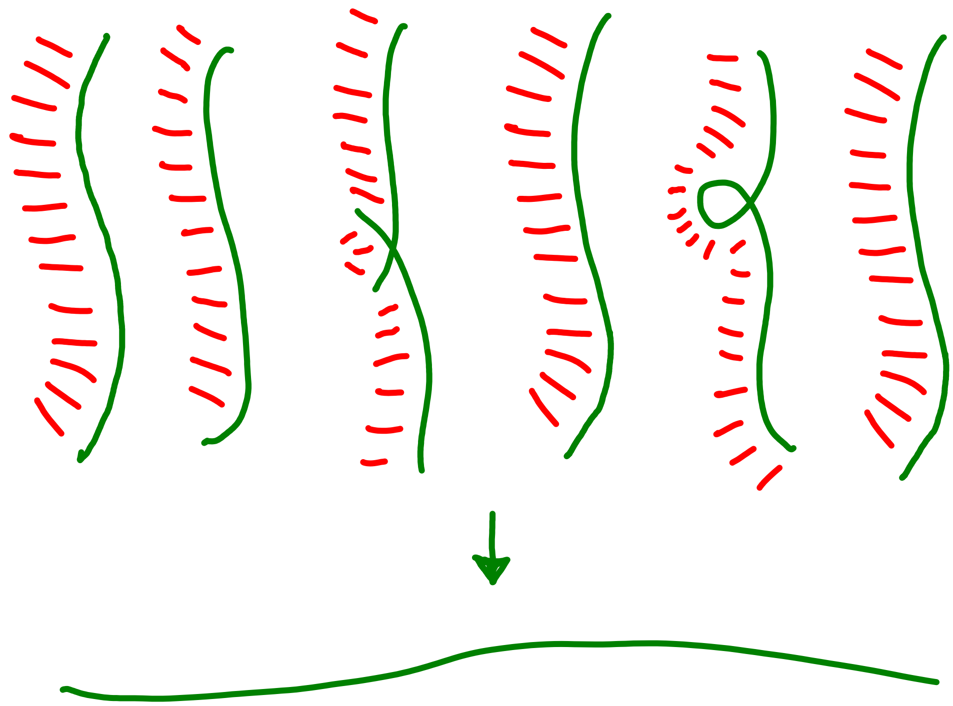
Picard CohFTs



R. Pandharipande

Columbia Math Physics

15 December 2020

Revised after the lecture

[0] Introduction / Disclaimer

I will continue here a conversation started with Andrei in 2018 at MSRI

In the meantime there were some related developments in my group

- Relations on the moduli spaces $\overline{\mathcal{M}}_g(X, \beta)$ — Bae 2019, Bae - Lho 2020

- Pixton's formula on the Picard Stack — BHPSS 2020

These led me to view the Picard stack as a space we can really work with.

More recently: studying R-actions with Dimitri Zvonkine

Warning: mostly speculative and likely not completely correct

[I] Three types of moduli spaces and three types of CohFTs :

(i) $\bar{\mathcal{M}}_{g,n}$ Deligne-Mumford stable curves

$2g-2+n>0$

classical case

Even Case

Well-defined notion of a CohFT with unit :

- $(V, \eta, 1)$ $1 \in V$ distinguished element

finite dim $\mathbb{Q}$-vector space

Scalars often extended

nondegenerate symmetric 2-form on V

$\boxed{\text{Deligne Mumford CohFT}}$

- $\Omega_{g,n} \in H^*(\overline{\mathcal{M}}_{g,n}) \otimes (V^*)^n$

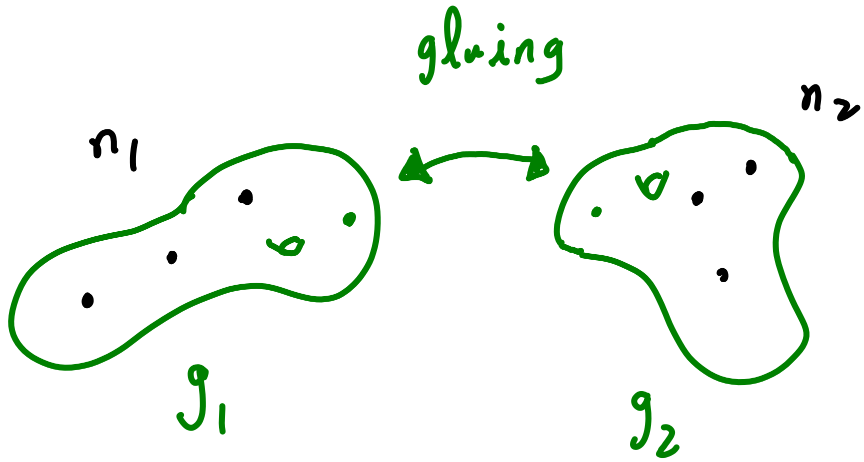
Symmetric with respect to Σ_n action

CohFT Axioms:

— Splitting axioms for

$$\overline{\mathcal{M}}_{g-1,n+2} \xrightarrow{\iota} \overline{\mathcal{M}}_{g,n},$$

$$\overline{\mathcal{M}}_{g_1,n_1+1} \times \overline{\mathcal{M}}_{g_2,n_2+1} \xrightarrow{\iota} \overline{\mathcal{M}}_{g,n},$$



$$\iota^* \Omega_{g,n}(v_1, \ldots, v_n)$$

$$\|$$

$$\sum_{i,j} \Omega_{g_1, n_1+1}(v_1, \ldots, v_{n_1}, e_i)\, g^{ij}\, \Omega_{g_2, n_2+1}(e_j, v_{n_1+1}, \ldots, v_n)$$

$g_{ij} := \eta(e_i, e_j)$

g^{ij} inverse

– Unit axiom for

$$\overline{\mathcal{M}}_{g,n+1} \xrightarrow{p} \overline{\mathcal{M}}_{g,n}$$

involves contraction

$$\Omega_{g,n+1}(v_1, \dots, v_n, 1) = p^* \Omega_{g,n}(v_1, \dots, v_n)$$

— Metric axiom

$$\Omega_{0,3}(v_1, v_2, 1) = \eta(v_1, v_2)$$

$$\cap$$

$$H^*(\bar{M}_{0,3}) \cong \mathbb{Q}$$

(ii) Artin stacks

Suggestions of the theory in Teleman's paper

$$\mathcal{M}_{g,n}$$

pointed, connected, nodal curves, no stability imposed

Notion of an Artin CohFT ?

Various Choices $\to$ H^* of Artin stack

$\to$ CH^* of Artin Stack

Kresch's Theory

$\to$ CH^*_{op} operational Chow

What is $CH^*_{op}(\mathcal{M}_{g,n})$?

A class $\gamma \in CH^k_{op}(\mathcal{M}_{g,n})$

acts on the Chow theory of

every family :

Let
$$\begin{tikzcd} C \arrow[d, "\pi"'] \\ S \arrow[u, bend right, "p_1, \dots, p_n"'] \end{tikzcd}$$
be

a family of genus g curves with n markings. Then

$$S \xrightarrow{f} \mathcal{M}_{g,n}$$

and $f^*(\gamma) : CH_r(S) \to CH_{r-k}(S)$

with a large number of compatibilities, see BHPSS

[Bae - Holmes - P - Schmitt - Schwarz]

An Artin CohFT with unit

consists of

- $(V, \eta, 1)$ $1 \in V$ distinguished element

finite dim $\mathbb{Q}$-Vectorspace

nondegenerate Symmetric 2-form on V

- $\Omega_{g,n} \in CH^*_{op}(\mathcal{M}_{g,n}) \otimes (V^*)^n$

Symmetric with respect to Σ_n action

Artin CohFT Axioms :

formally the same as before

- Splitting axioms for

$$\mathcal{M}_{g-1,n+2} \to \mathcal{M}_{g,n} ,$$

$$\mathcal{M}_{g_1,n_1+1} \times \mathcal{M}_{g_2,n_2+1} \to \mathcal{M}_{g,n} ,$$

- Unit axiom for

$$\mathcal{M}_{g,n+1} \xrightarrow{p} \mathcal{M}_{g,n} ,$$

$$\Omega_{g,n+1}(v_1, \dots, v_n, 1) = p^* \Omega_{g,n}(v_1, \dots, v_n)$$

Appears similar to the DM case

but is different : no contraction

– Metric axiom

$$\Omega_{0,3}(v_1, v_2, 1) = \eta(v_1, v_2)\cdot [\mathcal{M}_{0,3}]$$

can be refined to

$$\Omega_{0,2}(v_1, v_2) = \eta(v_1, v_2)\cdot [\mathcal{M}_{0,2}]$$

QUESTION (Andrei): CAN we consider η more generally in $CH^*_{op}(\mathcal{M}_{0,2})$?

ANS: Maybe

fundamental class $\mathrm{Id} \in CH^*_{op}(\mathcal{M}_{0,2})$

The 3 point metric axioms follows from the stronger 2 point axiom.

A basic difference between $\overline{M}_{g,n}$ and $\mathcal{M}_{g,n}$ is stabilization

An additional property for Artin CohFts:

- Stability:

an Artin CohFt Ω is Stable if the restriction of Ω to DM stable curves via

$$\overline{\mathcal{M}}_{g,n} \longrightarrow \mathfrak{M}_{g,n}$$

yields a DM CohFT.

[The issue is the unit, stability is not important for us today]

Every stable Artin CohFT yields a DM CohFT after pull-back along $\overline{M}_{g,n} \to \mathfrak{M}_{g,n}$

(iii) Picard stacks

$$\mathfrak{P}_{g,n} = \coprod_d \mathfrak{P}^d_{g,n}$$

pointed connected nodal curves with a line bundle and no stability imposed

degree of the line bundle

Notion of a Picard CohFT ?

We will use operational Chow

(but there choices as before).

Later we switch to H^* for classification

New direction (Main point today)

A Picard CohFT with unit

Consists of

- $(V, \eta, 1)$

$1 \in V$ distinguished element

finite dim $\mathbb{Q}$- Vectorspace

nondegenerate Symmetric 2-form on V

- $\Omega_{g,n} \in CH^*_{op}(P_{g,n}) \otimes (V^*)^n$

We can separate the degrees

$$\Omega_{g,n} = \bigoplus_d \Omega^d_{g,n}$$

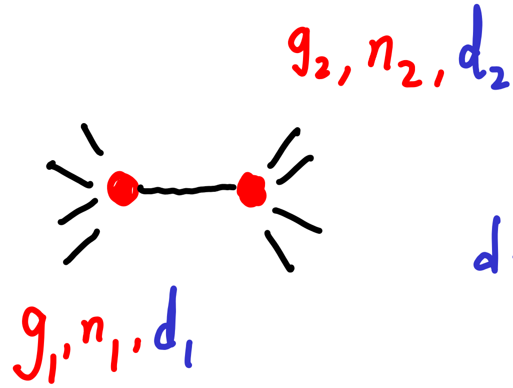
$$\Omega^d_{g,n} = CH^*_{op}(P^d_{g,n}) \otimes (V^*)^n$$

Picard CohFT Axioms :

Splitting axioms should be approached with more care since there are no gluing maps.

To the graph

$$\Gamma(g_1,n_1,d_1 \mid g_2,n_2,d_2) =$$



$d = d_1 + d_2$

We associate an Artin stack Pic_Γ

objects :

$$\begin{tikzcd} & & \mathcal{L} \arrow[dl] \\ & C \arrow[d, shift left] & \\ & S \arrow[u, shift left, "p_1,\dots,p_n"] & \end{tikzcd}$$

line bundle

which are families of pointed curves with line bundles with dual graph

$$\Gamma(g_1,n_1,d_1 \mid g_2,n_2,d_2)$$

or a refinement.

in particular,
there exists
a nodal section
corresponding to
the edge of

$$
\begin{tikzcd}
& & \mathcal{L} \arrow[dl] \\
& C \arrow[d] & \\
& S \arrow[u, bend left=40, "p_1, \dots, p_n"] \arrow[u, bend right=40, "\text{node}"'] &
\end{tikzcd}
$$

$$\Gamma(g_1, n_1, d_1 \mid g_2, n_2, d_2)$$

We have maps

$$
\begin{tikzcd}
& \mathrm{Pic}_\Gamma \arrow[r, "\iota_\Gamma"] \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \mathrm{Pic}^{d}_{g,n} \\
\mathrm{Pic}^{d_1}_{g_1, n_1+1} & & \mathrm{Pic}^{d_2}_{g_2, n_2+1}
\end{tikzcd}
\qquad d = d_1 + d_2
$$

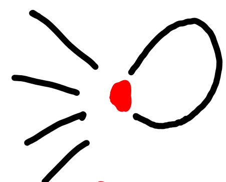
Let e_i be a $\mathbb{Q}$-basis of V

$\eta(e_i, e_j) = g_{ij}$, $\quad g^{ij}$ is the inverse

- Splitting axioms:

$$i_r^* \Omega^{d}_{g,n}(\dots) = \sum_{ij} \pi_1^* \Omega^{d_1}_{g_1,n_1+1}(\dots, e_i)\, g^{ij}\, \pi_2^* \Omega^{d_2}_{g_2,n_2+1}(e_j, \dots)$$

and the parallel splitting for the graph:



g, n

Warning: slightly more care is needed here [the ordering at the node branches]

- Unit axiom is unchanged:

$$p^{d}_{g,n+1} \xrightarrow{p} p^{d}_{g,n}$$

forgetful map, no stabilization

$$\Omega^{d}_{g,n+1}(v_1,\dots,v_n,1) = p^* \Omega^{d}_{g,n}(v_1,\dots,v_n)$$

- metric axiom

$$\Omega^{0}_{0,2}(v_1,v_2) = \eta(v_1,v_2)\cdot[\mathfrak{P}^{0}_{0,2}]$$

fundamental class

$\mathrm{Id} \in CH^*_{op}(\mathfrak{P}^{0}_{0,3})$

- Stability can also be defined in the same way.

For stable Picard theories:

$$\overline{\mathcal{M}}_{g,n} \to \mathfrak{M}_{g,n} \xrightarrow{\text{Trivial bundle}} \mathfrak{P}_{g,n}$$

DM CohFT $\leftarrow$ Artin CohFT $\leftarrow$ Pic CohFT

[II] Action of the Givental group

(i) For DM CohFTs,

the standard case:

$$R(z) \in \mathrm{End}(V)[[z]]$$

$$R(z) = \underset{\text{Id}}{\mathbb{1}} + R_1 z + R_2 z^2 + \cdots$$

$$R(z) \cdot R^*(-z) = \mathbb{1}$$

Symplectic condition

adjoint with respect to η

Such $R(z)$ form a group,

the Givental group.

Theorem (Givental):

Let Ω be a DM CohFT on $(V, \eta, 1)$.

Let $R(z)$ be an element of the Givental group.

then there is new CohFT on $(V, \eta, 1)$ defined by

basic actions

$$R.\Omega = RT\Omega$$

$$T(z) = z \cdot 1 - z R^{-1}(z)(1) \in V[[z]]$$

$\uparrow$ V $\uparrow$ V

(ii) For Artin CohFTs,

Essentially in Teleman

Claim: the R-action holds with few changes:

Let Ω be an Artin CohFT on $(V, \eta, 1)$.

Let $R(z)$ be an element of the Givental group.

then there is new CohFT on $(V, \eta, 1)$ defined by

$$R.\Omega = RT\Omega, \quad T(z) = z \cdot 1 - z R^{-1}(z)(1)$$

(iii) In the Picard case

There is a richer R-action

Class of the universal line, related to the Θ divisor

$$R(z) \in \mathrm{End}(V)[[L]][[z]]$$

$$R(z) = \underset{\uparrow \text{Id}}{1} + R_1 z + R_2 z^2 + \cdots$$

Better: $R_0 = \mathrm{Id} \bmod L$

$R_i \in \mathrm{End}(V)[[L]]$

$$R(z) \cdot R^*(-z) = 1$$

adjoint with respect to η

Claim: There is an R-action on Picard CohFTs*

No unit axiom

Let Ω be a Picard CohFT* on (V, η).

Let $R(z)$ be an element of the richer Givental group.

then there is new CohFT

$$R\Omega$$

on (V, η).

How do we define the action?

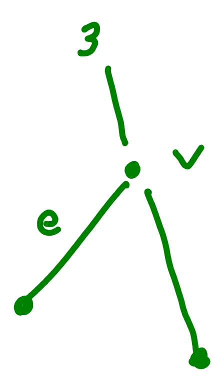
What is L?

$$[R\Omega]_{g,n}(v_1,\dots,v_n) = \sum_{\Gamma \in G^{Pic}_{g,n}} \frac{1}{|Aut(\Gamma)|}\, Cont_\Gamma(v_1,\dots,v_n)$$

in $CH^*_{op}(\mathfrak{P}_{g,n}) \otimes (V^*)^n$

all graph corresponding to strata of $\mathfrak{P}_{g,n}$.

∞-sum, no problem for $CH^*_{op}(\mathfrak{P}_{g,n})$

$\Gamma =$ 

$g(v)$

$n(v)$

$d(v)$

What is $Cont_\Gamma$?

- place $\Omega^{d(v)}_{g(v),n(v)}$ at each vertex v of Γ

- place $R^{-1}(\psi_\ell)v_\ell$ at every leg ℓ of Γ

- at every edge e of Γ, place

$$\frac{\eta^{-1} - R^{-1}(\psi_e')\,\eta^{-1}\,R^{-1}(\psi_e'')^t}{\psi_e' + \psi_e''}$$

The main new point is that

$$R(z) \in \mathrm{End}(V)[[L]][[z]]$$

so there are L's everywhere.

What is L?

Given a half edge of Γ,

L corresponds to

$$c_1(\mathcal{L}_p)$$

where $\mathcal{L}_p \to \overline{\mathcal{M}}_{g,n}$ is the universal line bundle at the marking (or node) associated to the half-edge.

where are the kappas?

We also need the translation T-action.

Let $T = T_1 z + T_2 z^2 + \cdots$

where $T_1 \in V \otimes_{\mathbb{Q}} L\,\mathbb{Q}[[L]]$

$T_{i\geq 2} \in V \otimes_{\mathbb{Q}} \mathbb{Q}[[L]]$

Comment of Bae: Should include $T_0 = L^2 + \ldots$

Ans: I agree, but Zvonkine says makes actions subtler, but we are considering the form

Let Ω be a Picard CohFT* on (V, η).

No unit axiom

Claim: There is a T-action on Picard CohFTs*

$$\mathrm{T}\Omega_{g,n}(v_1, \dots, v_n) =$$

there are L's here

$$\sum_{m \geq 0} \frac{1}{m!} (\pi_m)_\star \Omega_{g,n+m}(v_1, \dots, v_n, \mathrm{T}(\psi_{n+1}), \dots, \mathrm{T}(\psi_{n+m}))$$

∞-sum, no problem

requires explanation for $m \geq 1$

$$\begin{array}{c} \mathcal{L} \\ \swarrow \\ \mathcal{C}_{g,n} \\ \downarrow \\ \mathcal{P}_{g,n} \end{array}$$

universal bundle

universal curve

Define [$m \geq 1$]

$$\pi_m : \mathcal{P}_{g,n,m} \longrightarrow \mathcal{P}_{g,n}$$

Constructed by adding m stable points to the universal curve $\mathcal{C}_{g,n}$

[Note $\mathcal{P}_{g,n,m} \neq \mathcal{P}_{g,n+m}$]

Since
$$\underbrace{\mathcal{P}_{g,n,m} \to}_{\text{Contraction Map}} \underbrace{\mathcal{C}_{g,n} \times \cdots \times \mathcal{C}_{g,n}}_{m \text{ factors}},$$

Products over $\mathcal{P}_{g,n}$

$\mathcal{L}_j \to \mathcal{P}_{g,n,m}$ is defined for each marking $1 \leq j \leq m$

by pulling back $\mathcal{L} \to \mathcal{C}_{g,n}$ from the corresponding factor

$$T(\psi_{n+j}) = \cdots + L^a \psi_{n+j}^b + \cdots$$

We set $L \to c_1(\mathcal{L}_j)$ so

$$T(\psi_{n+j}) = \cdots + c_1(\mathcal{L}_j)^a \psi_{n+j}^b + \cdots$$

push down to richer K classes

for example

$$\begin{tikzcd} & \mathcal{L} \arrow[dl] \\ \mathcal{C}_{g,n} \arrow[d, "\pi"'] & \\ \mathcal{M}_{g,n} & \end{tikzcd}$$

universal bundle

universal curve

$$\pi_*\left(c_1(\mathcal{L})^2\right) \overset{\text{def}}{=} H \in CH^1_{op}(\mathcal{M}_{g,n})$$

[III] Semisimplicity

Let Ω be a Picard CohFT* on (V, η).

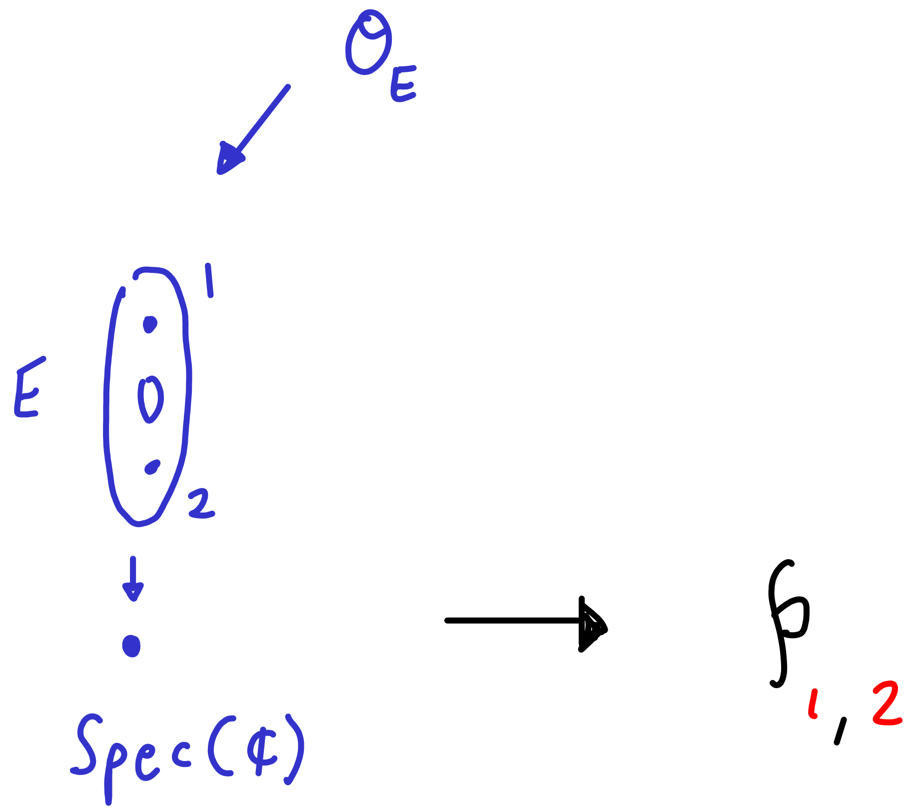
No unit axiom

Then Ω is <u>semisimple</u> if the 2-tensor

$$V \otimes V \to \mathbb{Q}$$

defined by a fixed 2 pointed elliptic curve with the trivial line bundle is nondegenerate

[Equivalent to standard definition in the DM case]



$$\Omega_{1,2}(V_1, V_2) : CH(\cdot) \to CH(\cdot)$$

action on $[\cdot]$ yields an number in $\mathbb{Q}$.

Using metric, we turn $V \otimes V \to \mathbb{Q}$ into $V \to V$ and nondegerate means invertible.

Precise statement

Claim: Classification of

Semisimple Picard CohFT*s

Now use H^*

Let Ω be a Picard CohFT* on (V, η).

STEP (i): Define the Ω^{top}, the topological part of Ω.

STEP (ii): $\exists$ R, T such that

$$\Omega = RT\,\Omega^{top}$$

Parallel to DM Givental-Teleman classification

What is Ω^{top} ?

$$\Omega^{top}_{g,n}(v_1, \ldots, v_n) =$$

Component of $\Omega_{g,n}(v_1, \ldots, v_n)$

in $H^0(P_{g,n})$

Switch now to cohomology

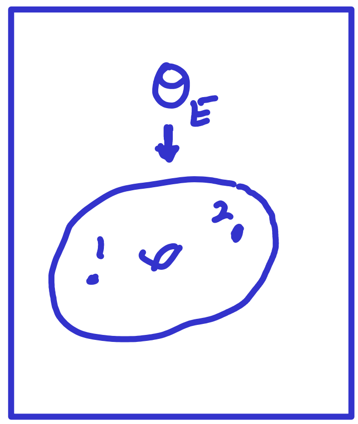
$$\| $$

$$\bigoplus_d H^0(P^d_{g,n})$$

Check: Ω^{top} is in fact a Pic CohFT*

Proof: To be pursued in the Spring!

Method: use



to increase genus. Use cohomological stability of universal Jacobian

parallel to Mumford's Conjecture [Madsen-Weiss]

$$\mathrm{Jac} \downarrow \mathcal{M}_g$$

Ebert
Randal-Williams

Hope for the best using Teleman's strategy

Adding the unit is an important second layer in the classification.

Claim: Classification of

Semisimple Picard CohFTs

with unit

use H^*

Let Ω be a Picard CohFT on $(V, \eta, 1)$.

$\exists\, R$ such that

$$\Omega = RT\,\Omega^{top}$$

May need modification for L, work in progress

with $T(z) = z \cdot 1 - z\,R^{-1}(z)(1)$

[IV] Other groups

The entire discussion of

$P_{g,n}$ and Picard CohFTs

may be viewed as associated

to the group $\mathbb{C}^*$

- There is no further difficulty in considering

$$T = (\mathbb{C}^*)^r$$

Later, $(\mathbb{C}^*)^2$ will be of particular interest

Then the Picard stack

is replaced by

$\mathrm{Bun}^{T}_{g,n}$

Artin stack of Principal T-bundles on connected, pointed, nodal curves.

just an r-tuple of Line bundles

The R-matrix changes now

$$R(z) \in \mathrm{End}(V)[[L_1, L_2, \dots, L_r]][[z]]$$

It is better to write

$$R(z) \in \mathrm{End}(V) \otimes_{\mathbb{Q}} \hat{H}^*_T [[z]]$$

Completed T-equivariant cohomology of a point

As far as the proof of the classification of semisimple Bun^T CohFTs an issue which emerges is the promotion of the

Ebert, Randal-Williams stability

results to r-tuples of

line bundle.

Hope:
$r>1$ not
serious
complication

- The next most natural group to consider is

$$G = GL_r(\mathbb{C})$$

So then we have Bun^{GL_r} stacks, Bun^{GL_r} CohFTs, and

$$R(z) \in \mathrm{End}(V) \otimes_{\mathbb{Q}} \hat{H}^*_{GL_r}[[z]]$$

Completed GL_r-equivariant cohomology of a point

$$\hat{H}^*_{GL_r} = \mathbb{Q}[[c_1, c_2, \ldots, c_r]]$$

For the theory, we require some cohomological stability of

$$\begin{array}{c} \mathrm{Bun}_g^{GL_r} \\ \downarrow \\ \mathcal{M}_g \end{array}$$

Specifically, we require

$$\lim_{g\to\infty} H^*\left(\mathrm{Bun}_g^{GL_r}\right)$$

$$\|$$

$$\mathbb{Q}\left[\kappa_{a_1,b_1,\dots,b_r}\right]$$

No claim for finite g, only stable limit

Question (Andrei)

where $\kappa_{a_1,b_1,\dots,b_r} = \pi_* \, \omega_\pi^{a_1} \, c_1^{b_1} \cdots c_r^{b_r}$,

map induced by universal bundle

$$\begin{tikzcd} \mathcal{C} \arrow[d, "\pi"'] \arrow[r] & BGL_r \\ \mathrm{Bun}^{GL_r} & \end{tikzcd}$$

- Let G be any reductive (connected) algebraic group, then we have

Bun^G stacks, Bun^G CohFTs,

$$R(z) \in \mathrm{End}(V) \otimes_{\mathbb{Q}} \hat{H}^*_G [[z]]$$

and we require stability

$$\lim_{g \to \infty} H^*(\mathrm{Bun}^G_g) = \mathbb{Q}[\kappa_{a,v}]$$

Weil invariant

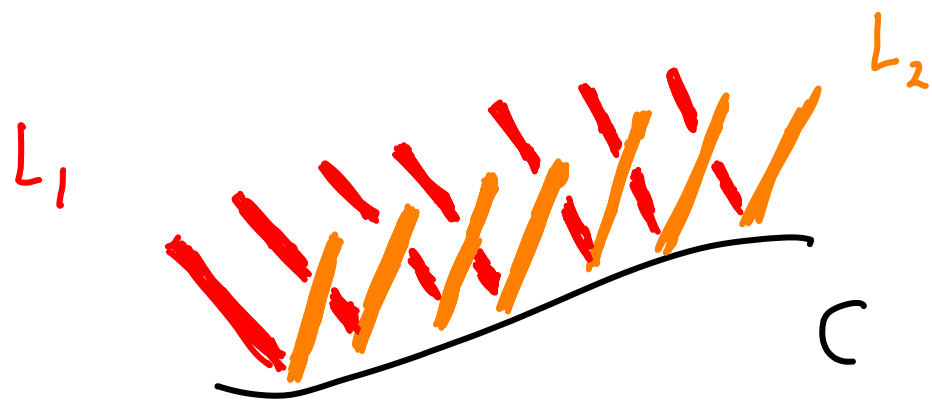
index for $t \subset \mathfrak{g}$ max torus

Question: Are such stability results known?

[V] Examples / Applications

There are two motivating constructions

(i) Local 3fold theory of curves



Also could be A_N-surfaces

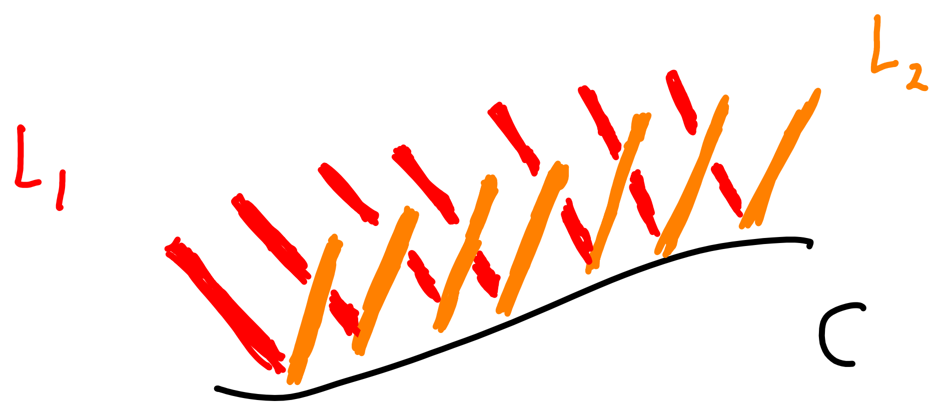
(ii) Twisted theory

$$\mathbb{C}^* \curvearrowright X$$

nonsingular projective variety with a $\mathbb{C}^*$-action

Local 3fold theory of curves

By now there is ~20 year history,
mostly about a fixed local curve



But for a Pic CohFT, we
let C, L_1, L_2 all vary freely.

The theory is defined via
stable pairs.

We fix integer an $D \geq 1$

Define $(V, \eta, 1)$

unit in cohomology

$H^*_T(\mathrm{Hilb}(\mathbb{C}^2))_{loc}$

T-equivariant Poincaré pairing

D points

Basis indexed by partitions of D

Next we must define

$\Omega^{d_1, d_2}_{g,n}(v_1, \ldots, v_n)$ acting on cycles

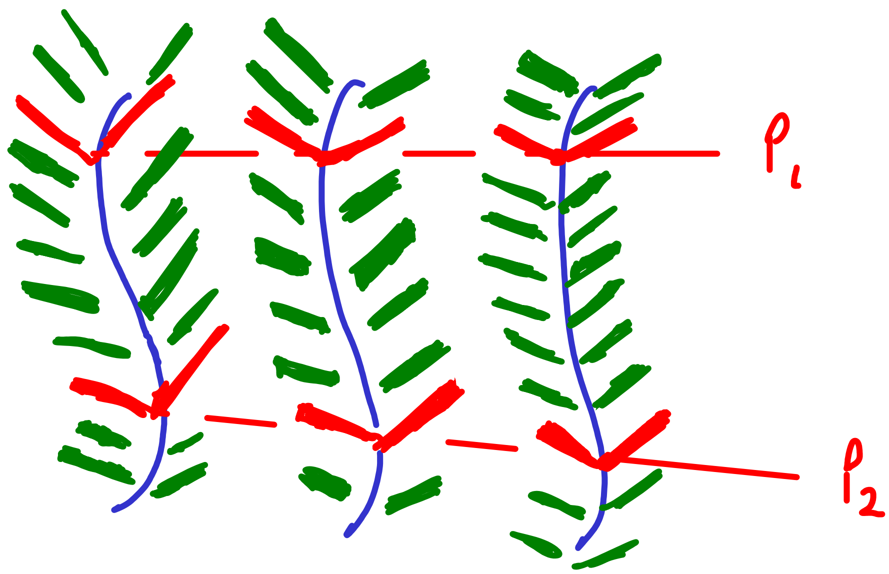
So we start with

$$\begin{array}{c} \mathcal{L}_1 \searrow \quad \swarrow \mathcal{L}_2 \\ \mathcal{C} \\ p_i \uparrow \downarrow \\ S \end{array}$$

irreducible dimension s

Construct a family of local 3 folds

which is relative to the sections p_i



Consider stable pairs in the fibers

with boundary conditions specified by v_i,

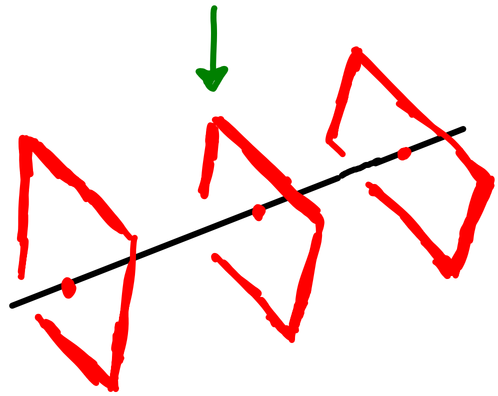
Then push the virtual class down

to S and sum over χ as

usual with q.

$$\mathcal{L}_1 \oplus \mathcal{L}_2 \cong \mathcal{O}_C \oplus \mathcal{O}_C$$

Basic Case



$g(C) = 0$

$QH^*(\mathrm{Hilb}(\mathbb{C}^2))$

Okounkov - P

A_n- Case

Maulik-Oblomkov

Local GW theory of Curves Bryan - P

QH^* of Nakajima Quiver Varieties

Maulik-Okounkov

Crepant Resolution QH^*

Bryan-Graber

GW/DT/PT for toric 3-fold

MOOP

descendent correspondence

P- Pixton

OOP19, OOP20

Complete intersections/ Good degenerations

P-Pixton

What about letting C vary but holding $\mathcal{L}_1 \oplus \mathcal{L}_2 = \mathcal{O}_C \oplus \mathcal{O}_C$?

Full crepant Resolution Conjecture $\mathrm{Hilb}(\mathbb{C}^2) \to \mathrm{Sym}(\mathbb{C}^2)$

Higher genus GW theory of $\mathrm{Hilb}(\mathbb{C}^2)$

GW/DT/PT for families of local curves

P- HH Tseng

Proof uses:
- classification of DM CohFT
- Control of the R-matrix
- Exact analytic continuation results Okounkov – P

$\Downarrow$

Bezrukavnikov - Okounkov

What does Pic CohFT yield?

Provides divide / conquer strategy

$$\Omega^{d_1,d_2}_{g,n}$$

- We know topological part (for almost 20 years!)

- When $\mathcal{L}_1 \oplus \mathcal{L}_2 \cong \mathcal{O} \oplus \mathcal{O}$ the theory is controlled $\Rightarrow$ $R\big|_{\substack{L_1=0 \\ L_2=0}}$

- when $\mathbb{P}^1$, but $\mathcal{L}_1, \mathcal{L}_2$ free, we have a lot of information from T-equivariant theory

- further L_i information in constant maps

Try to determine L_i dependence in R-matrix

work in progress...

(ii) Twisted theory

$$\mathbb{C}^* \curvearrowright X$$

nonsingular projective variety with a $\mathbb{C}^*$-action

Just a sketch of the idea:

We start with

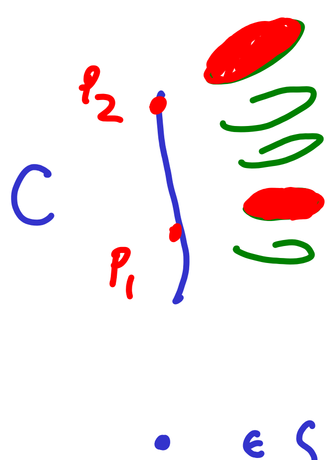
$$\begin{array}{c} \mathcal{L}_1 \\ \swarrow \\ \mathcal{C} \\ p_i \uparrow \downarrow \\ S \end{array}$$

family of $\wp_{g,n}$

irreducible dimension s

$$\begin{array}{c} \mathcal{L}_1^* \times_{\mathbb{C}^*} X \\ \swarrow \\ \mathcal{C} \\ p_i \uparrow \downarrow \\ S \end{array}$$

fiber picture



X_C is a fibration over C

Relative to fibers

Unit

$$(V, n, 1)$$

$H^{\bullet}_{\mathbb{C}^*}(X)_{loc}$ Poincaré pairing

relative condition in V

$$\sum_{\substack{\beta \in H_2(X_C) \\ \beta \text{ projects to } [C]}} q^{\beta} \langle \gamma_1, \ldots, \gamma_r \rangle_{g,n,\beta}$$

Push down to get a cycle class on S $\Rightarrow$ defines $\Omega^{d}_{g,n}$

- like GW theory when $\mathcal{L} = \mathcal{O}$
- like quasimaps for $\mathcal{L}$

Again : Using classification, hope to determine the full theory $\Omega^{d}_{g,n}$ using divide / conquer

[VI] Geometry of the Picard stack.

$\mathfrak{P}_{g,n} \longleftarrow$ in either H^* or CH^*_{op} there is a tautological ring

$$R^*_{g,n} \subset CH^*_{op}(\mathfrak{P}_{g,n})$$

see Bae Holmes P Schmitt Schwarz [BHPSS]

$A = (a_1, \ldots, a_n)$

$DR_{g,A} \in CH^g_{op}(\mathcal{P}_{g,n})$

The main result of BHPSS is a proof of an analogue of Pixton's formula for $DR_{g,A}$ in $R^*(\mathcal{P}_{g,n})$

Clader-Janda
Bae

Pixton's formula comes with Pixton's relations

$\Rightarrow$ very nice set of tautological relations

Using the developments for $\overline{\mathcal{M}}_{g,n}$ as motivation

Question: Is there a Pic CohFT approach to Pixton's relations on $\mathcal{P}_{g,n}$?

Another direction:

Are there other essential relations in $R^*(\mathcal{M}_{g,n})$?

Or can we generate all relations already from playing with Pixton's relations?

What about relations for other groups G?

$$\mathrm{Bun}^T_{g,n} \to \mathrm{Bun}^G_{g,n}$$

max torus

$$R^*(\mathrm{Bun}^T_{g,n}) \hookleftarrow R^*(\mathrm{Bun}^G_{g,n})$$

Weil invariant part

Many Pixton relations from each map

$$\mathrm{Bun}^T_{g,n} \to \mathrm{Bun}^{\mathbb{C}^*}_{g,n}$$

$\boxed{\text{Are there other relations?}}$

Enough Speculation!

The End