3$ surfaces of degree 2ℓ. The space $\mathcal{F}_{2\ell}$ is a nonsingular Deligne-Mumford stack of dimension 19. We consider the Chow ring $\mathsf{A}^*(\mathcal{F}_{2\ell})$, which will be taken with $\mathbb{Q}$-coefficients throughout this paper. The Chow of ring admits a tautological subring

$$\mathsf{R}^*(\mathcal{F}_{2\ell}) \subset \mathsf{A}^*(\mathcal{F}_{2\ell}),$$

which was defined in [MOP] and will be reviewed in Section 2.

We focus here on the moduli space $\mathcal{F}_2$ of $K3$ surfaces of degree $2\ell = 2$. The generic $K3$ surface $(S, L) \in \mathcal{F}_2$ is a double cover

$$\epsilon : S \to \mathbb{P}^2$$

Date: April 2024.

branched along a nonsingular sextic curve with quasi-polarization $L = \epsilon^* \mathcal{O}_{\mathbb{P}^2}(1)$. The geometry of $\mathcal{F}_2$ is therefore closely related to the classical geometry of sextic plane curves.

Theorem 1. *The following results hold for the Chow ring* $\mathsf{A}^*(\mathcal{F}_2)$*:*

(i) *The Chow ring is tautological,*

$$\mathsf{R}^*(\mathcal{F}_2) = \mathsf{A}^*(\mathcal{F}_2)\,,$$

and is generated by 4 elements $\alpha_1, \alpha_2, \beta, \gamma$ *in degrees* $1, 1, 2, 3$ *respectively.*

(ii) $\mathsf{R}^{17}(\mathcal{F}_2) = \mathbb{Q}$ *and* $\mathsf{R}^{18}(\mathcal{F}_2) = \mathsf{R}^{19}(\mathcal{F}_2) = 0$.

(iii) *The dimensions are given by*

$$\sum_{k=0}^{19} t^k \dim \mathsf{R}^k(\mathcal{F}_2) = 1{+}2t + 3t^2 + 5t^3 + 6t^4 + 8t^5 + 10t^6 + 12t^7 + 13t^8 + \\ + 14t^9 + 12t^{10} + 10t^{11} + 8t^{12} + 6t^{13} + 5t^{14} + 3t^{15} + 2t^{16} + t^{17}.$$

(iv) *The cycle class map is an isomorphism onto the even cohomology:*

$$\forall k \geq 0\,, \quad \mathsf{R}^k(\mathcal{F}_2) \cong H^{2k}(\mathcal{F}_2)\,.$$

Theorem 1 is the first complete Chow calculation for the moduli spaces $\mathcal{F}_{2\ell}$ of quasi-polarized $K3$ surfaces. There are several immediate connections and consequences:

• The generators (i) of $\mathsf{R}^*(\mathcal{F}_2)$ are *not* all divisor classes. Indeed, the Chow Betti numbers given in (iii) grow too quickly to be generated by the two divisor classes.

• The socle and vanishing results of (ii) were proposed earlier as analogues of Faber's conjectures [Fa] for the tautological ring of the moduli space of curves $\mathcal{M}_g$:

$$\mathsf{R}^{>g-2}(\mathcal{M}_g) = 0\,, \quad \mathsf{R}^{g-2}(\mathcal{M}_g) = \mathbb{Q}\,.$$

Conjecture 2 (Oprea–Pandharipande (2015))**.** *Let* Γ *be the Picard lattice for a K3 surface such that* $d = 20 - \operatorname{rank}(\Gamma) > 3$. *For the moduli space* $\mathcal{F}_\Gamma$ *of* Γ*-polarized* $K3$ *surfaces, we have*

$$\mathsf{R}^{d-2}(\mathcal{F}_\Gamma) \cong \mathbb{Q} \quad \textit{and} \quad \mathsf{R}^{d-1}(\mathcal{F}_\Gamma) = \mathsf{R}^{d}(\mathcal{F}_\Gamma) = 0\,.$$

In cohomology with $\mathbb{Q}$-coefficients, the vanishing part of Conjecture 2 is established in [Pe1]. For the hyperbolic lattice U, the moduli space $\mathcal{F}_U$ corresponds to elliptic $K3$ surfaces with section. The socle and Chow vanishing properties of Conjecture 2 for $\mathcal{F}_U$ are established in [CK]. The moduli space $\mathcal{F}_2$ is the first rank 1 case where Conjecture 2 is proven.

• The Chow Betti number calculation (iii) shows that the pairing into $\mathsf{R}^{17}(\mathcal{F}_2)$ is not perfect because the middle dimensions are not equal,

$$\dim \mathsf{R}^8(\mathcal{F}_2) \neq \dim \mathsf{R}^9(\mathcal{F}_2)\,.$$

In fact, the kernel of the pairing into $\mathsf{R}^{17}(\mathcal{F}_2)$ is large. A full characterization of the kernel is presented in Section 7. The pairing turns out not to be perfect in any degree other than 0 and 17.

• A construction of the moduli space $\mathcal{F}_2$ as an open subset of a weighted blowup of the moduli space of sextic curves in $\mathbb{P}^2$ is given in [S], see [La, Lo] for a summary. A partial desingularization of the full GIT compactification of the space of sextics is obtained in [KL1] and requires three further blowups. Using all four blowups, the cohomology of $\mathcal{F}_2$ was studied in [KL1, KL2]. A main result of [KL2] is the calculation of the Poincaré polynomial[1]

$$\sum_{k=0}^{38} q^k \dim H^k(\mathcal{F}_2) = 1 + 2q^2 + 3q^4 + 5q^6 + 6q^8 + 8q^{10} + 10q^{12} + 12q^{14} + 13q^{16} + 14q^{18} + 12q^{20} + 10q^{22} + 8q^{24} + 6q^{26} + q^{27} + 5q^{28} + 3q^{30} + q^{31} + 2q^{32} + 2q^{33} + q^{34} + 3q^{35}.$$

While there are odd cohomology classes, the dimensions of the even cohomology of $\mathcal{F}_2$ match the Chow Betti numbers (iii) as required by (iv).

• By a result of [BLMM], the even cohomology $H^{2k}(\mathcal{F}_{2\ell})$ for $k \leq 4$ is tautological for all $\ell \geq 1$. Isomorphism (iv) is a much stronger property which holds for the moduli space $\mathcal{F}_2$.

1.2. **Plan of the paper.** Definitions and basic results related to tautological classes on moduli spaces of $K3$ surfaces are reviewed in Section 2. Our approach to the geometry of $\mathcal{F}_2$ relies upon Shah's blowup construction [S] which is discussed in Section 3. Part (i) of Theorem 1 is proven in Section 3.2.

Shah describes $\mathcal{F}_2$ as an open subset of a weighted blowup of the space of sextic plane curves. The heart of our Chow calculation for $\mathcal{F}_2$ is presented in Sections 4–6, where relations obtained from the removal of various loci are determined. Parts (ii) and (iii) of Theorem 1 are proven in Section 6.3. The complete Chow calculation of the moduli space of elliptic $K3$ surfaces [CK] is used in the proof.

The Chow pairing into $\mathsf{R}^{17}(\mathcal{F}_2)$ is analyzed in Section 7. The kernel of the pairing is determined in Proposition 14. Part (iv) of Theorem 1, the isomorphism of the cycle class map onto the even cohomology, is proven in Section 8.

The Kirwan-Lee calculation of the Poincaré polynomial of $\mathcal{F}_2$ is discussed carefully in Appendix A. In particular, we explain how to correct the calculations in [KL1, KL2].

1.3. **Future directions.** While complete Chow calculations for the moduli spaces $\mathcal{F}_{2\ell}$ will likely become intractable for large ℓ, the study of $\mathcal{F}_4$ should be possible as there is a parallel (though more complicated) construction starting from the moduli of quartic surfaces, see [LOG].

Another direction of study is to find structure in the tautological ring $\mathsf{R}^*(\mathcal{F}_{2\ell})$ beyond Conjecture 2. The parallel direction in the study of the moduli space of curves has led to the surprising

[1]The value of the Poincaré polynomial given in [KL2] is

$$1 + 2q^2 + 3q^4 + 5q^6 + 6q^8 + 8q^{10} + 10q^{12} + 12q^{14} + 13q^{16} + 14q^{18} + 12q^{20} + 10q^{22} + 8q^{24} + 6q^{26} + q^{27} + 5q^{28} + 3q^{30} + q^{31} + 2q^{32} + q^{33} + 3q^{35},$$

which differs from the statement above by $q^{33} + q^{34}$. We will explain the necessary correction in Appendix A.

discovery of uniform sets of tautological relations, see [P] for a survey. Whether any analogues of the Faber-Zagier and Pixton relations hold for the moduli of $K3$ surfaces is an interesting question.

Finding algebraic cycle classes on $\mathcal{F}_{2\ell}$ which are non-tautological in cohomology is another open direction. Since such classes for the moduli of curves and abelian varieties can be constructed using the geometry of Hurwitz covers of higher genus curves [COP2, GP, Z], a simple idea for $K3$ surfaces is the following. Let $\mathcal{B}_g \subset \mathcal{F}_{2\ell}$ be the closure of the locus of $K3$ surfaces for which there exists a nonsingular linear section (of genus $\ell + 1$) admitting a degree 2 map to a genus $g \geq 1$ curve. A reasonable expectation is that the image in cohomology of the algebraic cycle class

$$[\mathcal{B}_g] \in \mathsf{A}^*(\mathcal{F}_{2\ell})$$

is non-tautological for appropriate choices of g and ℓ.

Acknowledgments. We thank Dan Abramovich, Veronica Arena, Adrian Clingher, Carel Faber, Frances Kirwan, Bochao Kong, Hannah Larson, Zhiyuan Li, Alina Marian, Davesh Maulik, Stephen Obinna, Dan Petersen, and Burt Totaro for many related conversations over the years. We are grateful to the referee for several comments and corrections.

An earlier attempt to calculate the Chow ring of $\mathcal{F}_2$ with Qizheng Yin and Fei Si did not succeed (due to the geometric and computational complexity of the approach). A different path using the results of [CK] is taken here which leads to several simplifications. We thank Qizheng Yin and Fei Si for the previous collaboration.

S.C. was supported by a Hermann-Weyl-Instructorship from the Forschungsinstitut für Mathematik at ETH Zürich. D.O. was supported by NSF-DMS 1802228. R.P. was supported by SNF-200020-182181, ERC-2017-AdG-786580-MACI, and SwissMAP.

This project has received funding from the European Research Council (ERC) under the European Union Horizon 2020 research and innovation program (grant agreement No. 786580).

2. Tautological classes

We review here the definition of the tautological rings of the moduli spaces $\mathcal{F}_{2\ell}$. Consider the universal $K3$ surface and quasi-polarization

$$\pi : \mathcal{S} \to \mathcal{F}_{2\ell}\,, \quad \mathcal{L} \to \mathcal{S}\,.$$

The most basic tautological classes are:

- *Hodge classes.* The dual Hodge bundle is defined as the pushforward

$$\mathbb{E}^\vee = \mathbf{R}^2\pi_*\mathcal{O}_\mathcal{S} \to \mathcal{F}_{2\ell}\,.$$

Let $\lambda \in \mathsf{A}^1(\mathcal{F}_{2\ell})$ denote the first Chern class

$$\lambda = c_1(\mathbb{E}) = -c_1\left(\mathbf{R}^2\pi_*\mathcal{O}_\mathcal{S}\right)\,.$$

The ring $\Lambda^*(\mathcal{F}_{2\ell}) \subset \mathsf{A}^*(\mathcal{F}_{2\ell})$ generated by powers of λ was studied in [GK]. It is shown there that

$$\Lambda^*(\mathcal{F}_{2\ell}) = \mathbb{Q}[\lambda]/(\lambda^{18}) .$$

The Chern classes of the tangent bundle of $\mathcal{F}_{2\ell}$ belong to the ring Λ^*. In fact, by [GK] (see the paragraph following Proposition 3.2) we have

$$\mathrm{ch}(T\mathcal{F}_{2\ell}) = -1 + 21e^{-\lambda} - e^{-2\lambda} .$$

• *Noether-Lefschetz classes.* Let Γ be an even lattice of signature $(1, r-1)$ for an integer $r \leq 19$. Consider the moduli space $\mathcal{F}_\Gamma$ parametrizing Γ-polarized $K3$ surfaces:

$$\iota : \Gamma \to \mathrm{Pic}(S) ,$$

with the image of ι containing a quasi-polarization of the surface S. Upon fixing a primitive $v \in \Gamma$ with $v^2 = 2\ell$ mapping to the quasipolarization, there is a forgetful map

$$\mathcal{F}_\Gamma \to \mathcal{F}_{2\ell} ,$$

whose image determines a Noether-Lefschetz cycle in $\mathcal{F}_{2\ell}$. Define

$$\mathsf{NL}^*(\mathcal{F}_{2\ell}) \subset \mathsf{A}^*(\mathcal{F}_{2\ell})$$

to be the $\mathbb{Q}$-subalgebra generated by all Noether-Lefschetz cycle classes. A more extensive discussion of the Noether-Lefschetz cycles can be found for instance in [MP].

• *Kappa classes.* Let T_π^{rel} be the relative tangent bundle of the universal surface $\pi : \mathcal{S} \to \mathcal{F}_{2\ell}$. The first Chern class is related to the Hodge class,

$$c_1\left(T_\pi^{\mathrm{rel}}\right) = -\pi^*\lambda .$$

Define $t = c_2\left(T_\pi^{\mathrm{rel}}\right)$. By [GK, Proposition 3.1], the pushforwards $\kappa_{0,n} = \pi_*(t^n)$ are contained in Λ^*. More general kappa classes can be defined by including the class of the quasi-polarization $c_1(\mathcal{L})$ and considering the pushforwards

$$\kappa_{m,n} = \pi_*\left(c_1(\mathcal{L})^m \cdot t^n\right) .$$

These classes depend upon the normalization of $\mathcal{L}$ by line bundles pulled back from $\mathcal{F}_{2\ell}$. By defining canonical normalizations for *admissible* $\mathcal{L}$, the ambiguity can be removed, see [PY].

More generally, we define enriched kappa classes over $\mathcal{F}_\Gamma$ and consider their pushforwards to $\mathcal{F}_{2\ell}$. After picking a basis B for Γ, we obtain line bundles $\mathcal{L}_1, \ldots, \mathcal{L}_r \to \mathcal{F}_\Gamma$, and we set

$$\kappa^{\Gamma,\mathsf{B}}_{a_1,\ldots,a_r,b} = \pi_*\left(c_1(\mathcal{L}_1)^{a_1} \cdot \ldots \cdot c_1(\mathcal{L}_r)^{a_r} \cdot c_2(T_\pi^{\mathrm{rel}})^b\right) . \tag{1}$$

The tautological ring of the moduli space $\mathcal{F}_{2\ell}$,

$$\mathsf{R}^*(\mathcal{F}_{2\ell}) \subset \mathsf{A}^*(\mathcal{F}_{2\ell}) ,$$

is defined in [MOP] to be the $\mathbb{Q}$-subalgebra generated by the pushforwards of all enriched kappa classes (1) for all possible Γ.

By definition, $\mathsf{R}^*(\mathcal{F}_{2\ell}) \supset \mathsf{NL}^*(\mathcal{F}_{2\ell})$. The following isomorphism was proven in [PY].

Theorem 3 (Pandharipande–Yin (2020))**.** *For all $\ell \geq 1$, we have $\mathsf{R}^*(\mathcal{F}_{2\ell}) = \mathsf{NL}^*(\mathcal{F}_{2\ell})$.*

Stronger results hold for divisor classes. In codimension 1, the isomorphism

$$\mathrm{Pic}(\mathcal{F}_{2\ell})_{\mathbb{Q}} = \mathsf{NL}^1(\mathcal{F}_{2\ell})$$

was conjectured in [MP] and proven in [BLMM]. Combined with [B], this isomorphism determines the Picard rank. The integral Picard group has recently been considered in [LFV]. Furthermore, bases for the rational Picard group for small ℓ are given in [OG] and [GLT].

For $\mathcal{F}_2$, consider the divisor Ell of elliptic $K3$ surfaces with a section and the divisor Sing of $K3$ surfaces for which the quasi-polarization fails to be ample. These are Noether-Lefschetz divisors corresponding to the lattices

$$\begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix},$$

respectively. By [OG], we have

$$\mathrm{Pic}(\mathcal{F}_2)_{\mathbb{Q}} = \langle [\mathsf{Ell}]\,, [\mathsf{Sing}] \rangle$$

which is consistent with part (iii) of Theorem 1. In fact, the arguments of [OG] or alternatively [MOP, Proposition 1] imply that λ is a combination of $[\mathsf{Ell}]$ and $[\mathsf{Sing}]$ with nonzero coefficients. Thus, we also have

$$\mathrm{Pic}(\mathcal{F}_2)_{\mathbb{Q}} = \langle \lambda, [\mathsf{Ell}] \rangle\,. \tag{2}$$

3. Shah's construction

3.1. **Overview.** We review here the construction of $\mathcal{F}_2$ described in [S]. We start with the moduli space of sextics with suitably restricted singularities. The moduli space $\mathcal{F}_2$ is then obtained as an open subset of a weighted blowup of the locus of triple conics. Using the construction, we will prove part (i) of Theorem 1.

3.2. **ADE sextics.** By a result of [Ma], a quasi-polarized $K3$ surface (S, L) of degree 2 must take one of the following two geometric forms:

(a) S is the resolution of a double cover $\epsilon : \widehat{S} \to \mathbb{P}^2$ branched along a sextic curve C with $L = \epsilon^*\mathcal{O}_{\mathbb{P}^2}(1)$. For S to be a nonsingular $K3$ surface, C must have ADE singularities [BHPV].

(b) S is an elliptic fibration $S \to \mathbb{P}^1$ with fiber class f, section σ, and $L = \sigma + 2f$.

Surfaces (S, L) of form (b) constitute the Noether-Lefschetz divisor Ell in $\mathcal{F}_2$. The complement $\mathcal{F}_2 \smallsetminus \mathsf{Ell}$ is a gerbe banded by a finite group over the moduli space of plane sextics with ADE singularities.

We will construct the moduli space of plane sextics with ADE singularities as a quotient stack. Let $V = \mathbb{C}^3$, and let

$$X = \mathbb{P}(\mathrm{Sym}^6 V^*) = \mathbb{P}^{27}$$

be the space of all sextic curves in $\mathbb{P}(V)$. A sextic $C \subset \mathbb{P}^2$ has at worst ADE singularities if and only if the following three conditions are satisfied simultaneously

(i) C is reduced,

(ii) C does not contain a consecutive triple point,

(iii) C does not contain a quadruple point.

By definition, a consecutive triple point $p \in C$ is a triple point such that the strict transform of C in the blowup of $\mathbb{P}^2$ at p continues to admit a triple point. In local coordinates (x, y) around p, the equation for C lies in the ideal $(x, y^2)^3$. We write NR, CTP, QP for the three loci failing conditions (i)-(iii) respectively. We have a morphism

$$\mathcal{F}_2 \smallsetminus \mathsf{Ell} \to \left[(X \smallsetminus (\mathsf{NR} \cup \mathsf{CTP} \cup \mathsf{QP}))\, /\, \mathrm{PGL}(V)\right] , \tag{3}$$

of degree $1/2$ (because of the extra $\mathbb{Z}_2$ automorphisms of the $K3$ surfaces). We also have a degree $1/3$ morphism

$$\left[(X \smallsetminus (\mathsf{NR} \cup \mathsf{CTP} \cup \mathsf{QP}))\, /\, \mathrm{SL}(V)\right] \to \left[(X \smallsetminus (\mathsf{NR} \cup \mathsf{CTP} \cup \mathsf{QP}))\, /\, \mathrm{PGL}(V)\right] . \tag{4}$$

induced by the map $\mathrm{SL}(V) \to \mathrm{PGL}(V)$. Both (3) and (4) yield isomorphisms on Chow groups with $\mathbb{Q}$-coefficients. The group $G = \mathrm{SL}(V)$ will be geometrically simpler for us to work with because $\mathrm{BSL}(V)$ carries a universal vector bundle.

The locus NR is reducible with components corresponding to the images of the maps

$$m_r : \mathbb{P}(\mathrm{Sym}^{6-2r} V^*) \times \mathbb{P}(\mathrm{Sym}^r V^*) \to X, \quad m_r(f, g) = fg^2$$

for $r = 1, 2, 3$. Let $\mathsf{ML} \subset \mathsf{NR}$ denote the irreducible component of NR corresponding to sextics containing a multiple line (the image of m_1). The locus ML has codimension 11 in X. The other components of NR are the double conic locus (the image of m_2) and the double cubic locus (the image of m_3). These have codimensions 17 and 18 in X. We will consider these two components together, thus writing DC for their union. The quasi-projective locus of cubes of *nonsingular* conics will be denoted TC. It is easy to see that the loci

$$\mathsf{QP},\ \mathsf{CTP},\ \mathsf{TC} \subset X$$

have codimensions 8, 9 and 22 respectively.

Proof of Theorem 1 (i). Throughout the paper, we identify Chow groups of quotient stacks with the corresponding equivariant Chow groups [EG, Kr].

We first consider the Chow classes in

$$\mathsf{A}^*(\mathcal{F}_2 \smallsetminus \mathsf{Ell}) = \mathsf{A}^*_G(X \smallsetminus (\mathsf{NR} \cup \mathsf{CTP} \cup \mathsf{QP})) .$$

Note that $\mathsf{A}^*_G = \mathbb{Z}[c_2, c_3]$, where c_2, c_3 are the second and third Chern classes of the universal bundle over $\mathrm{B}G$. The A^*_G-algebra $\mathsf{A}^*_G(X)$ is generated by H, the equivariant hyperplane class. Since we have a surjection

$$\mathsf{A}^*_G(X) \to \mathsf{A}^*_G(X \smallsetminus (\mathsf{NR} \cup \mathsf{CTP} \cup \mathsf{QP})) ,$$

we see that $\mathsf{A}^*(\mathcal{F}_2 \smallsetminus \mathsf{Ell})$ is generated by the images of H, c_2, c_3. By (2), $\mathsf{Pic}(\mathcal{F}_2)$ is generated by λ and $[\mathsf{Ell}]$. The class H is the restriction of a linear combination $a\lambda + b[\mathsf{Ell}]$ to $\mathcal{F}_2 \smallsetminus \mathsf{Ell}$, and therefore must be a multiple of λ. Thus, the ring $\mathsf{A}^*(\mathcal{F}_2 \smallsetminus \mathsf{Ell})$ is generated by λ, c_2, c_3.

Over the locus $\mathcal{F}_2 \smallsetminus \mathsf{Ell}$, the classes c_2, c_3 are the Chern classes of a rank 3 vector bundle $\mathcal{W}$ which can be explicitly described. Let

$$\pi : (\mathcal{S}, \mathcal{L}) \to \mathcal{F}_2$$

be the universal surface and the universal polarization. For surfaces in the locus $\mathcal{F}_2 \smallsetminus \mathsf{Ell}$, the quasi-polarization $L \to S$ is globally generated inducing a morphism $S \to \mathbb{P}(H^0(S, L))$. Then c_2, c_3 are the Chern classes of the bundle[2]

$$\mathcal{W} = \pi_*(\mathcal{L}) \otimes \det(\pi_*(\mathcal{L}))^{-\frac{1}{3}} \, .$$

A Grothendieck-Riemann-Roch calculation shows that c_2, c_3 are the restrictions of tautological classes on $\mathcal{F}_2 \smallsetminus \mathsf{Ell}$. The Grothendieck-Riemann-Roch calculation will be made more precise in Remark 4 below.

As shown in [CK, Theorem 4.1], the Chow classes on Ell are polynomials in (the restrictions of) λ and a certain codimension 2 class. In fact, up to a μ_2-banded gerbe, there is a quotient presentation

$$\mathsf{Ell} = [Z/\operatorname{SL}_2] \to \operatorname{BSL}_2$$

and the codimension 2 class is obtained by pullback from the base. As we will see below, the stack $\mathcal{F}_2$ also admits a quotient presentation

$$\mathcal{F}_2 = [W/G] \to \mathrm{B}G$$

hence the class c_2 also makes sense on $\mathcal{F}_2$ by pullback from the base. Furthermore, there is a natural morphism

$$\operatorname{SL}_2 \to G\,, \quad g \mapsto \operatorname{Sym}^2 g$$

compatible with the above maps. Geometrically, this corresponds to the fact that the linear series $|L|$ induces a map from S to a nonsingular conic in $\mathbb{P}^2$ in the elliptic case. Consequently, the restriction of c_2 from $\mathcal{F}_2$ to Ell can be chosen to be the degree 2 generator on Ell. Thus $\mathsf{A}^*(\mathsf{Ell})$ is generated by λ and c_2. Using the push-pull formula, we conclude that the image of $\mathsf{A}^*(\mathsf{Ell})$ in $\mathsf{A}^*(\mathcal{F}_2)$ is generated by $[\mathsf{Ell}]$, $\lambda[\mathsf{Ell}]$, $c_2[\mathsf{Ell}]$.

By excision, there is an exact sequence

$$\mathsf{A}^{*-1}(\mathsf{Ell}) \to \mathsf{A}^*(\mathcal{F}_2) \to \mathsf{A}^*(\mathcal{F}_2 \smallsetminus \mathsf{Ell}) \to 0\,.$$

As a result, λ, $[\mathsf{Ell}]$, c_2 and c_3 suffice to generate the ring $\mathsf{A}^*(\mathcal{F}_2)$. We also conclude that the entire Chow ring is tautological.

□

[2] The third root $\det(\pi_*(\mathcal{L}))^{-\frac{1}{3}}$ can either be viewed formally for the Chern class calculation or can be viewed as an actual line bundle on the fiber product of (3) and (4).

Remark 4. The proof of Theorem 1 (i) shows that we can choose the degree 1 generators to be

$$\alpha_1 = \lambda, \quad \alpha_2 = [\mathsf{Ell}]\ .$$

By [MOP, Proposition 1], the unique normalization-independent codimension 1 combination of κ classes can be written as

$$\kappa_{1,1} - 4\kappa_{3,0} = 18\alpha_1 - 12\alpha_2\,.$$

By the explicit Grothendieck-Riemann-Roch calculation alluded to in the proof above, the degree 2 generator can be taken to be

$$\beta = (\kappa_{4,0} + \kappa_{2,1}) - \frac{1}{36}(2\kappa_{3,0} + \kappa_{1,1})^2\,,$$

which is also normalization independent. As for the degree 3 generator, Grothendieck-Riemann-Roch shows that we can take

$$\gamma = 6\kappa_{5,0} + 10\kappa_{3,1} + 3\kappa_{1,2}\,,$$

for any normalization of the quasi-polarization $\mathcal{L}$. Changing $\mathcal{L}$ modifies γ by monomials in terms of degree 1 and 2. A less elegant normalization-independent expression can also be written down.

We briefly explain how to arrive at the expression for β claimed above. Note that by Grothendieck-Riemann-Roch, we have

$$\mathrm{ch}(\pi_*\mathcal{L}) = \pi_*\left(\exp(c_1(\mathcal{L}))\cdot \mathrm{Todd}(T_\pi^{\mathrm{rel}})\right)\,.$$

We expand

$$\exp(c_1(\mathcal{L})) = 1 + c_1(\mathcal{L}) + \frac{c_1(\mathcal{L})^2}{2} + \frac{c_1(\mathcal{L})^3}{6} + \frac{c_1(\mathcal{L})^4}{24} + \dots\,,$$

$$\mathrm{Todd}(T_\pi^{\mathrm{rel}}) = 1 - \frac{\lambda}{2} + \frac{\lambda^2 + t}{12} - \frac{\lambda t}{24} + \frac{-\lambda^4 + 4\lambda^2 t + 3t^2}{720} + \dots\,,$$

where $t = c_2(T_\pi^{\mathrm{rel}})$. From here, using the definition of the κ-classes and the fact that $\pi_*(t) = 24$, $\pi_*(t^2) = 88\lambda^2$ by [GK, Section 3], we immediately obtain

$$\mathrm{ch}_1(\pi_*\mathcal{L}) = \frac{2\kappa_{3,0} + \kappa_{1,1}}{12} - \frac{3\lambda}{2}, \quad \mathrm{ch}_2(\pi_*\mathcal{L}) = \frac{\kappa_{4,0} + \kappa_{2,1}}{24} - \frac{\lambda}{24}\cdot(2\kappa_{3,0} + \kappa_{1,1}) + \frac{7\lambda^2}{12}\,.$$

Recalling that

$$\mathcal{W} = \pi_*\left(\mathcal{L}\right) \otimes \det(\pi_*\left(\mathcal{L}\right))^{-\frac{1}{3}}\,,$$

we find $c_1(\mathcal{W}) = 0$ and

$$-c_2(\mathcal{W}) = \mathrm{ch}_2(\mathcal{W}) = \mathrm{ch}_2(\pi_*(\mathcal{L})) - \frac{1}{6}\left(\mathrm{ch}_1(\pi_*(\mathcal{L}))\right)^2 = \frac{1}{24}\left((\kappa_{4,0} + \kappa_{2,1} - \frac{1}{36}(2\kappa_{3,0} + \kappa_{1,1})^2\right) + \frac{5\lambda^2}{24}\,.$$

Therefore $\beta = -24c_2(\mathcal{W}) - 5\lambda^2$ can be chosen to be the degree 2 generator, replacing the generator $c_2(\mathcal{W})$ which arises in the proof above. The calculation for γ is similar.

3.3. **Blowing up the triple conic locus.** We would like to study the whole moduli space $\mathcal{F}_2$, not just $\mathcal{F}_2 \smallsetminus \mathsf{Ell}$. Shah showed how to construct $\mathcal{F}_2$ as a GIT quotient [S]. The discussion below is just an adaptation of his work in the language of stacks.

The sextic given by the triple nonsingular conic, $\delta = (x_1x_3 - x_2^2)^3$, plays a special role. In fact, in the quotient $[X/G]$, all surfaces in the divisor Ell correspond to the orbit of δ. Following [S], in order to construct the moduli space $\mathcal{F}_2$, we blow up the orbit of δ and remove further loci from the blowup. We make this more precise.

Elliptic surfaces (S, L) in Ell can also be exhibited as resolutions of certain branched double covers. In this case, the linear series $|L| = |\sigma + 2f|$ has fixed part σ, and $|L - \sigma|$ induces the map $S \to Q \subset \mathbb{P}^2$, where $Q = \{x_1x_3 - x_2^2 = 0\} \simeq \mathbb{P}^1$ is the nonsingular conic. However, the linear series $|2L|$ has no fixed components and induces a morphism

$$S \to \widehat{S} \to \mathbb{P}^5 .$$

Here $S \to \widehat{S}$ contracts all nonsingular rational curves on which L restricts trivially. Under the Veronese embedding

$$\mathsf{v} : \mathbb{P}^2 \to \mathbb{P}^5,$$

the conic Q corresponds to a rational normal curve $\mathsf{v}(Q)$ contained in a hyperplane section $\mathbb{P}^4$ of $\mathbb{P}^5$. The cone over $\mathsf{v}(Q)$ is denoted $\Sigma_4^0 \subset \mathbb{P}^5$. This cone is a flat deformation of the Veronese surface $\mathsf{v}(\mathbb{P}^2)$. Blowing up the vertex, we obtain the Hirzebruch surface Σ_4. The surface S is the resolution of the double cover

$$\widehat{S} \to \Sigma_4^0$$

ramified over the vertex of the cone and over a curve D. For S to be a $K3$ surface, D must be a cubic section of $\Sigma_4^0 \subset \mathbb{P}^5$ not passing through the vertex, and must have ADE singularities.

Following [La, T], we note the following uniform description of all surfaces in $\mathcal{F}_2$ as resolutions of branched double covers $\widehat{S}$. Indeed, in all cases, $\widehat{S}$ arises as a complete intersection

$$\{z^2 - f_6(x_1, x_2, x_3, y) = f_2(x_1, x_2, x_3, y) = 0\} \subset \mathbb{P}(1, 1, 1, 2, 3) ,$$

where f_2, f_6 have the indicated weighted degrees. When $f_2(0, 0, 0, 1) \neq 0$, we can change coordinates so that $f_2(x_1, x_2, x_3, y) = y$ and we recover the sextic double planes. When $f_2(0, 0, 0, 1) = 0$, and $f_2(x_1, x_2, x_3, 0)$ is non degenerate, we may assume $f_2(x_1, x_2, x_3, y) = x_1x_3 - x_2^2$ and we can write

$$f_6(x, y) = y^3 + yg_4(x) + g_6(x) .$$

This corresponds to the branched double covers of Σ_4^0. Indeed, we have $\Sigma_4^0 = \mathbb{P}(1, 1, 4)$, see [D], and the latter is cut out by $x_1x_3 - x_2^2 = 0$ in $\mathbb{P}(1, 1, 1, 2)$.

We return to the moduli space $\mathcal{F}_2$. There is a G-equivariant cubing map

$$m_3 : \mathbb{P}\,\mathrm{Sym}^2 V^* \to X .$$

Let $\Delta_2 \subset \mathbb{P}\,\mathrm{Sym}^2 V^*$ denote the divisor parametrizing singular conics. The induced map

$$\mathbb{P}\,\mathrm{Sym}^2 V^* \smallsetminus \Delta_2 \to X \smallsetminus (\mathsf{QP} \cup \mathsf{ML})$$

is a G-equivariant closed embedding whose image is the locus TC parametrizing cubes of nonsingular conics. A local calculation shows that

$$\mathsf{TC} \subset \mathsf{CTP}. \tag{5}$$

All nonsingular conics are equivalent under the action of G. Let $W = \mathbb{C}^2$ as an SL_2 representation. We can identify $V = \mathrm{Sym}^2 W$, as the conic is a $\mathbb{P}^1$ embedded in $\mathbb{P}^2$ via the Veronese map. By Luna's étale slice theorem, the quotient $[X/G]$ is locally identified around $[\mathsf{TC}/G]$ with a quotient by $\mathrm{SL}(W)$ of the normal slice to the orbit of the triple conic. The normal slice is the summand

$$\mathrm{Sym}^8 W^* \oplus \mathrm{Sym}^{12} W^* \subset \mathrm{Sym}^6(\mathrm{Sym}^2 W^*)\,.$$

This identification is more thoroughly explained in [S, Section 5] and [La, Lemma 4.9]. Next, we carry out the weighted blowup of the locus

$$\mathsf{TC} \subset X \smallsetminus (\mathsf{QP} \cup \mathsf{ML})\,,$$

and take the quotient.

We indicate the loci that need to be removed from the blowup to obtain $\mathcal{F}_2$. On the sextic ADE locus, in addition to the loci ML and QP already considered, we also need to remove the images of the double conic and double cubic locus DC and the consecutive triple point locus CTP.

Over the exceptional divisor $[\mathbb{P}_{(8,12)}(\mathrm{Sym}^8 W^* \oplus \mathrm{Sym}^{12} W^*)/\,\mathrm{SL}(W)]$, there is an open subset corresponding to elliptic K3 surfaces. By [M], a pair of binary forms $(A, B) \in \mathrm{Sym}^8 W^* \oplus \mathrm{Sym}^{12} W^*$ corresponds to an elliptic K3 surface if and only if

(i) $4A^3 + 27B^2$ is not identically zero.
(ii) For each point $q \in \mathbb{P}^1$, the order of vanishing of A at q is at most 3 or the order of vanishing of B at q is at most 5.

This description was used in the calculations of [CK]; these calculations play an important role below. We note that these are exactly the same requirements singled out in [S, Theorem 4.3]. Indeed, using the above notation, we can view g_4 and g_6 as binary forms A, B of degrees 8 and 12 on $\mathbb{P}^1$, respectively. From the exact description of $\widehat{S}$, we find that the branch curve takes the form $D = \{y^3 + yA + B = 0\}$. Condition (i) corresponds to non-reduced branch curves D; this is discussed in [S, Theorem 4.3, Case 2]. Condition (ii) ensures D has no consecutive triple points; this was noted in the proof of [S, Theorem 4.3, Case (1.i)] and can be seen as follows: locally near $y = u = 0$, the equation $y^3 + yA(u) + B(u)$ is in the ideal $(y, u^2)^3$ if and only if A, B vanish with order at least 4 and 6 at the origin. Condition (i) corresponds to the restriction of the strict transform of the locus DC to the exceptional divisor of the weighted blowup. Condition (ii) corresponds to the restriction of the strict transform of $[\mathsf{CTP}/G]$ to the exceptional divisor.

In conclusion, the moduli space $\mathcal{F}_2$ is obtained by removing the strict transform of (the images of) DC and CTP from the weighted blowup.

3.4. **Strategy.** The following procedure will be used to finish the proof of Theorem 1.

(i) We start with

$$\mathsf{A}_G^*(X) = \mathbb{Q}[H, c_2, c_3]/(p)\,,$$

where

$$p = H^{28} + c_1^G(\mathrm{Sym}^6 V^*)H^{27} + \cdots + c_{28}^G(\mathrm{Sym}^6 V^*)\,.$$

(ii) We impose relations by removing ML (the locus of multiple lines) and QP (the locus of quadruple points). That is, we compute the Chow ring $\mathsf{A}_G^*(Y)$ where

$$Y = X \smallsetminus (\mathsf{ML} \cup \mathsf{QP})\,.$$

This is carried out in Sections 4 and 5.

(iii) We perform the weighted blowup $\widetilde{Y}$ of Y at TC (the triple conic locus) and compute the corresponding Chow ring. This is carried out in Section 6.

(iv) Finally, we impose more relations by removing from $\widetilde{Y}$ the strict transforms of CTP (the consecutive triple point locus) and DC (the union of the double conic and double cubic locus). This is carried out in Section 6.

From the above discussion of Shah's work, steps (i)-(iv) account for all of the relations, and thus will finish the proof of Theorem 1(iii).

4. Relations from the locus of multiple lines

We impose here relations obtained by removing ML. We follow the notation from Section 3. We begin by studying the map

$$m_1 : \mathbb{P}\,\mathrm{Sym}^4 V^* \times \mathbb{P}V^* \to X = \mathbb{P}\,\mathrm{Sym}^6 V^*,$$

whose image is ML. Let h_1 and h_2 denote the hyperplane classes of $\mathbb{P}\,\mathrm{Sym}^4 V^*$ and $\mathbb{P}V^*$ pulled back to the product $\mathbb{P}\,\mathrm{Sym}^4 V^* \times \mathbb{P}V^*$. As before, let H be the hyperplane class of X.

For notational convenience, we will denote the map m_1 simply by m. Moreover, the same symbols m, h_1, h_2, H will be used to denote the corresponding maps and classes on the quotients by G.

Lemma 5. *The image of*

$$\mathsf{A}_G^*(\mathbb{P}\,\mathrm{Sym}^4 V^* \times \mathbb{P}V^*) \xrightarrow{m_*} \mathsf{A}_G^*(X)$$

is the ideal generated by the three classes

$$\sum_{j=0}^{11+i} \alpha_{i,j} H^j$$

for $0 \le i \le 2$, where the coefficients $\alpha_{ij} \in \mathsf{A}_G^*$ are recursively given by the formula

$$\alpha_{i,k} = \gamma_*((h_1 + 2h_2)^{27-k} \cdot h_2^i) - \sum_{j=k+1}^{11+i} \alpha_{i,j} s_{j-k}^G(\mathrm{Sym}^6 V^*)\,.$$

Here, $\gamma_* : \mathsf{A}_G^*(\mathbb{P}\,\mathrm{Sym}^4 V^* \times \mathbb{P}V^*) \to \mathsf{A}_G^*$ denotes the equivariant pushforward to a point.

Proof. Consider the pullback

$$m^* : \mathsf{A}^*_G(X) \to \mathsf{A}^*_G(\mathbb{P}\operatorname{Sym}^4 V^* \times \mathbb{P}V^*),$$

and note that $m^*H = h_1 + 2h_2$. By the projective bundle formula [F, Chapter 3], the $\mathsf{A}^*_G(X)$-module $\mathsf{A}^*_G(\mathbb{P}\operatorname{Sym}^4 V^* \times \mathbb{P}V^*)$ is generated by the classes $1, h_2, h_2^2$. Therefore, the image of

$$\mathsf{A}^*_G(\mathbb{P}\operatorname{Sym}^4 V^* \times \mathbb{P}V^*) \xrightarrow{m_*} \mathsf{A}^*_G(X)$$

is the ideal generated by the pushforwards $m_*(h_2^i)$ for $0 \le i \le 2$. We write

$$m_*(h_2^i) = \sum_{j=0}^{11+i} \alpha_{i,j} H^j \,,$$

where $\alpha_{i,j} \in \mathsf{A}_G^{11+i-j}$. We want to determine the coefficients $\alpha_{i,j}$. To pick out $\alpha_{i,k}$, multiply by H^{27-k}. We have

$$H^{27-k} \cdot m_*(h_2^i) = \sum_{j=0}^{11+i} \alpha_{i,j} H^{j+27-k} \,.$$

Consider the commutative diagram

$$\begin{array}{ccc} \left[\left(\mathbb{P}\operatorname{Sym}^4 V^* \times \mathbb{P}V^*\right)/G\right] & \xrightarrow{m} & [X/G] \\ & \searrow^{\gamma} & \downarrow \rho \\ & & \mathrm{B}G\,. \end{array}$$

Then,

$$\rho_*(H^{27-k} \cdot m_*(h_2^i)) = \rho_*\left(\sum_{j=0}^{11+i} \alpha_{i,j} H^{j+27-k}\right) = \sum_{j=0}^{11+i} \alpha_{i,j} s^G_{j-k}(\operatorname{Sym}^6 V^*)\,.$$

On the other hand, by the projection formula,

$$\rho_*(H^{27-k} \cdot m_*(h_2^i)) = \rho_* m_*(m^*H^{27-k} \cdot h_2^i) = \gamma_*((h_1 + 2h_2)^{27-k} \cdot h_2^i)\,.$$

Thus,

$$\sum_{j=0}^{11+i} \alpha_{i,j} s^G_{j-k}(\operatorname{Sym}^6 V^*) = \gamma_*((h_1 + 2h_2)^{27-k} \cdot h_2^i)\,.$$

Note that $s_0^G(\operatorname{Sym}^6 V^*) = 1$ and $s^G_{j-k}(\operatorname{Sym}^6 V^*) = 0$ for $j - k < 0$. Simplifying and rearranging, we see that

$$\alpha_{i,k} = \gamma_*((h_1 + 2h_2)^{27-k} \cdot h_2^i) - \sum_{j=k+1}^{11+i} \alpha_{i,j} s^G_{j-k}(\operatorname{Sym}^6 V^*)\,. \tag{6}$$

□

Remark 6. Equation (6) gives a recursion for computing $\alpha_{i,k}$. To this end, note that the pushforwards $\gamma_*(h_1^a \cdot h_2^b)$ can be immediately determined via the projective bundle geometry [F, Chapter 3]. Thus, one can express the classes $\alpha_{i,k}$ in terms of the generators H, c_2, c_3 of $\mathsf{A}^*_G(X)$. As a result, the image of m_* specified by Lemma 5, and consequently the relations obtained by removing the locus ML, can be made explicit.

We carried out this procedure in the Macaulay2 package Schubert2 [GS, GSSEC]. The interested reader can consult [COP1] for the implementation. For example, for $i = 0$, the polynomial $\sum_{j=0}^{11} \alpha_{0,j} H^j$ is given by

$$
\begin{aligned}
1555200c_2^4c_3 &+ 9552816c_2c_3^3 + (518400c_2^5 + 11162448c_2^2c_3^2)H + (5716656c_2^3c_3 + 56538324c_3^3)H^2 \\
&+ (712080c_2^4 + 8743140c_2c_3^2)H^3 + 3852036c_2^2c_3H^4 + (311700c_2^3 + 12450672c_3^2)H^5 \\
&- 519696c_2c_3H^6 + 107640c_2^2H^7 + 243324c_3H^8 - 9900c_2H^9 + 480H^{11}\,.
\end{aligned}
$$

The case $i = 1$ is given by

$$
\begin{aligned}
1866240c_2^3c_3^2 &+ 15431472c_3^4 + (362880c_2^4c_3 + 4968864c_2c_3^3)H + (17280c_2^5 + 5732856c_2^2c_3^2)H^2 \\
&+ (1278288c_2^3c_3 + 47364588c_3^3)H^3 - (74040c_2^4 + 7471926c_2c_3^2)H^4 \\
&+ 1636848c_2^2c_3H^5 - (51630c_2^3 - 4266918c_3^2)H^6 - 598968c_2c_3H^7 \\
&+ 36810c_2^2H^8 + 40392c_3H^9 - 2850c_2H^{10} + 30H^{12}\,.
\end{aligned}
$$

Finally, the case $i = 2$ is given by

$$
\begin{aligned}
1259712c_2^2c_3^3 &+ (1765152c_2^3c_3^2 + 42620256c_3^4)H - (565920c_2^4c_3 - 19960020c_2c_3^3)H^2 \\
&+ (61056c_2^5 + 7261812c_2^2c_3^2)H^3 - (426564c_2^3c_3 - 28062369c_3^3)H^4 \\
&- (15404c_2^4 + 8744085c_2c_3^2)H^5 + 1276371c_2^2c_3H^6 - (63167c_2^3 - 1147635c_3^2)H^7 \\
&- 218646c_2c_3H^8 + 12903c_2^2H^9 + 5139c_3H^{10} - 389c_2H^{11} + H^{13}\,.
\end{aligned}
$$

5. Relations from quadruple points

We impose here relations obtained by removing the locus QP of sextics with quadruple points. A sextic f has a quadruple point if locally analytically it lies in the ideal $(x, y)^4$.

Denote by $\pi : [\mathbb{P}V/G] \to \mathrm{B}G$ the universal $\mathbb{P}^2$-bundle and let z be the hyperplane class. Then $\pi_*\mathcal{O}(6) = \mathrm{Sym}^6 V^*$ as a G-equivariant bundle. Its projectivization is $[X/G]$ and the projectivization of $\pi^*\pi_*\mathcal{O}(6)$ is $[(\mathbb{P}V \times X)/G]$. Consider the rank 10 bundle of principal parts $P^3(\mathcal{O}(6))$ relatively over $[\mathbb{P}V/G] \to \mathrm{B}G$. It comes equipped with an equivariant evaluation map

$$
\pi^* \mathrm{Sym}^6 V^* = \pi^*\pi_*\mathcal{O}(6) \to P^3(\mathcal{O}(6))\,,
$$

which on fibers takes a sextic to its expansion along a third order neighborhood. This evaluation map is surjective because $\mathcal{O}(6)$ is 6-very ample and hence also 3-very ample. The kernel is thus an equivariant vector bundle K_{quad} of rank 18. The bundle K_{quad} parametrizes pairs (f, p) where f is a sextic with a quadruple or worse point at p. After projectivizing, the following diagram summarizes the above discussion:

$$
(7) \qquad
\begin{array}{ccccc}
[\mathbb{P}K_{\mathrm{quad}}/G] & \xrightarrow{\;j\;} & [(\mathbb{P}V \times X)/G] & \xrightarrow{\;\pi'\;} & [X/G] \\
& \searrow^{\rho''} & \downarrow{\scriptstyle \rho'} & & \downarrow{\scriptstyle \rho} \\
& & [\mathbb{P}V/G] & \xrightarrow{\;\pi\;} & \mathrm{B}G\,.
\end{array}
$$

The image of $\pi' \circ j$ is the locus $[\mathsf{QP}/G]$. Moreover, since $\pi' \circ j$ is proper, the induced map on Chow groups with $\mathbb{Q}$-coefficients

$$\mathsf{A}_*([\mathbb{P}K_{\text{quad}}/G]) \to \mathsf{A}_*([\mathsf{QP}/G])$$

is surjective. Indeed, one sees this by applying [V, Lemma 3.8] to the map

$$(\mathbb{P}K_{\text{quad}} \times U)/G \to (\mathsf{QP} \times U)/G\,,$$

where U is an open subset of a representation V of G on which G acts freely and the codimension of the complement of U in V is sufficiently large as in [EG, Definition-Proposition 1]. Therefore, we can compute the image of

$$\mathsf{A}_*([\mathsf{QP}/G]) \to \mathsf{A}_*([X/G])$$

by computing the image of

$$\mathsf{A}_*([\mathbb{P}K_{\text{quad}}/G]) \to \mathsf{A}_*([X/G])\,.$$

Lemma 7. *The ideal of relations obtained from removing the locus of sextics with quadruple points is generated by the classes*

$$\sum_{i=0}^{10} \rho^*\pi_*\left(z^j \cdot c_i^G(P^3(\mathcal{O}(6)))\right) \cdot H^{10-i}\,,$$

where $0 \le j \le 2$ *and* z *is the hyperplane class of* π*.*

Proof. From the explicit calculation of the Chow ring of projective bundles it follows that every class $\alpha \in \mathsf{A}_G^*(\mathbb{P}K_{\text{quad}})$ is a pullback of a class $\beta \in \mathsf{A}_G^*(\mathbb{P}V \times X)$. Then, by the projection formula,

$$j_*\alpha = j_*j^*\beta = [\mathbb{P}K_{\text{quad}}]^G \cdot \beta\,.$$

Because $\mathbb{P}K_{\text{quad}}$ is linearly embedded in $\mathbb{P}V \times X$, its equivariant fundamental class is given by

$$[\mathbb{P}K_{\text{quad}}]^G = c_{10}^G(\rho'^*P^3(\mathcal{O}(6)) \otimes \mathcal{O}_{\rho'}(1)) = \sum_{i=0}^{10} \rho'^* c_i^G(P^3(\mathcal{O}(6))) \cdot \pi'^* H^{10-i}\,.$$

Every class $\beta \in \mathsf{A}_G^*(\mathbb{P}V \times X)$ is of the form

$$\beta = \beta_0 + \beta_1 z + \beta_2 z^2\,,$$

where $\beta_i \in \mathsf{A}_G^*(X)$ and z is the hyperplane class of the projective bundle $[(\mathbb{P}V \times X)/G] \to [X/G]$. Hence the ideal generated by pushforwards of classes on $[\mathbb{P}K_{\text{quad}}/G]$ is just the ideal generated by the classes

$$\pi'_*\left(z^j \cdot \left(\sum_{i=0}^{10} \rho'^* c_i^G(P^3(\mathcal{O}(6))) \cdot \pi'^* H^{10-i}\right)\right) = \sum_{i=0}^{10} \rho^*\pi_*(z^j \cdot c_i^G(P^3(\mathcal{O}(6)))) \cdot H^{10-i}$$

for $0 \le j \le 2$, where to obtain the equality we have used [F, Proposition 1.7]. ∎

Remark 8. The 3 relations provided by Lemma 7 can be written explicitly in terms of the generators H, c_2, c_3 of $\mathsf{A}_G^*(X)$. To this end, we note that

- the equivariant Chern classes of the jet bundle $P^3(\mathcal{O}(6))$ are computed using the filtration by the vector bundles $\mathcal{O}(6) \otimes \operatorname{Sym}^k \Omega_\pi$ for $0 \le k \le 3$

- the pushforwards $\pi_*(z^j)$ are immediately found using the projective bundle geometry.

The $j=0$ case yields

$$-157464c_2c_3^2 - 236196c_2^2c_3H - 61020c_2^3H^2 + 434484c_3^2H^2 + 382725c_2c_3H^3 + 76545c_2^2H^4 - 66339c_3H^5 - 13230c_2H^6 + 405H^8\,.$$

The $j=1$ case yields

$$51840c_2^3c_3 - 122472c_3^3 + (17280c_2^4 - 539460c_2c_3^2)H - 446148c_2^2c_3H^2 - (91320c_2^3 - 339309c_3^2)H^3 + 244215c_2c_3H^4 + 39690c_2^2H^5 - 17577c_3H^6 - 2880c_2H^7 + 30H^9\,.$$

Finally, the case $j=2$ yields

$$209952c_2^2c_3^2 + (253152c_2^3c_3 - 338256c_3^3)H + (61056c_2^4 - 812592c_2c_3^2)H^2 - 475308c_2^2c_3H^3 - (76460c_2^3 - 178632c_3^2)H^4 + 104733c_2c_3H^5 + 13293c_2^2H^6 - 3267c_3H^7 - 390c_2H^8 + H^{10}\,.$$

6. The weighted blowup

6.1. **Overview.** Lemmas 5 and 7 allow us to compute the Chow ring of the stack

$$[(X \setminus (\mathsf{ML} \cup \mathsf{QP}))\,/G]\,.$$

The next step in the procedure is to perform the weighted blowup along the locus TC.

6.2. **The Chow ring of the weighted blowup.** First, we discuss the intersection theory of weighted blowups in general, following Arena and Obinna [AO].

Let $i : Z \hookrightarrow Y$ be a closed embedding of codimension d, with Z and Y nonsingular. Let N be the normal bundle weighted by a $\mathbb{G}_m$-action. Let $P_N(T)$ be the $\mathbb{G}_m$-equivariant top Chern class $c_d^{\mathbb{G}_m}(N)$, where T is the equivariant parameter. It is a polynomial in T of degree d. In the case where all the weights are 1, this is simply the Chern polynomial for the normal bundle. We have

$$\mathsf{A}^*(N) = \mathsf{A}^*(Z)[T]/(P_N(T))$$

where T is set to the hyperplane class of the weighted projective bundle $N \to Z$. Let $f : \widetilde{Y} \to Y$ denote the weighted blowup along Z, and $j : \widetilde{Z} \hookrightarrow \widetilde{Y}$ the exceptional divisor. By [AO, Theorem 5.5], there is a commutative diagram

$$\begin{array}{ccc} \mathsf{A}^*(\widetilde{Z}) & \xrightarrow{j_*} & \mathsf{A}^*(\widetilde{Y}) \\ {\scriptstyle f^!}\uparrow & & \uparrow{\scriptstyle f^*} \\ \mathsf{A}^*(Z) & \xrightarrow{i_*} & \mathsf{A}^*(Y) \end{array}$$

where $f^!(\alpha) = g^*\alpha \cdot \delta$. Here, $g : \widetilde{Z} \to Z$ is the weighted projective bundle structure on the exceptional divisor and

$$\delta = \frac{P_N(T) - P_N(0)}{T}\,.$$

Furthermore, we have the following structure theorem for the Chow groups of weighted blowups.

Theorem 9 (Arena-Obinna [AO])**.** *In the above setting, there is a split exact sequence (of Chow groups with $\mathbb{Q}$-coefficients)*

$$0 \to \mathsf{A}^{k-d}(Z) \xrightarrow{(f^!,-i_*)} \mathsf{A}^{k-1}(\widetilde{Z}) \oplus \mathsf{A}^k(Y) \xrightarrow{(j_*,f^*)} \mathsf{A}^k(\widetilde{Y}) \to 0\,.$$

When all the weights are all 1, the class δ is the top Chern class of the excess bundle for the usual blowup, and Theorem 9 recovers [F, Proposition 6.7(e)].

We apply these results to our situation. Set $Y = X \smallsetminus (\mathsf{ML} \cup \mathsf{QP})$. We have the diagram

$$\begin{array}{ccc}
\mathsf{A}^*_G(\widetilde{\mathsf{TC}}) & \xrightarrow{j_*} & \mathsf{A}^*_G(\widetilde{Y}) \\
{\scriptstyle f^!}\big\uparrow & & {\scriptstyle f^*}\big\uparrow \\
\mathsf{A}^*_G(\mathsf{TC}) & \xrightarrow{i_*} & \mathsf{A}^*_G(Y)
\end{array}$$

and for each k, we obtain split exact sequences

$$0 \to \mathsf{A}^{k-22}_G(\mathsf{TC}) \to \mathsf{A}^{k-1}_G(\widetilde{\mathsf{TC}}) \oplus \mathsf{A}^k_G(Y) \to \mathsf{A}^k_G(\widetilde{Y}) \to 0\,.$$

We now want to impose relations by removing the strict transform of CTP, which we will denote by $\widetilde{\mathsf{CTP}}$.

6.3. **Relations from consecutive triple points.** Recall that we say a sextic f has a consecutive triple point if analytically locally it lies in the ideal $(x, y^2)^3$. We begin by constructing the relevant bundle of principal parts.

Note that the local equation is in the ideal $(x, y^2)^3$ if and only if the coefficients in the Taylor expansion of the monomials in the set $S = \{1, x, y, x^2, xy, y^2, x^2y, xy^2, y^3, xy^3, y^4, y^5\}$ all vanish. To record the data of these monomial coefficients, we use the machinery of refined principal parts bundles as in [CL, Section 3.2]. The universal $\mathbb{P}^2$ bundle is denoted by $\pi : [\mathbb{P}V/G] \to \mathrm{B}G$. Set T to be the tangent bundle of $\mathbb{P}V$. The set S is admissible in the sense of [CL, Definition 3.7]. There is a rank 12 bundle on the domain of $a : [\mathbb{P}T/G] \to [\mathbb{P}V/G]$ denoted $P^S(\mathcal{O}(6))$ and an evaluation morphism

$$a^*\pi^* \operatorname{Sym}^6 V = a^*\pi^*\pi_*\mathcal{O}(6) \to a^*P^5(\mathcal{O}(6)) \to P^S(\mathcal{O}(6))\,, \tag{8}$$

where the first map is the usual fifth order principal parts evaluation pulled back to $\mathbb{P}T$, and the second map truncates the Taylor series along the monomials in S. The composite is surjective, as the first map is surjective because $\mathcal{O}(6)$ is 5-very ample, and the second map is surjective by definition. Let K_{ctp} denote the kernel of the morphism (8). It is a G-equivariant vector bundle of rank 16, parametrizing pairs (f, p, v) where f is a sextic with a consecutive triple point at p in the tangent direction $v \in T_p\mathbb{P}V$. The following diagram summarizes the situation:

$$\begin{array}{ccccccc}
[\mathbb{P}K_{\text{ctp}}/G] & \xrightarrow{\iota} & [(\mathbb{P}T \times X)/G] & \xrightarrow{a'} & [(\mathbb{P}V \times X)/G] & \xrightarrow{\pi'} & [X/G] \\
& {\scriptstyle \rho_3}\searrow & \big\downarrow{\scriptstyle \rho_2} & & \big\downarrow{\scriptstyle \rho_1} & & \big\downarrow{\scriptstyle \rho} \\
& & [\mathbb{P}T/G] & \xrightarrow{a} & [\mathbb{P}V/G] & \xrightarrow{\pi} & \mathrm{B}G\,.
\end{array} \tag{9}$$

The image of $\pi' \circ a' \circ \iota$ is the locus CTP. By an abuse of notation, we continue to denote by $\mathbb{P}K_{\text{ctp}}$ the pullback of $\mathbb{P}K_{\text{ctp}}$ along the open inclusion $Y = X \smallsetminus (\mathsf{QP} \cup \mathsf{ML}) \subset X$.

Let J denote the incidence variety in $\mathbb{P}T \times \mathsf{TC}$, parametrizing a triple nonsingular conic together with a tangent direction at a point of the conic. The tangent direction is uniquely determined by the point since the conic is nonsingular. Therefore, J is isomorphic to the universal nonsingular conic of dimension 6. Recall from (5) that $J \subset \mathbb{P}K_{\text{ctp}}$, corresponding to the fact that triple conics have consecutive triple points everywhere. Let $\widetilde{\mathbb{P}K_{\text{ctp}}}$ be the blowup of $\mathbb{P}K_{\text{ctp}}$ along J. We denote the corresponding exceptional divisor by E. Then by excision and Theorem 9, which we can apply because J and $\mathbb{P}K_{\text{ctp}}$ are nonsingular, we have the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathsf{A}_G^{k-21}(J) & \longrightarrow & \mathsf{A}_G^{k-10}(E) \oplus \mathsf{A}_G^{k-9}(\mathbb{P}K_{\text{ctp}}) & \longrightarrow & \mathsf{A}_G^{k-9}(\widetilde{\mathbb{P}K_{\text{ctp}}}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathsf{A}_G^{k-22}(\mathsf{TC}) & \longrightarrow & \mathsf{A}_G^{k-1}(\widetilde{\mathsf{TC}}) \oplus \mathsf{A}_G^{k}(Y) & \longrightarrow & \mathsf{A}_G^{k}(\widetilde{Y}) & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathsf{A}_G^{k-1}(\widetilde{\mathsf{TC}} \smallsetminus \operatorname{Im}(E)) \oplus \mathsf{A}_G^{k}(Y \smallsetminus \mathsf{CTP}) & \longrightarrow & \mathsf{A}_G^{k}(\widetilde{Y} \smallsetminus \widetilde{\mathsf{CTP}}) & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0\,. & &
\end{array}$$

Here $\operatorname{Im}(E)$ is the image of E under the map to $\widetilde{\mathsf{TC}}$ induced by diagram 9 and the various blowups. When $k \leq 19$, $\mathsf{A}_G^{k-22}(\mathsf{TC}) = \mathsf{A}_G^{k-21}(J) = 0$, so we have an equality of Poincaré polynomials

$$\sum_{k=0}^{19} t^k \cdot \dim \mathsf{A}_G^{k}(\widetilde{Y} \smallsetminus \widetilde{\mathsf{CTP}}) = \sum_{k=0}^{19} t^k \cdot \left(\dim \mathsf{A}_G^{k-1}(\widetilde{\mathsf{TC}} \smallsetminus \operatorname{Im}(E)) + \dim \mathsf{A}_G^{k}(Y \smallsetminus \mathsf{CTP})\right). \tag{10}$$

By the discussion in Section 3, we can identify $[\widetilde{\mathsf{TC}}/G]$ with the quotient stack

$$[\mathbb{P}_{(8,12)}(\operatorname{Sym}^8 W^* \oplus \operatorname{Sym}^{12} W^*)/\operatorname{SL}(W)]\,.$$

Under this identification, $[\operatorname{Im}(E)/G]$ parametrizes the locus of forms (A, B) such that there exists a point $q \in \mathbb{P}^1$ where the order of vanishing of A at q is at least 4 and the order of vanishing of B at q is at least 6. The relations obtained from removing this locus were calculated in [CK, Proposition 3.4]. As a result, we obtain the following Poincaré polynomial (see [CK, Theorem 1.2]):

$$\begin{aligned}\sum_{k=0}^{19} t^k \cdot \dim \mathsf{A}_G^{k-1}(\widetilde{\mathsf{TC}} \smallsetminus \operatorname{Im}(E))) &= t + t^2 + 2t^3 + 2t^4 + 3t^5 + 3t^6 + 4t^7 + 4t^8 + 5t^9 + \\ &\quad + 4t^{10} + 4t^{11} + 3t^{12} + 3t^{13} + 2t^{14} + 2t^{15} + t^{16} + t^{17}\,.\end{aligned} \tag{11}$$

To compute $\sum_{k=0}^{19} t^k \cdot \dim \mathsf{A}_G^k(\widetilde{Y} \smallsetminus \widetilde{\mathsf{CTP}})$, it suffices to compute $\sum_{k=0}^{19} t^k \cdot \dim \mathsf{A}_G^k(Y \smallsetminus \mathsf{CTP})$ by equation (10), which we do with the following Lemma. By the same argument given before Lemma 7, the image of $\mathsf{A}_*([\mathsf{CTP}/G]) \to \mathsf{A}_*([X/G])$ is equal to the image of $\mathsf{A}_*([\mathbb{P}K_{\text{ctp}}/G]) \to \mathsf{A}_*([X/G])$.

Lemma 10. *The ideal of relations obtained from removing the locus of sextics with consecutive triple points is generated by the classes*

$$\sum_{k=0}^{12} \rho^* \pi_* a_* (\tau^j z^i \cdot c_k^G(P^S(\mathcal{O}(6)))) \cdot H^{12-k} \,,$$

where $0 \le i \le 2$*,* $0 \le j \le 1$*,* z *is the hyperplane class on* π *and* τ *is the hyperplane class of* a*.*

Proof. Every class $\alpha \in \mathsf{A}^*_G(\mathbb{P}K_{\mathrm{ctp}})$ is pulled back from a class $\beta \in \mathsf{A}^*_G(\mathbb{P}T \times X)$, so by the projection formula

$$\iota_* \alpha = \iota_* \iota^* \beta = [\mathbb{P}K_{\mathrm{ctp}}]^G \cdot \beta \,.$$

The map $\pi' \circ a'$ is a composition of two projective bundles. We denote by $\tau := c_1(\mathcal{O}_{a'}(1))$ and $z = c_1(\mathcal{O}_{\pi'}(1))$. The class β can be represented as a polynomial with coefficients $\beta_{ij} \in \mathsf{A}^*_G(X)$:

$$\beta = \sum_{0 \le i \le 2,\ 0 \le j \le 1} \beta_{ij} \tau^j z^i \,,$$

where we have omitted some pullbacks to declutter the notation. Thus the image of the pushforward

$$(\pi' \circ a' \circ \iota)_* : \mathsf{A}^*_G(\mathbb{P}K_{\mathrm{ctp}}) \to \mathsf{A}^*_G(X)$$

is the ideal generated by

$$(\pi' \circ a')_* \big([\mathbb{P}K_{\mathrm{ctp}}]^G \cdot \tau^j z^i \big)$$

for $0 \le j \le 1$ and $0 \le i \le 2$. Because $\mathbb{P}K_{\mathrm{ctp}}$ is linearly embedded in $\mathbb{P}T \times X$, its equivariant fundamental class is given by

$$[\mathbb{P}K_{\mathrm{ctp}}]^G = c_{12}^G(\rho_2^* P^S(\mathcal{O}(6)) \otimes \mathcal{O}_{\rho_2}(1)) = \sum_{k=0}^{12} \rho_2^* \, c_k^G(P^S(\mathcal{O}(6))) \cdot (\pi' \circ a')^* H^{12-k} \,.$$

The result now follows by another application of the projection formula. □

Remark 11. The 6 relations provided by Lemma 10 can be written explicitly in terms of the generators H, c_2, c_3 of $\mathsf{A}^*_G(X)$, similarly to the ones in Lemmas 5 and 7. The equivariant Chern classes of $P^S(\mathcal{O}(6))$ are computed using that $P^S(\mathcal{O}(6))$ is filtered by tensor products of symmetric powers of the tautological bundles on $\mathbb{P}T$ and the bundle $\mathcal{O}(6)$. More explicitly, we have the tautological sequence

$$0 \to \mathcal{O}_{\mathbb{P}T}(-1) \to a^* T \to Q \to 0 \,.$$

We set $\Omega_x = \mathcal{O}_{\mathbb{P}T}(1)$ and $\Omega_y = Q^*$. Then for each monomial $x^i y^j$ in S, the filtration will have successive quotients $\mathrm{Sym}^i \, \Omega_x \otimes \mathrm{Sym}^j \, \Omega_y \otimes \mathcal{O}(6)$. For more details on this filtration, see [CL, Section 3.2].

We implemented the calculation in the Macaulay2 package Schubert2 [GS, GSSEC]. When $i = j = 0$, we obtain the relation

$$\begin{aligned} -36288 c_2^3 c_3 - 244944 c_3^3 - (254592 c_2^4 + 2610792 c_2 c_3^2) H - 1154736 c_2^2 c_3 H^2 \\ + (848280 c_2^3 + 4719870 c_3^2) H^3 + 883548 c_2 c_3 H^4 - 588546 c_2^2 H^5 + 61236 c_3 H^6 \\ + 118620 c_2 H^7 - 4362 H^9 \,. \end{aligned}$$

When $i = 1$ and $j = 0$, we have

$$34560c_2^5 + (233280c_2^2c_3^2 + 83376c_2^3c_3 - 2350296c_3^3)H - (623856c_2^4 + 6328044c_2c_3^2)H^2 - 1198476c_2^2c_3H^3 + (810240c_2^3 + 2435751c_3^2)H^4 - 180306c_2c_3H^5 - 316413c_2^2H^6 + 84186c_3H^7 + 28950c_2H^8 - 381H^{10}\,.$$

When $i = 2$ and $j = 0$, we have

$$70848c_3c_2^4c_3 + 478224c_2c_3^3 + (254592c_2^5 + 2960712c_2^2c_3^2)H + (525240c_2^3c_3 - 4865832c_3^3)H^2 - (847664c_2^4 + 6647562c_2c_3^2)H^3 - 93798c_2^2c_3H^4 + (589876c_2^3 + 567756c_3^2)H^5 - 395280c_2c_3H^6 - 117822c_2^2H^7 + 26550c_3H^8 + 4432c_2H^9 - 14H^{11}\,.$$

When $i = 0$ and $j = 1$, we have

$$-69120c_2^5 - 466560c_2^2c_3^2 - (616896c_2^3c_3 - 1032264c_3^3)H + (1181376c_2^4 + 10152540c_2c_3^2)H^2 + 2659392c_2^2c_3H^3 - (1697460c_2^3 + 5233653c_3^2)H^4 - 441774c_2c_3H^5 + 623943c_2^2H^6 - 30942c_3H^7 - 56070c_2H^8 + 831H^{10}\,.$$

When $i = j = 1$, we have

$$31104c_2^4c_3 + 209952c_2c_3^3 - (463104c_2^5 + 4805568c_2^2c_3^2)H - (1385856c_2^3c_3 - 5604552c_3^3)H^2 + (1718688c_2^4 + 12248496c_2c_3^2)H^3 + 845640c_2^2c_3H^4 - (1212072c_2^3 + 1992276c_3^2)H^5 + 343116c_2c_3H^6 + 225864c_2^2H^7 - 30564c_3H^8 - 9024c_2H^9 + 48H^{11}\,.$$

When $i = 2$ and $j = 1$, we have

$$69120c_2^6 + 575424c_2^3c_3^2 + 734832c_3^4 + (149472c_2^4c_3 - 4974696c_2c_3^3)H - (1180896c_2^5 + 12642480c_2^2c_3^2)H^2 - (953352c_2^3c_3 - 7605414c_3^3)H^3 + (1698256c_2^4 + 7821144c_2c_3^2)H^4 - 773712c_2^2c_3H^5 - (623858c_2^3 + 346977c_3^2)H^6 + 263682c_2c_3H^7 + 55773c_2^2H^8 - 7110c_3H^9 - 896c_2H^{10} + H^{12}\,.$$

Proof of Theorem 1 (ii) and (iii). The quotient presentation of $\mathcal{F}_2$ in Section 3 gives

$$\mathsf{A}^*(\mathcal{F}_2) = \mathsf{A}^*_G(\widetilde{Y} \smallsetminus (\widetilde{\mathsf{CTP}} \cup \widetilde{\mathsf{DC}}))\,.$$

In the ring $\mathsf{A}^*_G(X) = \mathbb{Q}[H, c_2, c_3]/(p)$, where

$$p = H^{28} + c_1^G(\mathrm{Sym}^6 V^*)H^{27} + \cdots + c_{28}^G(\mathrm{Sym}^6 V^*)\,,$$

we form the ideal of relations I generated by the relations from Lemmas 5, 7, and 10. There are 10 such relations that we need to account for. Using the Macaulay2 package Schubert2 [GS, GSSEC] we find that

$$\mathsf{A}^*_G(X)/I = \mathsf{A}^*_G(Y \smallsetminus \mathsf{CTP})$$

has the Poincaré polynomial

$$\sum_{k=0}^{19} t^k \cdot \dim \mathsf{A}^k_G(Y \smallsetminus \mathsf{CTP}) = 1+t+2t^2+3t^3+4t^4+5t^5+7t^6+8t^7+9t^8+9t^9+8t^{10} + 6t^{11}+5t^{12}+3t^{13}+3t^{14}+t^{15}+t^{16}\,. \tag{12}$$

Using equations (10) and (11), we see that

$$\sum_{k=0}^{19} t^k \cdot \dim \mathsf{A}^k_G(\widetilde{Y} \smallsetminus \widetilde{\mathsf{CTP}}) = 1+2t+3t^2+5t^3+6t^4+8t^5+10t^6+12t^7+13t^8+14t^9+12t^{10} + 10t^{11}+8t^{12}+6t^{13}+5t^{14}+3t^{15}+2t^{16}+t^{17}\,.$$

Note that this polynomial is precisely the polynomial in the statement of Theorem 1. The space $\mathcal{F}_2$ is an open substack of $[(\widetilde{Y} \smallsetminus \widetilde{\mathsf{CTP}})/G]$ whose complement $\widetilde{\mathsf{DC}}$ has components of codimension 17 and 18. By excision, it follows that the Poincaré polynomial of $\mathcal{F}_2$ agrees with that of $[(\widetilde{Y} \smallsetminus \widetilde{\mathsf{CTP}})/G]$, except for possibly the coefficient of t^{17}. We know, however, that $\dim \mathsf{A}^{17}(\mathcal{F}_2) \geq 1$ because $\lambda^{17} \neq 0$ [GK]. Therefore, we see that $\dim \mathsf{A}^{17}(\mathcal{F}_2) = 1$. □

Remark 12. The relations among the generators of $\mathsf{A}^*(\mathcal{F}_2)$ can be explicitly obtained from the proof of Theorem 1(ii) and (iii) above. Theorem 9 determines the relations in $\mathsf{A}^*_G(\widetilde{Y})$. They are calculated in Lemmas 5 and 7. The removal of the locus $\widetilde{\mathsf{CTP}}$ imposes the final additional relations, which are calculated in Lemma 10 and [CK].

Remark 13. The proof shows that the inclusion $r : \mathsf{Ell} \to \mathcal{F}_2$ induces an isomorphism between the top nonvanishing Chow groups

$$r_* : \mathsf{A}^{16}(\mathsf{Ell}) \xrightarrow{\sim} \mathsf{A}^{17}(\mathcal{F}_2)\,.$$

We will use the isomorphism in Section 7.

7. The tautological ring is not Gorenstein

Because of the asymmetry in the Poincaré polynomial in Theorem 1(iii), the tautological ring $\mathsf{R}^*(\mathcal{F}_2)$ cannot be Gorenstein. In fact, we will show that it fails to be Gorenstein in *every* degree except for 0 and 17.

By the discussion in Section 3, we have morphisms

$$\mathcal{F}_2 \xrightarrow{\zeta} [\widetilde{Y}/G] \xrightarrow{f} [Y/G]\,,$$

where the complement of the image of ζ is the union of $\widetilde{\mathsf{DC}}$ and $\widetilde{\mathsf{CTP}}$. Recalling the inclusion $r : \mathsf{Ell} \to \mathcal{F}_2$, we obtain a commutative diagram of pullback maps

$$\begin{array}{ccc} \mathsf{A}^k(\mathsf{Ell}) & \xleftarrow{r^*} & \mathsf{A}^k(\mathcal{F}_2) \\ \uparrow & & \uparrow \zeta^* \\ \mathsf{A}^k_G(\widetilde{\mathsf{TC}}) & \xleftarrow{j^*} & \mathsf{A}^k_G(\widetilde{Y}) \\ g^* \uparrow & & \uparrow f^* \\ \mathsf{A}^k_G(\mathsf{TC}) & \xleftarrow[\iota^*]{} & \mathsf{A}^k_G(Y)\,. \end{array}$$

Proposition 14. *For each $k \geq 0$, the kernel of the pairing map*

$$\mathsf{A}^k(\mathcal{F}_2) \times \mathsf{A}^{17-k}(\mathcal{F}_2) \to \mathsf{A}^{17}(\mathcal{F}_2) \cong \mathbb{Q}$$

is generated by classes of the form $\zeta^* f^* \ker(\iota^*)$. More precisely, the ideal in $\mathsf{A}^*(\mathcal{F}_2)$ of classes that pair to zero with every class in complementary degree is generated by H and c_3.

Proof. We first compute $\ker(\iota^*)$. We note that

$$\iota^* : \mathsf{A}^*_G(Y) \to \mathsf{A}^*_G(\mathsf{TC}), \quad \iota^* H = 0, \quad \iota^* c_2 = 4c_2, \quad \iota^* c_3 = 0\,.$$

Indeed, for the vanishing of $\iota^* H$, we recall that $\mathsf{TC} = \mathbb{P}^5 \setminus \Delta_2$, where Δ_2 is the divisor of singular conics. The vanishing of $\iota^* c_3$ is a consequence of the fact that the stabilizer of the orbit TC is PSL_2. In fact,

$$\mathsf{A}^*_G(\mathsf{TC}) = \mathsf{A}^*_G(G/\mathsf{PSL}_2) = \mathsf{A}^*_{\mathsf{PSL}_2}(\mathrm{pt}) = \mathbb{Q}[c_2].$$

Furthermore, using the map $\mathsf{PSL}_2 \to G$, $g \to \mathrm{Sym}^2 g$, we see that $\iota^* c_2 = 4c_2$. These remarks show

$$\ker \iota^* = \langle H, c_3 \rangle\,.$$

Suppose that $\alpha \in \ker(\iota^*)$ is a class of codimension k. We claim that $\zeta^* f^* \alpha$ belongs to the kernel of the pairing. To this end, consider an arbitrary class $\zeta^* \gamma \in \mathsf{A}^{17-k}(\mathcal{F}_2)$. From the diagram above and Theorem 9, we can write

$$\gamma = f^* \gamma_1 + j_* \gamma_2 \in \mathsf{A}^{17-k}_G(\widetilde{Y})\,.$$

Then

$$f^* \alpha \cdot \gamma = f^* \alpha \cdot (f^* \gamma_1 + j_* \gamma_2) = f^*(\alpha \cdot \gamma_1) + f^* \alpha \cdot j_* \gamma_2\,.$$

We have

$$f^* \alpha \cdot j_* \gamma_2 = j_*(j^* f^* \alpha \cdot \gamma_2) = j_*(g^* \iota^* \alpha \cdot \gamma_2) = 0\,.$$

Next, we will show that the composition

$$\mathsf{A}^{17}_G(Y) \xrightarrow{f^*} \mathsf{A}^{17}_G(\widetilde{Y}) \xrightarrow{\zeta^*} \mathsf{A}^{17}(\mathcal{F}_2) \tag{13}$$

is identically zero, and hence $\zeta^* f^*(\alpha \cdot \gamma_1) = 0$. By Theorem 9, we have an isomorphism

$$\mathsf{A}^{17}_G(\widetilde{Y}) \cong \mathsf{A}^{17}_G(Y) \oplus \mathsf{A}^{16}_G(\widetilde{\mathsf{TC}})\,,$$

and under this isomorphism f^* is simply the inclusion into the first factor. Note that the restriction $\mathsf{A}^{17}_G(\widetilde{Y}) \xrightarrow{\zeta^*} \mathsf{A}^{17}(\mathcal{F}_2)$ factors through $\mathsf{A}^{17}_G(\widetilde{Y} \smallsetminus \widetilde{\mathsf{CTP}})$. By the same argument leading to equation (10) (and using the same notation), we have

$$\mathsf{A}^{17}_G(\widetilde{Y} \smallsetminus \widetilde{\mathsf{CTP}}) \cong \mathsf{A}^{17}_G(Y \smallsetminus \mathsf{CTP}) \oplus \mathsf{A}^{16}_G(\widetilde{\mathsf{TC}} \smallsetminus \mathrm{Im}(E))\,.$$

Moreover, the restriction map

$$\mathsf{A}^{17}_G(\widetilde{Y}) \cong \mathsf{A}^{17}_G(Y) \oplus \mathsf{A}^{16}_G(\widetilde{\mathsf{TC}}) \to \mathsf{A}^{17}_G(\widetilde{Y} \smallsetminus \widetilde{\mathsf{CTP}}) \cong \mathsf{A}^{17}_G(Y \smallsetminus \mathsf{CTP}) \oplus \mathsf{A}^{16}_G(\widetilde{\mathsf{TC}} \smallsetminus \mathrm{Im}(E))$$

is simply the sum of the restrictions on each summand. But $\mathsf{A}^{17}_G(Y \smallsetminus \mathsf{CTP}) = 0$, as calculated in (12). Thus, the map (13) is identically zero. We conclude that every class in $\zeta^* f^* \ker(\iota^*)$ is in the kernel of the pairing.

Now suppose we have a class

$$\beta \in \mathsf{A}^k(\mathcal{F}_2)/(\zeta^* f^* \ker(\iota^*)), \quad \beta \neq 0, \quad 0 < k < 17\,.$$

We show such a class is not in the kernel of the intersection pairing by exhibiting an element supported on Ell which pairs with β non-trivially.

Let $\epsilon \in \mathsf{A}^{16-k}(\mathsf{Ell})$, $\epsilon \neq 0$ be arbitrary. Then

$$\beta \cdot r_* \epsilon = r_* \left(r^* \beta \cdot \epsilon\right).$$

By Remark 13, the map

$$r_* : \mathsf{A}^{16}(\mathsf{Ell}) \to \mathsf{A}^{17}(\mathcal{F}_2)$$

is an isomorphism. Thus

$$\beta \cdot r_* \epsilon \neq 0 \iff r^* \beta \cdot \epsilon \neq 0\,.$$

Because $A^*(\mathsf{Ell})$ is Gorenstein with socle in codimension 16 [CK], we can pick ϵ such that $r^*\beta \cdot \epsilon \neq 0$ so long as $r^*\beta \neq 0$.

We show "by hand" that $\beta \neq 0$ implies $r^*\beta \neq 0$. First, we pick a basis for $\mathsf{A}^k(\mathcal{F}_2)/(\zeta^* f^* \ker(\iota^*))$. We then show that every basis element pulls back to a distinct basis element in $\mathsf{A}^k(\mathsf{Ell})$ under r^*. These bases are obtained from a computer calculation, using the relations found in [CK] and in Lemmas 5, 7, and 10, as well as the fact that

$$\mathsf{A}^k(\mathcal{F}_2) = \mathsf{A}^k_G(Y \smallsetminus \mathsf{CTP}) \oplus \mathsf{A}^{k-1}(\mathsf{Ell})\,.$$

For example, in codimension 7, we have a basis for

$$\mathsf{A}^7(\mathcal{F}_2) = \mathsf{A}^7_G(Y \smallsetminus \mathsf{CTP}) \oplus \mathsf{A}^6(\mathsf{Ell})$$

is given by

$$\{H^7, H^5\, c_2, H^4 c_3, H^3\, c_2^2, H^2\, c_2 c_3, H\, c_2^3, H\, c_3^2, c_2^2\, c_3, [\mathsf{Ell}]\, \lambda^6, [\mathsf{Ell}]\, \lambda^4\, c_2, [\mathsf{Ell}]\, \lambda^2\, c_2^2, [\mathsf{Ell}]\, c_2^3\}\,,$$

where the first 8 basis elements come from $\mathsf{A}^7_G(Y \smallsetminus \mathsf{CTP})$ and the latter 4 come from $\mathsf{A}^6(\mathsf{Ell})$. Thus, a basis for $\mathsf{A}^7(\mathcal{F}_2)/(\zeta^* f^* \ker(\iota^*))$ is given by

$$\{[\mathsf{Ell}]\, \lambda^6, [\mathsf{Ell}]\, \lambda^4 c_2, [\mathsf{Ell}]\, \lambda^2 c_2^2, [\mathsf{Ell}]\, c_2^3\}\,.$$

Under r^*, $[\mathsf{Ell}]$ maps to a non-zero multiple of λ by the self-intersection formula, λ on $\mathcal{F}_2$ restricts to λ on Ell, and c_2 on $\mathcal{F}_2$ restricts on Ell to a non-zero multiple of the class denoted by c_2 in [CK]. A basis for $\mathsf{A}^7(\mathsf{Ell})$ is given by

$$\{\lambda^7, \lambda^5 c_2, \lambda^3 c_2^2, \lambda c_2^3\},$$

so by inspection the basis elements for $\mathsf{A}^7(\mathcal{F}_2)/(\zeta^* f^* \ker(\iota^*))$ map to nonzero multiples of distinct basis elements for $\mathsf{A}^7(\mathsf{Ell})$. We repeat this argument in every codimension $0 < k < 17$, completing the proof. □

8. The cycle map

We present here the proof of Theorem 1(iv). Throughout Section 8, H_* will denote rational Borel–Moore homology [BM]. In general, for any scheme or Deligne-Mumford stack M, the group $H_k(M)$ carries a mixed Hodge structure and an increasing weight filtration with weights between $-k$ and 0. The cycle map takes values in the lowest weight piece of the Hodge structure

$$\mathsf{cl} : \mathsf{A}_k(M) \to W_{-2k} H_{2k}(M)\,.$$

If M is nonsingular, we identify cohomology and Borel–Moore homology, but singular spaces will also enter the discussion.

We seek to show that the cycle map

$$\mathsf{cl} : \mathsf{A}_k(\mathcal{F}_2) \to H_{2k}(\mathcal{F}_2)$$

is an isomorphism. Using the expressions for the Poincaré polynomial calculated in [KL2] and Appendix A together with the Chow Betti numbers from Theorem 1 (iii), it suffices to prove that the cycle map is injective. We will prove below the following related injectivity.

Lemma 15. *The cycle map*

$$\mathsf{cl} : \mathsf{A}^G_k(Y \smallsetminus \mathsf{CTP}) \to W_{-2k} H^G_{2k}(Y \smallsetminus \mathsf{CTP})$$

is injective.

Assuming Lemma 15 for now, let $\alpha \in \mathsf{A}_k(\mathcal{F}_2)$ be so that $\mathsf{cl}(\alpha) = 0$. We wish to show $\alpha = 0$. If α is not in the kernel of the intersection pairing in $\mathsf{A}^*(\mathcal{F}_2)$, we can find a class α' of complementary degree so that $\alpha \cdot \alpha' \neq 0$. In particular, we may assume $\alpha \cdot \alpha' = \lambda^{17}$ since the latter generates $\mathsf{A}^{17}(\mathcal{F}_2)$. Then,

$$0 = \mathsf{cl}(\alpha) \cdot \mathsf{cl}(\alpha') = \mathsf{cl}(\lambda)^{17}.$$

However, the same argument used in Chow in [GK] shows that in cohomology we also have $\lambda^{17} \neq 0$, yielding a contradiction.

Thus α must be in the kernel of the intersection paring. In particular $2 < k < 19$. By Proposition 14, we can write

$$\alpha = \zeta^* \beta, \quad \beta \in \langle H, c_3 \rangle\,.$$

Here, we recall that $\zeta : \mathcal{F}_2 \hookrightarrow [\widetilde{Y}/G]$ with complement $\widetilde{\mathsf{DC}}$ and $\widetilde{\mathsf{CTP}}$, and β is a Chow class on $[\widetilde{Y}/G]$.

Consider the diagram

$$\begin{array}{ccccccc}
\mathsf{A}_k^G(\widetilde{\mathsf{CTP}}) & \xrightarrow{\eta_*} & \mathsf{A}_k^G(\widetilde{Y}) & \xrightarrow{\zeta^*} & \mathsf{A}_k^G(\widetilde{Y} \smallsetminus (\widetilde{\mathsf{CTP}} \cup \widetilde{\mathsf{DC}})) & \longrightarrow & 0 \\
\downarrow \mathsf{cl} & & \downarrow \mathsf{cl} & & \downarrow \mathsf{cl} & & \\
W_{-2k}H_{2k}^G(\widetilde{\mathsf{CTP}}) & \xrightarrow{\eta_*} & W_{-2k}H_{2k}^G(\widetilde{Y}) & \xrightarrow{\zeta^*} & W_{-2k}H_{2k}^G(\widetilde{Y} \smallsetminus (\widetilde{\mathsf{CTP}} \cup \widetilde{\mathsf{DC}})) & \longrightarrow & 0\,.
\end{array}$$

For the second excision sequence, exactness to the right follows since we keep track of the Hodge weights. For both exact sequences, we may ignore the two dimensional set $\widetilde{\mathsf{DC}}$ on the left terms for dimension reasons for $2 < k < 19$. Since

$$\zeta^* \left(\mathsf{cl}(\beta)\right) = \mathsf{cl}\left(\zeta^*(\beta)\right) = \mathsf{cl}(\alpha) = 0\,,$$

it follows that over $\widetilde{Y}$, we have

$$\mathsf{cl}(\beta) = \eta_* \left(\gamma\right)$$

where γ is an equivariant Borel–Moore homology class on the locus $\widetilde{\mathsf{CTP}}$. Using the blowdown map $f : \widetilde{Y} \to Y$, we obtain

$$f_* \,\mathsf{cl}(\beta) = f_*\, \eta_* \left(\gamma\right),$$

where the right hand side is a Borel–Moore class on CTP. The restriction $f_* \,\mathsf{cl}(\beta) = \mathsf{cl}\left(f_*\beta\right)$ thus vanishes in the Borel–Moore homology of $[(Y \smallsetminus \mathsf{CTP})\,/G]$, so by Lemma 15, we conclude

$$f_* \left(\beta\right) = 0$$

in $\mathsf{A}_k^G(Y \smallsetminus \mathsf{CTP})$. By excision, we can find a class δ such that on Y we have

$$f_* \left(\beta\right) = \overline{\eta}_* \left(\delta\right),$$

where $\overline{\eta} : \mathsf{CTP} \cap Y \hookrightarrow Y$. In particular

$$f_*(\beta - f^*\overline{\eta}_* \left(\delta\right)) = 0$$

in $\mathsf{A}_k^G(Y)$, hence

$$\beta - f^*\overline{\eta}_* \left(\delta\right)$$

is a class supported on the exceptional divisor of the blowup $f : \widetilde{Y} \to Y$ by excision applied to the embedding of the exceptional divisor in $\widetilde{Y}$. Restrict the class $\beta - f^*\overline{\eta}_* \left(\delta\right)$ to $\mathcal{F}_2$ via ζ, and note that $f^*\overline{\eta}_* \left(\delta\right)$ restricts trivially since we removed the strict transform $\widetilde{\mathsf{CTP}}$. We conclude that

$$\alpha = r_* \left(\epsilon\right),$$

for a class ϵ on Ell, where as usual $r : \mathsf{Ell} \to \mathcal{F}_2$ denotes the inclusion. We claim however that in this case α cannot be in the kernel of the intersection pairing unless $\alpha = 0$.

To see this last statement, recall from [CK] that $\mathsf{A}^*(\mathsf{Ell})$ is Gorenstein. If $\epsilon \neq 0$, we can find a complementary class ϵ' with

$$\epsilon \cdot \epsilon' = \lambda^{16}\,.$$

The pullback

$$r^* : \mathsf{A}^*(\mathcal{F}_2) \to \mathsf{A}^*(\mathsf{Ell})$$

is surjective, since two of the ring generators λ, c_2 on the left hand side are sent to the ring generators λ, $4c_2$ on the right hand side. Thus, we may write

$$\epsilon' = r^* \xi \, .$$

Since α is in the kernel of the pairing, we have

$$0 = \alpha \cdot \xi = r_* (\epsilon) \cdot \xi = r_* (\epsilon \cdot r^* \xi) = r_* (\epsilon \cdot \epsilon') = r_* (\lambda^{16}) \, .$$

This contradicts Remark 13.

Proof of Lemma 15. For simplicity, write

$$Z = \mathsf{ML} \cup \mathsf{QP} \cup \mathsf{CTP} \subset X$$

where as before X denotes the projective space of sextics. Then $Y \smallsetminus \mathsf{CTP} = X \smallsetminus Z$. We need to establish the injectivity of the map

$$\mathsf{cl} : \mathsf{A}_k^G(X \smallsetminus Z) \to W_{-2k} H_{2k}^G(X \smallsetminus Z) \, .$$

Consider the following excision diagram

$$\begin{array}{ccccccc} \mathsf{A}_k^G(Z) & \longrightarrow & \mathsf{A}_k^G(X) & \longrightarrow & \mathsf{A}_k^G(X \smallsetminus Z) & \longrightarrow & 0 \\ \downarrow \mathsf{cl} & & \downarrow \mathsf{cl} & & \downarrow \mathsf{cl} & & \\ W_{-2k} H_{2k}^G(Z) & \longrightarrow & W_{-2k} H_{2k}^G(X) & \longrightarrow & W_{-2k} H_{2k}^G(X \smallsetminus Z) & \longrightarrow & 0 \, . \end{array} \tag{14}$$

We first claim that the middle cycle map is an isomorphism. Indeed, recall that $G = \mathrm{SL}(V)$ and let $K = \mathrm{GL}(V)$. We have an isomorphism

$$\mathsf{A}_k^K(X) \to H_{2k}^K(X) \, .$$

This follows by explicitly computing both sides. In fact, both sides agree with the cohomology of the bundle

$$X_K = \mathbb{P}(\mathrm{Sym}^6 E^*) \to BK \, ,$$

where $E \to \mathrm{BGL}(V)$ is the universal bundle. To go further, we use the terminology of [To, Section 4]. There, two properties are singled out: the weak property is the statement that the cycle map is an isomorphism, while the strong property requires additional assumptions about odd cohomology, which vanishes for X_K. In other words, $\mathbb{P}(\mathrm{Sym}^6 E^*) \to BK$ satisfies the strong property. To pass to the group G, we note that the mixed space $X_G \to X_K$ is a $\mathbb{C}^*$-bundle obtained from the total space of the determinant line bundle $F = \det \mathrm{pr}^* \mathrm{Sym}^6 E^*$ on $\mathbb{P}(\mathrm{Sym}^6 E^*)$ and removing the zero section. By homotopy equivalence, F also satisfies the strong property since X_K does, and the zero section satisfies it as well. The complement satisfies the weak property by [To, Lemma 6], as claimed.

To show the rightmost cycle map is injective in the diagram (14), it suffices to show the leftmost cycle map is surjective. For simplicity, write

$$Z_1 = \mathsf{ML}\,, \quad Z_2 = \mathsf{QP}\,, \quad Z_3 = \mathsf{CTP}$$

for the three components of Z, and write T_1, T_2, T_3 for the nonsingular spaces that dominate them

$$T_1 = \mathbb{P}(V^*) \times \mathbb{P}(\mathrm{Sym}^4 V^*)\,, \quad T_2 = \mathbb{P}K_{\mathrm{quad}}\,, \quad T_3 = \mathbb{P}K_{\mathrm{ctp}}\,.$$

By Mayer–Vietoris in both Chow [F, Example 1.3.1(c)] and Borel–Moore homology [BM, Theorem 3.10] and [PS, Theorem 5.35 and Remark 5.36], we have a diagram

$$\begin{array}{ccccc} \mathsf{A}_k^G(Z_1) \oplus \mathsf{A}_k^G(Z_2) \oplus \mathsf{A}_k^G(Z_3) & \longrightarrow & \mathsf{A}_k^G(Z) & \longrightarrow & 0 \\ \downarrow{\scriptstyle \mathsf{cl}} & & \downarrow{\scriptstyle \mathsf{cl}} & & \\ W_{-2k}H_{2k}^G(Z_1) \oplus W_{-2k}H_{2k}^G(Z_2) \oplus W_{-2k}H_{2k}^G(Z_3) & \longrightarrow & W_{-2k}H_{2k}^G(Z) & \longrightarrow & 0\,. \end{array}$$

The surjectivity of the second row follows since the $(2k-1)$st Borel–Moore homology groups has no Hodge pieces of weight $-2k$. Therefore, to complete the proof we need to check the surjectivity of the cycle map on the left.

For each $1 \le i \le 3$, we form the diagram

$$\begin{array}{ccccc} \mathsf{A}_k^G(T_i) & \longrightarrow & \mathsf{A}_k^G(Z_i) & \longrightarrow & 0 \\ \downarrow{\scriptstyle \mathsf{cl}} & & \downarrow{\scriptstyle \mathsf{cl}} & & \\ W_{-2k}H_{2k}^G(T_i) & \longrightarrow & W_{-2k}H_{2k}^G(Z_i) & \longrightarrow & 0\,. \end{array}$$

Surjectivity of the first row is standard (and, in fact, is not necessary for us), while surjectivity of the second row is found in [Le, Lemma A.4] or [Pe2]. The final step is then to prove that the cycle map on the left is surjective. The left cycle map is an isomorphism by the same argument used for X using the explicit description of T_1, T_2, T_3 as iterated projective bundles over projective spaces. $\square$

Appendix A. The Poincaré polynomial of the moduli space

A.1. **The results of Kirwan and Lee.** We discuss here the calculation of the Poincaré polynomial of $\mathcal{F}_2$ in [KL1, KL2]. The value of the Poincaré polynomial given in [KL2, Theorem 3.1] is

$$1 + 2q^2 + 3q^4 + 5q^6 + 6q^8 + 8q^{10} + 10q^{12} + 12q^{14} + 13q^{16} + 14q^{18} + 12q^{20} + 10q^{22} + 8q^{24} + 6q^{26} + q^{27} + 5q^{28} + 3q^{30} + q^{31} + 2q^{32} + q^{33} + 3q^{35}\,. \tag{15}$$

However, the above polynomial is incompatible with the geometry of the moduli space. Indeed, the projective Bailey-Borel compactification

$$\mathcal{F}_2 \hookrightarrow \overline{\mathcal{F}}^{\mathrm{BB}}$$

has a 1-dimensional boundary. Using this observation, it was shown in [GK] that

$$\lambda^{17} \neq 0 \in H^{34}(\mathcal{F}_2). \tag{16}$$

In fact, intersecting two general hyperplane sections of $\overline{\mathcal{F}}^{\mathrm{BB}}$ gives a compact 17-dimensional subvariety of $\mathcal{F}_2$ on which λ^{17} is non-zero by the ampleness of λ. However, this contradicts the vanishing $H^{34}(\mathcal{F}_2) = 0$ implied by (15).

The value of the Poincaré polynomial used throughout our paper is

$$\begin{aligned} &1 + 2q^2 + 3q^4 + 5q^6 + 6q^8 + 8q^{10} + 10q^{12} + 12q^{14} + 13q^{16} + 14q^{18} + 12q^{20} \\ &\quad + 10q^{22} + 8q^{24} + 6q^{26} + q^{27} + 5q^{28} + 3q^{30} + q^{31} + 2q^{32} + 2q^{33} + q^{34} + 3q^{35}\,, \end{aligned} \tag{17}$$

which differs from (15) by $q^{33} + q^{34}$. The correction is aligned with the non-vanishing (16) of cohomology in degree 34.

The main error in [KL2] is in the proof, but not the statement, of Proposition 3.2. First, in [KL2, Lemma 5.6], Kirwan and Lee claim to describe the image of a certain map τ_2^*, but actually only describe a proper subspace of the image. This impacts the proof, but again not the statement, of [KL2, Lemma 5.7]. The inaccurate claim in the proof is used on [KL2, page 581] to study the kernel of another map χ^4, ultimately leading to the erroneous Poincaré polynomial (15).

In Section A.4, we explain how to derive the correct Poincaré polynomial (17) using the statement of [KL2, Proposition 3.2] together with the non-vanishing (16). The latter fact was not used by Kirwan-Lee. In order to explain the issues regarding the proof of [KL2, Proposition 3.2], a lengthier discussion of Kirwan-Lee's beautiful but intricate argument is required. A correct derivation is explained in Section A.5 after we describe the geometric set-up and a few intermediate results in Sections A.2 and A.3.

A.2. **Kirwan's desingularization.** The approach in [KL1, KL2] starts with the GIT quotient of the space of sextics

$$\overline{\mathcal{F}}^{\mathrm{GIT}} = X /\!\!/ G\,,$$

where $X = \mathbb{P}^{27}$ and $G = \mathrm{SL}_3$. Kirwan's partial desingularization

$$\overline{\mathcal{F}}^{\mathrm{K}} = \widetilde{X} /\!\!/ G$$

arises as a composition of four (weighted) blowups. It is obtained by first blowing up X^{ss} along the orbits whose stabilizers have the highest dimension, deleting the unstable strata in the blowup, and then repeating the same procedure to the resulting space. The partial desingularization $\overline{\mathcal{F}}^{\mathrm{K}}$ possesses only finite quotient singularities, whereas the singularities of $\overline{\mathcal{F}}^{\mathrm{GIT}}$ are more complicated. One of the main results of [KL1] is the calculation of the Betti numbers of $\overline{\mathcal{F}}^{\mathrm{K}}$:

$$\begin{aligned} &1 + 5q^2 + 11q^4 + 18q^6 + 25q^8 + 32q^{10} + 40q^{12} + 48q^{14} + 55q^{16} + 60q^{18} + 60q^{20} + 55q^{22} \\ &\quad + 48q^{24} + 40q^{26} + 32q^{28} + 25q^{30} + 18q^{32} + 11q^{34} + 5q^{36} + q^{38}\,. \end{aligned} \tag{18}$$

This is used in [KL2] to compute the Betti numbers of $\mathcal{F}_2$, viewing the latter as an open in $\overline{\mathcal{F}}^{\mathrm{K}}$.[3]

While the Chow groups of $\overline{\mathcal{F}}^{\mathrm{K}}$ are not needed for our paper, in the spirit of Section 8, we expect that the cycle map

$$\mathsf{A}^*(\overline{\mathcal{F}}^{\mathrm{K}}) \to H^{2*}(\overline{\mathcal{F}}^{\mathrm{K}})$$

is an isomorphism.

A.3. **Shah's compactification.** The first step of the desingularization procedure yields Shah's compactification

$$\overline{\mathcal{F}}^{\mathrm{Sh}} = X_1 /\!\!/ G\,.$$

Here X_1 is the weighted blowup of the triple conic locus TC in the locus of semistable sextics:

$$\pi : X_1 \to X^{ss}\,.$$

Indeed, TC is the orbit with the largest stabilizer, namely $R_0 = \mathrm{SO}_3 = \mathrm{PSL}_2$.

Three further (unweighted) blowups are necessary to arrive at $\overline{\mathcal{F}}^{\mathrm{K}}$, see [KL1, page 504]. The first of the remaining three blowups has as center the orbit $G\Delta$, where the reducible sextic $\Delta = (xyz)^2$ is invariant under the maximal torus R_1 in G. The final two blowups have as centers the orbits $G\widehat{Z}^{ss}_{R_2}$ and $G\widehat{Z}^{ss}_{R_3}$, where $\widehat{Z}^{ss}_{R_2}$, $\widehat{Z}^{ss}_{R_3}$ are the loci of semistable points fixed by two specific rank 1 tori R_2, R_3:

$$R_2 = \mathrm{diag}\,\langle \lambda^{-2}, \lambda, \lambda\rangle, \quad R_3 = \mathrm{diag}\,\langle \lambda, \lambda^{-1}, 1\rangle, \quad \lambda \in \mathbb{C}^*\,.$$

In fact, the locus $G\widehat{Z}^{ss}_{R_3}$ (which will be relevant below) lies over the locus of products of three conics tangent at 2 points [KL1, Section 5.3].

There is a blowdown map

$$\overline{\mathcal{F}}^{\mathrm{K}} \to \overline{\mathcal{F}}^{\mathrm{Sh}}\,,$$

which is an isomorphism over the stable locus X_1^s/G of the Shah space. The moduli space

$$\mathcal{F}_2 \hookrightarrow X_1^s/G$$

[3] There are a few minor typos in the proof of [KL1, Theorem 1.3]. On the table in [KL1, page 499], the locus labelled $(1,0)$ corresponds to the stratum of unstable sextics of the form $\ell^5 m$ where ℓ, m are distinct lines. This stratum contributes

$$\frac{q^{46}}{(1-q^2)^2}\,.$$

As a result, formula [KL1, Section 2.4, (1)] should read

$$\frac{1-q^{50}}{(1-q^2)(1-q^4)(1-q^6)} - \frac{q^{20}-q^{28}}{(1-q^2)^3}\,.$$

Similarly, formula [KL1, Section 4.2, (2)] should be

$$\frac{q^{22}-q^{42}}{(1-q^2)^2}\,.$$

Additionally, there is a misprint in [KL1, Section 5.2, (1)] which should read

$$\frac{1+q^2}{(1-q^2)^2}(q^{16}+q^{32}-2q^{38})\,.$$

There are a few other small misprints but they do not affect the general argument.

is obtained by removing the union of a line and a surface

$$Z = (A \setminus \Delta) \cup (B \setminus (B \cap D))\,.$$

The locus A is a projective line passing through the point $\Delta = (xyz)^2$ and corresponds to the double cubic locus in X. The surface B corresponds to the double conic + conic locus, and $B \cap D$ is a curve in B.

The Poincaré polynomial of X_1^s/G is computed in [KL2, Proposition 3.2]:

$$\begin{aligned}(19)\quad & 1 + 2q^2 + 3q^4 + 5q^6 + 6q^8 + 8q^{10} + 10q^{12} + 12q^{14} + 13q^{16} + 14q^{18} + 12q^{20} \\ & \qquad + 10q^{22} + 8q^{24} + 6q^{26} + q^{27} + 5q^{28} + 3q^{30} + q^{31} + 2q^{32} + q^{33} + q^{34} + q^{35}.\end{aligned}$$

This calculation is very important for the overall argument.[4]

A.4. **The Poincaré polynomial of $\mathcal{F}_2$.** We confirm the Poincaré polynomial (17) relying on equation (19). Just as on [KL2, page 580], we use the relative homology sequence for the pair $(X_1^s/G, \mathcal{F}_2)$. The difference is that we take into account that $H^{34}(\mathcal{F}_2) \neq 0$, thus leading to the different result (17).

First, we note the Gysin isomorphism

$$H_i(X_1^s/G, \mathcal{F}_2) = H_c^{38-i}(Z)\,.$$

In fact, we have

$$H_c^0(Z) = 0\,, \quad H_c^2(Z) = \mathbb{Q} \oplus \mathbb{Q}\,, \quad H_c^4(Z) = \mathbb{Q}\,,$$

see [KL2, (6.3)]. The relative homology sequence

$$\ldots \to H_i(\mathcal{F}_2) \to H_i(X_1^s/G) \to H_c^{38-i}(Z) \to H_{i-1}(\mathcal{F}_2) \to \ldots$$

immediately yields isomorphisms

$$H_i(\mathcal{F}_2) = H_i(X_1^s/G), \quad i \leq 32, \quad i = 37, \quad i = 38\,.$$

Expressions (15), (17), (19) all agree in degrees ≤ 32 and $i = 37$, $i = 38$. Furthermore,

$$0 \to H_{36}(\mathcal{F}_2) \to H_{36}(X_1^s/G) \to \mathbb{Q} \oplus \mathbb{Q} \to H_{35}(\mathcal{F}_2) \to H_{35}(X_1^s/G) \to 0\,.$$

Using (19), we have

$$H_{36}(X_1^s/G) = 0\,, \quad H_{35}(X_1^s/G) = \mathbb{Q}\,.$$

Therefore

$$H_{36}(\mathcal{F}_2) = 0\,, \quad H_{35}(\mathcal{F}_2) = \mathbb{Q} \oplus \mathbb{Q} \oplus \mathbb{Q}\,,$$

also in agreement with both (15) and (17).

[4] The intersection homology of the Shah compactification was computed in [KL1, Theorem 1.2]. On general grounds, see [K3, Remark 3.4], the Betti numbers of X_1^s/G agree with the intersection homology Betti numbers in degree less than roughly the dimension (up to a correction dictated by the unstable strata). In our case, this confirms the Poincaré polynomial of X_1^s/G in degrees ≤ 16. However, the remaining Betti numbers cannot be immediately derived from [KL1].

However, discrepancies appear in degrees 33 and 34. We have

$$0 \to H_{34}(\mathcal{F}_2) \to H_{34}(X_1^s/G) \to \mathbb{Q} \to H_{33}(\mathcal{F}_2) \to H_{33}(X_1^s/G) \to 0\,.$$

Using $H_{34}(X_1^s/G) = \mathbb{Q}$ from (19) and the fact that $H_{34}(\mathcal{F}_2) \neq 0$ as noted above, it follows that the first map must be an isomorphism and

$$H_{34}(\mathcal{F}_2) = \mathbb{Q}\,.$$

Using (19) one more time, we have $H_{33}(X_1^s/G) = \mathbb{Q}$, hence

$$H_{33}(\mathcal{F}_2) = \mathbb{Q} \oplus \mathbb{Q}\,.$$

This confirms equation (17).

A.5. **Further discussion.** To give further credence to (17), we also identify the faulty reasoning in [KL2]. To this end, we need to zoom in on the argument. We will explain that while (19) records the correct Poincaré polynomial of X_1^s/G, there are some errors in the derivation. The necessary corrections impact the last page of [KL2], and thus the final result.

The strategy used to establish (19) is as follows:

(i) a lower bound on the Betti numbers of X_1^s/G is obtained from the Poincaré polynomial of $\overline{\mathcal{F}}^{\mathrm{K}}$ in (18) together with the relative homology sequence for the pair

$$X_1^s/G \hookrightarrow \overline{\mathcal{F}}^{\mathrm{K}}\,.$$

The resulting lower bounds for Betti numbers of X_1^s/G are recorded in [KL2, (3.11), (3.12)].

(ii) Matching upper bounds are obtained in [KL2, Sections 4, 5]. The outcome is [KL2, Corollary 5.11].

In fact, steps (i) and (ii) are only carried out in degrees less or equal than 23, while the higher terms are determined in [KL2, Section 6].

Step (i) requires the calculation of the Poincaré polynomial of the complement[5] [KL2, Section 3]:

$$Q = \overline{\mathcal{F}}^{\mathrm{K}} \setminus X_1^s/G\,.$$

Step (ii) examines the kernel of the restriction map

$$\chi^* : H^*(\overline{\mathcal{F}}^{\mathrm{K}}) \to H^*(Q)\,.$$

This kernel is identified with the kernel of the restriction

$$\rho^* : H^*(\overline{\mathcal{F}}^{\mathrm{K}}) \to H^*(\widehat{E}_1 /\!\!/ G) \oplus H^*(E_2 /\!\!/ G) \oplus H^*(E_3 /\!\!/ G)\,.$$

Here, $\widehat{E}_1 /\!\!/ G$ is the strict transform of the exceptional divisor $E_1 /\!\!/ G$ of the second blowup (at Δ), and $E_2 /\!\!/ G$ and $E_3 /\!\!/ G$ are the exceptional divisors of the third and fourth blowup [KL2, page 567].

[5]The top term of the Poincaré polynomial of Q in [KL2, (3.8)] should be $3q^{36}$, taking into account the correction $3q^{30}$ versus $3q^{20}$ in [KL2, (3.5)] and using the correct sign for the contribution of $E_{T,2} /\!\!/ G$. Similarly, there is a misprint in the first formula in [KL2, page 569] which requires the coefficient 6 for q^{26}.

The correct identification of the kernel in codimension 4 is needed in [KL2, Section 6] to determine the Betti numbers of $\mathcal{F}_2$ in high degrees.

Before reviewing the analysis of the kernel of ρ^* in [KL2, Sections 4, 5], we need a few standard preliminaries. Consider the general setting of a G-equivariant blowup

$$p : \widetilde{M} \to M$$

of a nonsingular quasiprojective M along a nonsingular equivariant center N of codimension c, with exceptional divisor E. Note that the natural sequence

$$0 \to H^*_G(M) \to H^*_G(\widetilde{M}) \to H^*_G(E)/H^*_G(N) \to 0 \tag{20}$$

induces an additive identification [KL1, page 505]:

$$H^*_G(\widetilde{M}) = p^* H^*_G(M) \oplus H^*_G(E)/H^*_G(N). \tag{21}$$

Furthermore, $H^*_G(E)/H^*_G(N)$ has the additive basis

$$\zeta^k \cdot p^*\alpha, \quad 1 \le k \le c-1\,,$$

for classes α giving a basis of $H^*_G(N)$, and ζ denoting the hyperplane class of the projective bundle $E \to N$. For (21), the splitting

$$H^*_G(E)/H^*_G(N) \to H^*_G(\widetilde{M})$$

of the natural restriction map in (20) is not explicitly stated in [KL1]. However, a splitting can be specified on the additive basis

$$\zeta^k \cdot p^*\alpha \mapsto E^{k-1} \cdot j_!(p^*\alpha), \quad 1 \le k \le c-1, \tag{22}$$

with $j_!$ denoting the Gysin map for the closed immersion $E \to \widetilde{M}$. This convention is standard and is used for instance in [K3, page 495].

As an approximation of ρ, one constructs spaces dominating the cohomology groups of the domain and target of ρ. For the domain, the space $\overline{\mathcal{F}}^{\mathrm{K}}$ arises as a 4-step blowup, and each blowup contributes to cohomology via (21). Thus, the cohomology of $\overline{\mathcal{F}}^{\mathrm{K}}$ has 5 natural pieces, yielding generators[6]

$$p^* : F_1^* \oplus F_2^* \oplus F_3^* \oplus F_4^* \oplus F_5^* \twoheadrightarrow H^*(\overline{\mathcal{F}}^{\mathrm{K}}). \tag{23}$$

Similarly, there is a surjection

$$q^* : G_1^* \oplus G_2^* \oplus G_3^* \twoheadrightarrow H^*(\widehat{E}_1 /\!\!/ G) \oplus H^*(E_2 /\!\!/ G) \oplus H^*(E_3 /\!\!/ G). \tag{24}$$

The interested reader can consult [KL2, Section 4] for a more detailed discussion and notation. There is an induced map on generators

$$\sigma^* : F_1^* \oplus F_2^* \oplus F_3^* \oplus F_4^* \oplus F_5^* \to G_1^* \oplus G_2^* \oplus G_3^*\,,$$

which is an approximation of ρ^*. The kernel of σ^* is calculated first.

[6] This uses Kirwan surjectivity [K1]; we only obtain generators after the unstable loci are deleted.

By [KL2, page 576], the kernel of σ^* consists of pairs

$$(a,b) \in F_1^* \oplus F_2^*, \quad \tau_1^*(a) + \tau_2^*(b) = 0, \quad \tau_4^*(a) = 0, \quad \tau_6^*(a) = 0\,.$$

Here,

$$F_1^* = H^*(X) \otimes H^*(\mathrm{BSL}_3), \quad F_2^* = \widetilde{H}^*(\mathbb{P}^{21}) \otimes H^*(\mathrm{BSO}_3)\,.$$

The space F_1^* is the equivariant cohomology of the space of plane sextics. Additively, F_2^* can be identified with the equivariant cohomology of the exceptional divisor of the first blowup, modulo the equivariant cohomology of the center of the blowup. Indeed, the codimension of the triple conic orbit is $27 - 5 = 22$, and the normalizer is SO_3. The maps

$$\tau_1^* : F_1^* \to G_3^*, \quad \tau_2^* : F_2^* \to G_3^*$$

are introduced in [KL2, page 571], while τ_4^*, τ_6^* are constructed in [KL2, pages 572–573]. The target of τ_1^* and τ_2^* is the equivariant cohomology of the last exceptional divisor $G_3^* = H_G^*(E_3)$. However, the discussion of [KL2, (4.6)] shows that these maps factor through the equivariant cohomology $\widetilde{G}_3^* = H_G^*(G\widehat{Z}_{R_3}^{ss})$ of the center of the last blowup, followed by pullback:

$$\tau_1^* : F_1^* \to \widetilde{G}_3^*, \quad \tau_2^* : F_2^* \to \widetilde{G}_3^*\,.$$

A key step is to show

$$\tau_1^*(a) + \tau_2^*(b) = \tau_4^*(a) = \tau_6^*(a) = 0 \implies \tau_1^*(a) = \tau_2^*(b) = 0, \tag{25}$$

see [KL2, page 576]. In turn, this relies on [KL2, Lemma 5.6] which specifies the image of τ_2^*.

It is important to understand the map τ_2^*. This map arises from blowing up the codimension 18 orbit $G\widehat{Z}_{R_3}^{ss}$ after all the other blowups have been carried out. To explain the notation, $\widehat{Z}_{R_3}^{ss}$ consists of the semistable points fixed by the torus

$$R_3 = \mathrm{diag}\,\langle \lambda, \lambda^{-1}, 1\rangle \subset G = \mathrm{SL}_3\ .$$

This blowup is described in [KL1, Section 5.3]. It is noted in [KL2, page 570] that the cohomology of the center of the blowup is

$$\widetilde{G}_3^* = H_G^*(G\widehat{Z}_{R_3}^{ss}) \cong H_{N(R_3)}^*(\widehat{Z}_{R_3}^{ss}) \cong H^*(\widehat{Z}_{R_3}^{ss} /\!\!/ N(R_3)) \otimes H^*(BN_2)\,.$$

Here, $N(R_3)$ is the normalizer of R_3 in G inducing a residual action on $\widehat{Z}_{R_3}^{ss}$, and N_2 is the normalizer of the maximal torus in SL_2. The first isomorphism is a general fact which follows from [K2, Corollary 5.6], while the second isomorphism is explained in [KL1, Section 5.3].

The map τ_2^* can be described as taking classes on the exceptional divisor of the first blowup, viewing them as classes on the first blowup under the construction (22), then restricting to $\widehat{Z}_{R_3}^{ss}$, while switching from G-equivariance to $N(R_3)$-equivariance [KL2, page 571]. Now recall that the surface $\widehat{Z}_{R_3}^{ss} /\!\!/ N(R_3)$ carries an exceptional divisor θ obtained by blowing up the triple conic δ in $Z_{R_3}^{ss} /\!\!/ N(R_3)$, see [KL2, page 576]. Lemma 5.6 in [KL2] states that

$$\mathrm{Im}\ \tau_2^* \subset \widetilde{G}_3^* = H^*(\widehat{Z}_{R_3}^{ss} /\!\!/ N(R_3)) \otimes H^*(BN_2)$$

equals $\{\theta\} \otimes H^*(BN_2)$. This is incorrect, and it should be replaced by classes supported on θ, not the class of θ itself. This comes from the fact that in (22) self-intersections of the exceptional divisor also arise; see also (27) below. As a result, the conclusion

$$\tau_1^*(a) = \tau_4^*(a) = \tau_6^*(a) = \tau_2^*(b) = 0$$

in [KL2, page 576] does not hold.

However, the strategy of the argument is sound, and [KL2, Lemma 5.10] can be salvaged. We will give the details below. We first consider the equations

$$\tau_4^*(a) = \tau_6^*(a) = 0\,,$$

where we may assume a has degree ≤ 23, since the lemma only concerns such degrees. By [KL2, Lemma 5.4], the intersection $\mathrm{Ker}\ \tau_4^* \cap \mathrm{Ker}\ \tau_6^*$ in the polynomial ring

$$F_1^* = H^*(\mathbb{P}^{27}) \otimes H^*(\mathrm{BSL}_3) = \mathbb{Q}[H, c_2, c_3]/(H^{28})$$

is generated by two classes of degrees 4 and 14, namely

$$H^2 \ \text{ and } \ \alpha = H \cdot (4c_2^3 + 27c_3^2)\,.$$

We can explicitly list all classes in the intersection of the two kernels of τ_4^* and τ_6^*. The ranks of $\mathrm{Ker}\ \tau_4^* \cap \mathrm{Ker}\ \tau_6^*$ are given in each degree by

$$q^4 + q^6 + 2q^8 + 3q^{10} + 4q^{12} + 6q^{14} + 7q^{16} + 9q^{18} + 11q^{20} + 13q^{22}\,,$$

in agreement with [KL2, page 577]. For instance, in degree 4 (the simplest case), we have the unique class H^2. In degree 20 (the most involved case), we have the 11 classes

$$H^{10}, H^8c_2, H^7c_3, H^6c_2^2, H^5c_2c_3, H^4c_3^2, H^4c_2^3, H^3c_2^2c_3, H^2c_2^4, H^2c_2c_3^2, \alpha c_3\,,$$

and the classes a of degree 20 lie in the span of these terms. Similarly,

$$b \in F_2^* = \widetilde{H}^*(\mathbb{P}^{21}) \otimes H^*(\mathrm{BSO}_3)\,.$$

Let ζ be the hyperplane class on the first exceptional divisor. The ranks of F_2^* in degrees ≤ 23 are immediately calculated to be

$$q^2 + q^4 + 2q^6 + 2q^8 + 3q^{10} + 3q^{12} + 4q^{14} + 4q^{16} + 5q^{18} + 5q^{20} + 6q^{22}\,.$$

For example, in degree 4, we have the class ζ^2. In degree 20, we have the 5 classes

$$\zeta^{10},\ \zeta^8c_2,\ \zeta^6c_2^2,\ \zeta^4c_2^3,\ \zeta^2c_2^4\,,$$

and the classes b of degree 20 lie in the span of these terms. By the proof of [KL2, Lemma 5.4], the homomorphism

$$\tau_1^* : F_1^* \to \widetilde{G}_3^* = H^*(\widehat{Z}_{R_3}^{ss} /\!\!/ N(R_3)) \otimes H^*(BN_2)$$

is given by

(26) $$\tau_1^*(H) = C \otimes 1, \quad \tau_1^*(c_2) = -1 \otimes \xi + n([\mathrm{pt}] \otimes 1), \quad \tau_1^*(c_3) = C' \otimes \xi,$$

where n is an integer, ξ is the degree 4 generator of $H^*(BN_2)$, and C, C' are curves on the surface $\widehat{Z}^{ss}_{R_3} /\!\!/ N(R_3)$. In fact, by construction $C^2 = 1$. Similarly,

$$\tau_2^*(\zeta) = \theta \otimes 1, \quad \tau_2^*(c_2) = -1 \otimes \xi \implies \tau_2^*(\zeta^2) = -[\mathrm{pt}] \otimes 1, \quad \tau_2^*(\zeta^k) = 0, \quad k \geq 3. \tag{27}$$

The last vanishing can be seen using the construction (22) and the fact that we restrict to a surface $\widehat{Z}^{ss}_{R_3} /\!\!/ N(R_3)$. Now we can solve (25) in each degree using (26) and (27). For instance, in degree 4, we need

$$a \cdot \tau_1^*(H^2) + b \cdot \tau_2^*(\zeta^2) = 0 \implies a - b = 0$$

so the kernel is spanned by $H^2 + \zeta^2$. In degree 20, we write

$$a = a_1 \cdot H^{10} + a_2 \cdot H^8 c_2 + a_3 \cdot H^7 c_3 + a_4 \cdot H^6 c_2^2 + a_5 \cdot H^5 c_2 c_3 + a_6 \cdot H^4 c_3^2 + a_7 \cdot H^4 c_2^3$$
$$+ a_8 \cdot H^3 c_2^2 c_3 + a_9 \cdot H^2 c_2^4 + a_{10} \cdot H^2 c_2 c_3^2 + a_{11} \cdot \alpha c_3 ,$$
$$b = b_1 \cdot \zeta^{10} + b_2 \cdot \zeta^8 c_2 + b_3 \cdot \zeta^6 c_2^2 + b_4 \cdot \zeta^4 c_2^3 + b_5 \cdot \zeta^2 c_2^4 .$$

From here, using (26), (27), we find

$$\tau_1^*(a) + \tau_2^*(b) = 0 \iff a_9 - 4a_{11} \cdot (C.C') - b_5 = 0 .$$

This yields a 15-dimensional solution space. After solving (25) in each degree, we find the ranks of Ker σ^* in degree ≤ 23 are accurately recorded by [KL2, Lemma 5.7]:

$$q^4 + 2q^6 + 3q^8 + 5q^{10} + 6q^{12} + 8q^{14} + 10q^{16} + 12q^{18} + 15q^{20} + 17q^{22} .$$

However, the description of the kernel of σ^* in degree 4 needs to be corrected. As already mentioned, this impacts the argument in [KL2, page 581].

At this stage, thanks to [KL2, Lemma 5.7], we have complete knowledge of the kernel of σ^* in degree ≤ 23. The next results [KL2, 5.9 - 5.11] concern Ker ρ^* which is required in part (ii) above. No correction to the statements in [KL2] is needed here. However, the derivation of [KL2, Lemma 5.10] crucially uses [KL2, (5.1)]. This derivation requires a few modifications to the values in [KL2]. Up to order 23, we have:

$$\begin{aligned} \text{Ker } p^* &= q^{16} + 5q^{18} + 14q^{20} + 28q^{22} , \\ \text{Ker } q_{11}^* &= q^{18} + 3q^{20} + 6q^{22} , \\ \text{Ker } q_2^* &= q^{16} + 3q^{18} + 5q^{20} + 8q^{22} , \end{aligned} \tag{28}$$

while

$$\text{Ker } q_3^* = q^{18} + 5q^{20} + 10q^{22}$$

is correct in [KL2]. Here, p^* is introduced in (23), and q_{11}^*, q_2^*, q_3^* are certain components of the morphism (24). Thus, using [KL2, (5.9)], the expression

$$\text{Ker } q_{11}^* + \text{Ker } q_2^* + \text{Ker } q_3^* + \text{Ker } \sigma^* - \text{Ker } p^*$$

yields the upper bound for Ker ρ^* = Ker χ^* in [KL2, Lemma 5.10] to be

$$q^4 + 2q^6 + 3q^8 + 5q^{10} + 6q^{12} + 8q^{14} + 10q^{16} + 12q^{18} + 14q^{20} + 13q^{22} .$$

This completes step (ii), and also confirms [KL2, Proposition 3.2] and equation (19) along with it.

The method of computing of the ranks of Ker p^* and Ker q_2^* is described in [KL2, (5.1)], but the details are suppressed and the results are recorded imprecisely. For instance, Ker p^* receives the following 6 contributions:

- from the domain of p^*, the term $F_1^* = H^*(\mathbb{P}^{27}) \otimes H^*(\mathrm{BSL}_3)$ contributes
$$\frac{1-q^{56}}{1-q^2} \cdot \frac{1}{(1-q^4)(1-q^6)};$$
- next, $F_2^* = \widetilde{H}^*(\mathbb{P}^{21}) \otimes H^*(\mathrm{BSO}_3)$ contributes
$$\frac{q^2-q^{44}}{1-q^2} \cdot \frac{1}{1-q^4};$$
- the remaining pieces of the domain of p^* are found in [KL2, page 571]. We have $F_3^* = \widetilde{H}^*(\mathbb{P}^{20}) \otimes H^*(BN)$ which contributes
$$\frac{q^2-q^{42}}{1-q^2} \cdot \frac{1}{(1-q^4)(1-q^6)}.$$
Here N is the normalizer of the maximal torus R_1 in G (this is denoted N_3 in [KL2]);
- similarly for $F_4^* = H^*(\widehat{Z}_{R_2} /\!\!/ N(R_2)) \otimes H^*(B\mathbb{C}^*) \otimes \widetilde{H}^*(\mathbb{P}^{18})$ we get the contribution
$$(1+q^2) \cdot \frac{1}{1-q^2} \cdot \frac{q^2-q^{38}}{1-q^2}.$$
The first term is computed in [KL1, Section 5.1];
- for $F_5^* = H^*(\widehat{Z}_{R_3} /\!\!/ N(R_3)) \otimes H^*(BN_2) \otimes \widetilde{H}^*(\mathbb{P}^{17})$ we get
$$(1+3q^2+q^4) \cdot \frac{1}{1-q^4} \cdot \frac{q^2-q^{36}}{1-q^2}.$$
The first term was computed in [KL1, Section 5.3];
- for the target of p^*, the contribution of $H^*(\overline{\mathcal{F}}^{\mathrm{K}})$ is recorded in (18).

Putting these contributions together, we find that Ker p^* is given by (28), as claimed. The discrepancy with the value in [KL2, (5.1)] is $3q^{22} \mod q^{24}$.

Next, we examine $q_2^* : G_2^* \twoheadrightarrow H^*(E_2 /\!\!/ G)$. The dimension of the target is recorded in [KL2, (3.5)]:
$$(1+q^2)(1+2q^2+3q^4+4q^6+5q^8+6q^{10}+7q^{12}+8q^{14}+8q^{16}+8q^{18}+8q^{20}+7q^{22}),$$
up to order 23. By [KL2, (4.3)], the domain is
$$G_2^* = H^*(\widehat{Z}_{R_2} /\!\!/ N(R_2)) \otimes H^*(B\mathbb{C}^*) \otimes H^*(\mathbb{P}^{18}),$$
whose contribution equals
$$(1+q^2) \cdot \frac{1}{1-q^2} \cdot \frac{1-q^{38}}{1-q^2}.$$
Subtracting the two series above, we find the dimension of Ker q_2^* matching the last equation in (28). The value recorded [KL2, (5.1)] is different. The misprint likely originates with [KL1, Section 5.2].

Finally, we consider the map

$$q_1^* : G_1^* \to H^*(\widehat{E}_1 /\!\!/ G)$$

discussed in [KL2, (4.4)]. Here, $\widehat{E}_1$ is a blowup of E_1^{ss} described in [KL2, page 570]. As noted in [KL2, page 578], the map q_1^* has three components q_{11}^*, q_{12}^*, q_{13}^*, where

$$q_{11}^* : H_G^*(E_1) \to H^*(\widehat{E}_1 /\!\!/ G)\,.$$

Since $\widehat{E}_1$ has no strictly semistable points, we have

$$H^*(\widehat{E}_1 /\!\!/ G) = H_G^*(\widehat{E}_1^{ss})\,,$$

see also [KL2, page 570]. We factor q_{11}^* as the composition

$$H_G^*(E_1) \xrightarrow{f^*} H_G^*(E_1^{ss}) \xrightarrow{g^*} H_G^*(\widehat{E}_1) \xrightarrow{h^*} H_G^*(\widehat{E}_1^{ss})\,.$$

The first map f^* is surjective on general grounds [K1]. We will compute its kernel below. The middle map g^* is a pullback induced by a blowup so it is injective. The third map h^* removes unstable strata from the blowup $\widehat{E}_1$ to arrive at $\widehat{E}_1^{ss}$.

The calculation of the kernel of the surjection $f^* : H_G^*(E_1) \to H_G^*(E_1^{ss})$ is a matter of recording dimensions.

(a) For the domain, we have $H_G^*(E_1) = H^*(BN) \otimes H^*(\mathbb{P}^{20})$ [KL2, page 570]. The first factor comes from center of the blowup using the isomorphism $G\Delta = G/N$. We recall that N stands for the normalizer of the maximal torus in G. The projective space $\Sigma = \mathbb{P}^{20}$ corresponds to the projectivization of the normal bundle at Δ of the orbit $G\Delta$ in X. This contributes

$$\frac{1}{(1-q^4)(1-q^6)} \cdot \frac{1-q^{42}}{1-q^2}\,.$$

(b) For the target, on general grounds we have

$$H_G^*(E_1^{ss}) = H_G^*(G \times_N (\mathbb{P}^{20})^{ss}) = H_N^*((\mathbb{P}^{20})^{\mathrm{ss}})\,.$$

The N-equivariant Poincaré series of $(\mathbb{P}^{20})^{ss}$ was calculated in [KL1, Section 6.5], equation (1). Expanding up to order 23 we find

$$1 + q^2 + 2q^4 + 3q^6 + 4q^8 + 5q^{10} + 7q^{12} + 8q^{14} + 10q^{16} + 11q^{18} + 11q^{20} + 10q^{22}\,.$$

The kernel of f^* up to order 23 is determined from here by subtracting the two expressions (a) and (b). The answer reproduces the value claimed on the second line of (28).

Since g^* is injective, we have

$$\mathrm{Ker}\ g^* \circ f^* = \mathrm{Ker}\ f^*\,.$$

We furthermore claim this agrees with the kernel of $q_{11}^* = h^* \circ g^* \circ f^*$. To this end, it suffices to show

$$\mathrm{Ker}\ h^* \cap \mathrm{Im}\ g^* = 0. \tag{29}$$

Indeed, we need to rule out the situation that classes supported on unstable strata of $\widehat{E}_1$ might equal a class pulled back from E_1^{ss}. Should this happen, removing the unstable stratum from $\widehat{E}_1$

will also kill additional classes on E_1^{ss}, thus increasing the kernel of q_{11}^* when compared to the kernel of f^*. The discussion might have been clear to the authors of [KL1, KL2] and it was not recorded explicitly, but we indicate here a possible argument.

Let $\Sigma = \mathbb{P}^{20}$. By the above remarks (a) and (b), the maps f^*, g^* and h^* can be rewritten as

$$H_N^*(\Sigma) \xrightarrow{f^*} H_N^*(\Sigma^{ss}) \xrightarrow{g^*} H_N^*(\widehat{\Sigma}) \xrightarrow{h^*} H_N^*(\widehat{\Sigma}^{ss})$$

where $\widehat{\Sigma}$ is the blowup of Σ^{ss} along the N-orbits of the R_2-fixed and R_3-fixed loci. It is explained in [KL1, Section 4.3] that

$$\Sigma = \mathbb{P}(W) = \mathbb{P}^{20}$$

where W is the subspace of sextics spanned by the 21 monomials $x^i y^j z^{6-i-j}$ for $i, j \geq 0$, $i + j \leq 6$ and

$$(i,j) \notin \{(2,2),(2,1),(2,3),(3,1),(3,2),(1,3),(1,2)\}\,.$$

Using the terminology of [KL1, page 508], these monomials are obtained from the "Hilbert diagram" in [KL1, page 497] by removing the middle hexagon.

Recalling $R_2 = \mathrm{diag}\,\langle \lambda^{-2}, \lambda, \lambda\rangle$, it follows that the R_2-fixed locus is the projective line

$$\widehat{Z}_{R_2} = \mathbb{P}\langle x^2y^4, x^2z^4\rangle\,.$$

Let Q_2 is the normalizer of R_2 in N, which is easily computed to be isomorphic to the normalizer of the maximal torus in GL_2. We have $Q_2/R_2 = \mathbb{C}^* \rtimes \mathbb{Z}/2\mathbb{Z}$. It is easy to see that R_2 acts trivially on $\widehat{Z}^{ss}_{R_2} = \mathbb{P}^1$, and the $\mathbb{C}^*$-factor of Q_2/R_2 acts with equal opposite weights. On general grounds $\widehat{Z}^{ss}_{R_2}$ consists in the Q_2-semistable points of $\widehat{Z}_{R_2}$, see [K2, Remark 5.5]. It follows that the unstable points are x^2y^4 and x^2z^4, so $\widehat{Z}^{ss}_{R_2} = \mathbb{C}^*$ Thus, the equivariant cohomology of the orbit is

$$H_N^*(N\widehat{Z}^{ss}_{R_2}) = H^*_{Q_2}(\widehat{Z}^{ss}_{R_2}) = H^*(BR_2) = H^*(B\mathbb{C}^*)\,.$$

This is in agreement with [KL2, page 570].

We consider the blowup of Σ^{ss} along the orbit of $\widehat{Z}^{ss}_{R_2}$. The exceptional divisor F of the blowup is a $\mathbb{P}^{18}$-bundle over the base. We need to identify the unstable locus in the exceptional divisor. The weights of the representation of R_2 on $\mathbb{P}^{18}$ can be lifted from [KL1, Section 5.2]. Up to an overall factor of -3, they are $-2, -1, 1, 2, 3, 4$ with multiplicities $7, 4, 2, 3, 2, 1$. Thus, in suitable coordinates, the action is given by

$$\lambda \cdot [x : y : z : w : t : s] = [\lambda^{-2}x : \lambda^{-1}y : \lambda z : \lambda^2 w : \lambda^3 t : \lambda^4 s]\,,$$

where

$$(x, y, z, w, t, s) \in \mathbb{C}^7 \oplus \mathbb{C}^4 \oplus \mathbb{C}^2 \oplus \mathbb{C}^3 \oplus \mathbb{C}^2 \oplus \mathbb{C}$$

are not all zero. The unstable locus is easily seen to be the union

$$\mathbb{P}^{10} \sqcup \mathbb{P}^7\,,$$

corresponding to $z = w = t = s = 0$ and $x = y = 0$ respectively. This conclusion is in agreement with the Poincaré polynomial calculation in [KL1, Section 5.2 (1)]. Letting ϵ denote the equivariant parameter for R_2, and letting H denote the hyperplane class on $\mathbb{P}^{18}$, we compute the R_2-equivariant classes

$$\left[\mathbb{P}^{10}\right] = (H+\epsilon)^2(H+2\epsilon)^3(H+3\epsilon)^2(H+4\epsilon), \quad \left[\mathbb{P}^{7}\right] = (H-2\epsilon)^7(H-\epsilon)^4 .$$

Similar expressions hold for the classes of all equivariant linear subspaces of $\mathbb{P}^{10}$ or $\mathbb{P}^7$: the monomials in H and ϵ above will have different exponents. By inspection, nonzero combinations of such classes never come from the base of the blowup $H^*(BR_2) = \mathbb{Q}[\epsilon]$ by pullback. This is the key to establishing (29).

To this end, the reader may find the following diagram useful:

$$\begin{array}{ccccccc}
 & & & & H^*_N(\widehat{S}) & \xrightarrow{\overline{j}^*} & H^*_N(\widehat{S}\cap F) \\
 & & & & \big\downarrow{\scriptstyle i_!} & & \big\downarrow{\scriptstyle \overline{i}_!} \\
0 & \longrightarrow & H^*_N(\Sigma^{ss}) & \xrightarrow{g^*} & H^*_N(\widehat{\Sigma}) & \xrightarrow{j^*} & H^*_N(F)\,. \\
 & & & \searrow & \big\downarrow{\scriptstyle h^*} & & \\
 & & & & H^*_N(\widehat{\Sigma}^{ss}) & &
\end{array}$$

Here, $\widehat{S}$ is the unstable locus of $\widehat{\Sigma}$. The three-term column of the diagram is the Gysin sequence for the closed subvariety $\widehat{S} \subset \widehat{\Sigma}$. For the first term, the cohomology is shifted by codimension, but the notation does not indicate this explicitly. In fact, $\widehat{S}$ is not pure dimensional, the individual connected components need to be considered separately.

On general grounds [K1], the unstable locus admits a stratification by locally closed nonsingular subvarieties

$$\widehat{S} = \bigsqcup_{\beta} \widehat{S}_\beta\,.$$

The intersection

$$\widehat{S}\cap F = \bigsqcup_{\beta} (\widehat{S}_\beta \cap F)$$

is the union of unstable strata of the exceptional divisor F, see the proof of [K2, Proposition 7.4]. The inclusion $\overline{j}$ induces an isomorphism in cohomology

$$\overline{j}^* : H^*_N(\widehat{S}) \to H^*_N(\widehat{S}\cap F)\,.$$

Indeed, it is shown in the proof of [K2, Proposition 7.4] that for each individual stratum, the inclusion induces an isomorphism

$$\overline{j}^*_\beta : H^*_N(\widehat{S}_\beta) \to H^*_N(\widehat{S}_\beta \cap F)\,.$$

Comparing the spectral sequence of the two stratifications of $\widehat{S}$ and $\widehat{S}\cap F$ (or equivalently by comparing the Gysin sequences induced by adding the unstable strata one at a time), we conclude the same is true about the map $\overline{j}^*$.

By the above discussion and [K2, Lemma 7.8], we see that

$$\widehat{S}\cap F \to N\widehat{Z}_{R_2}$$

is a $\mathbb{P}^{10} \sqcup \mathbb{P}^7$-fibration contained in the $\mathbb{P}^{18}$-fibration $F \to N\widehat{Z}_{R_2}$. To establish (29), let

$$\alpha \in H_N^*(\widehat{\Sigma}), \quad \alpha \in \text{Ker } h^* \cap \text{Im } g^* .$$

Then, from the third-term column of the diagram, we have

$$\alpha = i_!(\gamma), \quad \gamma \in H_N^*(\widehat{S}) .$$

We compute

$$j^*\alpha = j^*i_!(\gamma) = \overline{i}_!\overline{j}^*(\gamma) .$$

The class $j^*\alpha$ must come from the base of the $\mathbb{P}^{18}$-fibration $F \to N\widehat{Z}_{R_2}$, since α is in the image of g^*. However, the argument in the paragraphs above shows that classes $\overline{j}^*(\gamma)$ supported on the unstable part $\widehat{S} \cap F$ do not come from the base, unless of course

$$\overline{j}^*(\gamma) = 0 .$$

Using that $\overline{j}^*$ is an isomorphism, we must have $\gamma = 0$, hence $\alpha = i_!(\gamma) = 0$ as claimed by (29).

A similar analysis applies to $R_3 = \text{diag } \langle \lambda, \lambda^{-1}, 1 \rangle$, so

$$\widehat{Z}_{R_3} = \mathbb{P}\langle x^3y^3, xyz^4, z^6 \rangle .$$

The blowup is a $\mathbb{P}^{17}$-bundle over the base, and the representation of R_3 is computed in [KL1, Section 5.4]. The unstable locus is similarly a projective bundle over the base. An analogous argument applies in this case as well.

References

[AO] V. Arena, S. Obinna, *The integral Chow ring of weighted blowups*, preprint `arXiv:2307.01459`.

[BHPV] W. Barth, K. Hulek, C. Peters, A. Van de Ven, *Compact Complex Surfaces*, Ergeb. Math. Grenzgeb., Vol. 4, Springer-Verlag, Berlin, 2004.

[BLMM] N. Bergeron, Z. Li, J. Millson, C. Moeglin, *The Noether-Lefschetz conjecture and generalizations*, Invent. Math. 208 (2017), 501 - 552.

[BM] A. Borel, J. Moore, *Homology theory for locally compact spaces*, Michigan Math. J. 7 (1960), 137 - 159.

[B] J. Bruinier, *On the rank of Picard groups of modular varieties attached to orthogonal groups,* Compositio Math. (2002), 49 - 63.

[CK] S. Canning, B. Kong, *The Chow rings of moduli spaces of elliptic surfaces over* $\mathbb{P}^1$, Algebraic Geometry, 10 (2023), 489 – 503.

[CL] S. Canning, H. Larson, *Chow rings of low-degree Hurwitz spaces*, J. Reine Angew. Math. 789 (2022), 103 - 152.

[COP1] S. Canning, D. Oprea, R. Pandharipande, *Degree2K3*, Github repository, available at `https://github.com/src2165/Degree2K3`.

[COP2] S. Canning, D. Oprea, R. Pandharipande, *Nontautological cycles on* $\mathcal{A}_g$, in preparation. Lecture available at `https://people.math.ethz.ch/~rahul/AbelianBerlin.pdf`.

[D] I. V. Dolgachev, *Weighted Projective Varieties*, Group Actions and Vector Fields (Vancouver, B.C., 1981), Lecture Notes in Math. 956, Springer-Verlag (1982), 34 - 71.

[EG] D. Edidin, W. Graham, *Equivariant intersection theory*, Invent. Math. 131 (1998), 595 - 634.

[Fa] C. Faber, *A conjectural description of the tautological ring of the moduli space of curves*, Moduli of curves and abelian varieties, Aspects Math., E33 (1999), 109 - 129.

[F] W. Fulton, *Intersection theory*, Springer-Verlag, Berlin, 1998.

[GK] G. van der Geer, T. Katsura, *Note on tautological classes of moduli of K3 surfaces*, Mosc. Math. J. 5 (2005), 775 - 779.

[GP] T. Graber, R. Pandharipande, *Constructions of nontautological classes on the moduli spaces of curves*, Michigan Math. J. 51 (2003), 93 - 109.

[GS] D. R. Grayson, M. E. Stillman, *Macaulay2, a software system for research in algebraic geometry*, available at `http://www.math.uiuc.edu/Macaulay2/`.

[GSSEC] D. R. Grayson, M. E. Stillman, S. A. Strømme, D. Eisenbud, C. Crissman, *Schubert2: characteristic classes for varieties without equations*, Version 0.7, Macaulay2 package available at `http://www.math.uiuc.edu/Macaulay2/`.

[GLT] F. Greer, Z. Li, Z. Tian, *Picard groups on moduli of K3 surfaces with Mukai models*, Int. Math. Res. Not. 16 (2015), 7238 - 7257.

[K1] F. Kirwan, *Cohomology of quotients in symplectic and algebraic geometry*, Mathematical Notes, Princeton University Press, Princeton, NJ, 1984.

[K2] F. Kirwan, *Partial desingularizations of quotients of nonsingular varieties and their Betti numbers*, Ann. of Math. 122 (1985), 41 - 85.

[K3] F. Kirwan, *Rational intersection cohomology of quotient varieties*, Invent. Math. 86 (1986), 471 - 505.

[KL1] F. Kirwan, R. Lee, *The cohomology of moduli spaces of K3 surfaces of degree 2. I.*, Topology 28 (1989), 495 - 516.

[KL2] F. Kirwan, R. Lee, *The cohomology of moduli spaces of K3 surfaces of degree 2. II.*, Proc. London Math. Soc. 58 (1989), 559 - 582.

[Kr] A. Kresch, *Cycle groups for Artin stacks*, Invent. Math. 138 (1999), 495 - 536.

[La] R. Laza, *The KSBA compactification for the moduli space of degree two K3 pairs*, J. Eur. Math. Soc. 18 (2016), 225 - 279.

[LOG] R. Laza, K. O'Grady, *Birational geometry of the moduli space of quartic K3 surfaces*, Compositio Math. 155 (2019), 1655 - 1710.

[Le] J. Lewis, *A Survey of the Hodge Conjecture*, CRM Monogr. Ser., 10 American Mathematical Society, Providence, RI, 1999.

[Lo] E. Looijenga, *New compactifications of locally symmetric varieties*, Proceedings of the 1984 Vancouver conference in algebraic geometry, CMS Conf. Proc., Amer. Math. Soc., 1986, 341 - 364.

[LFV] A. Di Lorenzo, R. Fringuelli, A. Vistoli, *Integral Picard group of moduli of polarized K3 surfaces*, preprint `arXiv:2305.07574`.

[MOP] A. Marian, D. Oprea, R. Pandharipande, *Segre classes and Hilbert schemes of points*, Ann. Sci. Éc. Norm. Supér. 50 (2017), 239 - 267.

[MP] D. Maulik, R. Pandharipande, *Gromov-Witten theory and Noether-Lefschetz theory*, in *A celebration of algebraic geometry*, 469 - 507, Clay Math. Proc., 18, Amer. Math. Soc., Providence, RI, 2013.

[Ma] A. Mayer, *Families of K-3 Surfaces*, Nagoya Math. J. 48 (1972), 1 - 17.

[M] R. Miranda, *The moduli of Weierstrass fibrations over* $\mathbf{P}^1$, Math. Ann. 255 (1981), 375 - 394.

[OG] K. O'Grady, *On the Picard group of the moduli space for* $K3$ *surfaces*, Duke Math J. 53 (1986), 117 - 124.

[P] R. Pandharipande, *A calculus for the moduli space of curves*, Algebraic geometry: Salt Lake City 2015, 459 - 487, Proc. Sympos. Pure Math., Amer. Math. Soc., Providence, RI, 2018.

[PY] R. Pandharipande, Q. Yin, *Relations in the tautological ring of the moduli space of K3 surfaces*, J. Eur. Math. Soc. 22 (2020), 213 - 252.

[PS] C. Peters, J. Steenbrink, *Mixed Hodge Structures*, Ergeb. Math. Grenzgeb., Vol 52, Springer–Verlag, Berlin, 2008.

[Pe1] D. Petersen, *A vanishing result for tautological classes on the moduli of K3 surfaces*, Amer. J. Math. 141 (2019), 733 - 736.

[Pe2] D. Petersen, answer to `https://mathoverflow.net/questions/142858/`.

[S] J. Shah, *A complete moduli space for K3 surfaces of degree 2*, Ann. of Math. 112 (1980), 485 - 510.

[T] A. Thompson, *Degenerations of K3 surfaces of degree two*, Trans. Amer. Math. Soc. 366 (2014), 219 - 243.

[To] B. Totaro, *Chow groups, Chow cohomology and linear varieties*, Forum of Mathematics, Sigma 2 (2014), Paper No. e17.

[V] A. Vistoli, *Intersection theory on algebraic stacks and on their moduli spaces*, Invent. Math. 97 (1989), 613-670.

[Z] J. van Zelm, *Nontautological bielliptic cycles*, Pacific J. Math., (2018), 495 - 504.

Department of Mathematics, ETH Zürich
Email address: `samir.canning@math.ethz.ch`

Department of Mathematics, University of California, San Diego
Email address: `doprea@math.ucsd.edu`

Department of Mathematics, ETH Zürich
Email address: `rahul@math.ethz.ch`