THE CANONICAL CLASS OF $\overline{M}_{0,n}(\mathbf{P}^r, d)$ AND ENUMERATIVE GEOMETRY

RAHUL PANDHARIPANDE

0. Summary

Let $\mathbb{C}$ be the field of complex numbers. Let the Severi variety

$$S(0,d) \subset \mathbf{P}\big(H^0(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(d))\big)$$

be the quasi-projective locus of irreducible, nodal rational curves. Let $\overline{S}(0,d)$ denote the closure of $S(0,d)$. Let $p_1, \ldots, p_{3d-2}$ be $3d-2$ general points in $\mathbf{P}^2$. Consider the subvariety $C_d \subset \overline{S}(0,d)$ corresponding to curves passing through all the points $p_1, \ldots, p_{3d-2}$. C_d is a complete curve in $\mathbf{P}\big(H^0(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(d))\big)$. Let N_d be the degree of C_d. N_d is determined by the recursive relation ([K-M], [R-T]):

$$N_1 = 1$$

$$\forall d > 1, \quad N_d = \sum_{i+j=d,\ i,j>0} N_i N_j \left(i^2 j^2 \binom{3d-4}{3i-2} - i^3 j \binom{3d-4}{3i-1} \right).$$

For $d \geq 3$, C_d is singular. The arithmetic genus g_d of C_d is determined by:

$$g_1 = 0,$$

$$g_2 = 0,$$

$$2g_d - 2 = \frac{6d^2+5d-15}{2d} N_d + \frac{1}{4d} \sum_{i=1}^{d-1} N_i N_{d-i} \Big(15i^2(d-i)^2 - 8di(d-i) - 4d \Big) \binom{3d-2}{3i-1}.$$

The last formula holds for $d \geq 3$. The geometric genus $\tilde{g}_d$ of the normalization $\tilde{C}_d$ is determined by ($d \geq 1$):

$$2\tilde{g}_d - 2 = -\frac{3d^2-3d+4}{2d^2} N_d + \frac{1}{4d^2} \sum_{i=1}^{d-1} N_i N_{d-i} (id - i^2) \Big((9d+4)i(d-i) - 6d^2 \Big) \binom{3d-2}{3i-1}.$$

These genus formulas are established by adjunction and intersection on Kontsevich's space of stable maps $\overline{M}_{0,n}(\mathbf{P}^r, d)$.

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1. The Canonical Class of $\overline{M}_{0,n}(\mathbf{P}^r, d)$

Let $\overline{M}_{0,n}(\mathbf{P}^r, d)$ be the coarse moduli space of degree d, Kontsevich stable maps from n-pointed, genus 0 curves to $\mathbf{P}^r$. Foundational treatments of $\overline{M}_{0,n}(\mathbf{P}^r, d)$ can be found in [Al], [P1], [K], and [K-M]. Only the case $r \geq 2$ will be considered here. Let $\mathcal{L}_p$ denote the line bundle obtained on $\overline{M}_{0,n}(\mathbf{P}^r, d)$ by the p^{th} evaluation map ($1 \leq p \leq n$). Let $\triangle$ be the set of boundary divisors. Let $\mathcal{H}$ denote the divisor of maps meeting a fixed codimension 2 linear space of $\mathbf{P}^r$. $\mathcal{H} = 0$ if $d = 0$. In [P2], it is shown the classes

$$\{\mathcal{L}_p\} \cup \triangle \cup \{\mathcal{H}\}$$

span $Pic(\overline{M}_{0,n}(\mathbf{P}^r, d)) \otimes \mathbb{Q}$.

The canonical class of the stack $\overline{\mathcal{M}}_{0,n}(\mathbf{P}^r, d)$ has the following coarse moduli interpretation. $\overline{M}_{0,n}(\mathbf{P}^r, d)$ is an irreducible variety with finite quotient singularities. When $r \geq 2$, the automorphism-free locus $\overline{M}^*_{0,n}(\mathbf{P}^r, d) \subset \overline{M}_{0,n}(\mathbf{P}^r, d)$ is a nonsingular, fine moduli space with codimension 2 complement except when ([P2])

$$[0, n, r, d] = [0, 0, 2, 2].$$

For $r \geq 2$ and $[0, n, r, d] \neq [0, 0, 2, 2]$, the first Chern class of the cotangent bundle to the moduli space $\overline{M}^*_{0,n}(\mathbf{P}^r, d)$ yields the canonical class in $Pic(\overline{M}_{0,n}(\mathbf{P}^r, d)) \otimes \mathbb{Q}$.

Let $P = \{1, 2, \ldots, n\}$ be the set of markings (P may be the empty set). The boundary components are in bijective correspondence with data of weighted partitions $(A \cup B, d_A, d_B)$ where:

(i.) $A \cup B$ is a partition of P.
(ii.) $d_A + d_B = d$, $d_A \geq 0$, $d_B \geq 0$.
(iii.) If $d_A = 0$ (resp. $d_B = 0$), then $|A| \geq 2$ (resp. $|B| \geq 2$).

Define $\mathcal{D}_{i,j}$ to be the reduced sum of boundary components with $d_A = i$ and $|A| = j$. Note $0 \leq i \leq d$ and $0 \leq j \leq n$. The divisors $\mathcal{D}_{0,0}$, $\mathcal{D}_{0,1}$, $\mathcal{D}_{d,n-1}$, $\mathcal{D}_{d,n}$ are equal to 0 by stability. Also, $\mathcal{D}_{i,j} = \mathcal{D}_{d-i,n-j}$.

Consider first the case $d = 0$. $\overline{M}_{0,n}(\mathbf{P}^r, 0) \cong \overline{M}_{0,n} \times \mathbf{P}^r$. It suffices to determine the canonical class of $\overline{M}_{0,n}$.

Proposition 1. *The canonical class* $\mathcal{K}_{\overline{M}}$ *of* $\overline{M}_{0,n}$ *is determined in* $Pic(\overline{M}_{0,n}) \otimes \mathbb{Q}$ *by:*

$$\mathcal{K}_{\overline{M}} = \sum_{j=2}^{[\frac{n}{2}]} \Big(\frac{j(n-j)}{n-1} - 2\Big)\mathcal{D}_{0,j}. \tag{1}$$

The canonical class has a different form in case $d > 0$, $n = 0$, $r \geq 2$.

Proposition 2. *The canonical class* $\mathcal{K}_{\overline{M}}$ *of* $\overline{M}_{0,0}(\mathbf{P}^r, d)$ *(*$d > 0, r \geq 2$*) is determined in* $Pic(\overline{M}_{0,0}(\mathbf{P}^r, d) \otimes \mathbb{Q}$ *by:*

$$\mathcal{K}_{\overline{M}} = -\frac{(d+1)(r+1)}{2d}\mathcal{H} + \sum_{i=1}^{[\frac{d}{2}]} \Big(\frac{(r+1)(d-i)i}{2d} - 2\Big)\mathcal{D}_{i,0}. \tag{2}$$

Finally, when $d > 0$, $n > 0$, $r \geq 2$, the form of the canonical class is the following:

Proposition 3. *The canonical class of* $\mathcal{K}_{\overline{M}}$ *of* $\overline{M}_{0,n}(\mathbf{P}^r, d)$ *(*$d > 0$*,* $n > 0$*,* $r \geq 2$*) is determined in* $Pic(\overline{M}_{0,n}(\mathbf{P}^r, d)) \otimes \mathbb{Q}$ *by:*

$$\mathcal{K}_{\overline{M}} = -\frac{(d+1)(r+1)d - 2n}{2d^2}\mathcal{H} - \sum_{p=1}^{n} \frac{2}{d}\mathcal{L}_p \tag{3}$$

$$+\sum_{i=0}^{[\frac{d}{2}]} \sum_{j=0}^{n} \Big(\frac{(r+1)(d-i)di + 2d^2 j - 4dij + 2ni^2}{2d^2} - 2\Big)\mathcal{D}_{i,j}.$$

Equation (1) can be derived from the explicit construction of $\overline{M}_{0,n}$ described in [F-M]. Equations (1-3) will be established here via intersections with curves.

2. Computing The Canonical Class

2.1. **Curves in $\overline{M}_{0,n}(\mathbf{P}^r, d)$.** By Proposition (2) of [P2], the canonical projection

$$Pic(\overline{M}_{0,n}(\mathbf{P}^r, d)) \otimes \mathbb{Q} \to Num(\overline{M}_{0,n}(\mathbf{P}^r, d)) \otimes \mathbb{Q}$$

is an isomorphism. Hence, the canonical class of $\overline{M}_{0,n}(\mathbf{P}^r, d)$ can be established via intersections with curves. Curves can easily be found in $\overline{M}_{0,n}(\mathbf{P}^r, d)$. The notation of [P2] is recalled here.

Let C be a nonsingular, projective curve. Let $\pi : S = \mathbf{P}^1 \times C \to C$. Select n sections $s_1, \ldots, s_n$ of π. A point $x \in S$ is an *intersection point* if two or more sections contain x. Let $\mathcal{N}$ be a line bundle on S of type (d, k). Let $z_l \in H^0(S, \mathcal{N})$ $(0 \leq l \leq r)$ determine a rational map $\mu : S - \to \mathbf{P}^r$ with simple base points. A point $y \in S$ is a *simple base point* of degree $1 \leq e \leq d$ if the blow-up of S at y resolves μ locally at y and the resulting map is of degree e on the exceptional divisor E_y. The set of *special points* of S is the union of the intersection points and the simple base points. Three conditions are required:

(1.) There is at most one special point in each fiber of π.
(2.) The sections through each intersection point x have distinct tangent directions at x.
(3.) (i.) $d = 0$. No $n-1$ sections pass through a point $x \in S$.
(ii.) $d > 0$. If at least $n-1$ sections pass through a point $x \in S$, then x is not a simple base point of degree d.

Condition (3.ii) implies there are no simple base points of degree d if $n = 0$ or 1. Let $\overline{S}$ be the blow-up of S at the special points. It is easily seen $\overline{\mu} : \overline{S} \to \mathbf{P}^r$ is Kontsevich stable family of n-pointed, genus 0 curves over C. Condition (2) ensures the strict transforms of the sections are disjoint. Condition (3) implies Kontsevich stability. There is a canonical morphism $\lambda : C \to \overline{M}_{0,n}(\mathbf{P}^r, d)$. Condition (1) implies C intersects the boundary components transversally.

2.2. $\overline{M}_{0,n}$**.** Curves in $\overline{M}_{0,n}$ are obtained by the above construction (omitting the map μ). Let $s_1, \ldots, s_n$ be n sections of $\pi : S = \mathbf{P}^1 \times C \to C$ satisfying $(1), (2), (3.i)$. For $1 \leq \alpha \leq n$, let s_α be of type $(1, \sigma_\alpha)$ on $S = P^1 \times C$. Let $\overline{\pi} : \overline{S} \to C$ be the blow-up of S as above. Let $s_1, \ldots, s_n$ also denote the transformed sections of $\overline{\pi}$. Let Q denote the points of C lying under the special points of S. There is a canonical sequence

$$0 \to R^1\overline{\pi}_*(\omega^*_{\overline{\pi}}(-\sum_1^n s_\alpha)) \to \lambda^*(T_{\overline{M}_{0,n}}) \to \bigoplus_{q\in Q} \mathbb{C}_q \to 0.$$

(See, for example, [K].) Hence $C \cdot \mathcal{K}_{\overline{M}} = -deg\big(R^1\overline{\pi}_*(\omega^*_{\overline{\pi}}(-\sum_1^n s_\alpha))\big) - C \cdot \sum_{j=2}^{[\frac{n}{2}]} \mathcal{D}_{0,j}$.

The degree of $R^1\overline{\pi}_*(\omega^*_{\overline{\pi}}(-\sum_1^n s_\alpha))$ is determined by the Grothendieck-Riemann-Roch formula. Let x_j for $2 \leq j \leq n-2$ be the number of intersection points of S which lie on exactly j sections. If $j \neq n/2$, $C \cdot \mathcal{D}_{0,j} = x_j + x_{n-j}$. If $j = n/2$, $C \cdot \mathcal{D}_{0,j} = x_j$. G-R-R yields:

$$deg\big(R^1\overline{\pi}_*(\omega^*_{\overline{\pi}}(-\sum_1^n s_\alpha))\big) = \sum_1^n 2\sigma_\alpha + \sum_2^{n-2}(1-j)x_j.$$

By the transverse intersection condition, the following relation must hold:

$$\sum_1^n \sigma_\alpha = \frac{1}{n-1}\sum_2^{n-2}\frac{j^2-j}{2}x_j.$$

Combining equations yields:

$$C \cdot \mathcal{K}_{\overline{M}} = \sum_2^{n-2}\Big(j - 2 - \frac{j^2-j}{n-1}\Big)x_j$$

$$= \sum_2^{n-2}\Big(\frac{j(n-j)}{n-1} - 2\Big)x_j.$$

Hence both sides of equation (1) have the same intersection numbers with C. Let D be any nonsingular curve in $\overline{M}_{0,n}$ which intersects the boundary transversely. The universal family over D can be blown-down to a projective bundle $\pi : T \to D$. The above calculation covers the case where $T = \mathbf{P}^1 \times D$. The general case (in which T is *any* $\mathbf{P}^1$-bundle) is identical. Since $A^1(\overline{M}_{0,n})$ is spanned by curves meeting the boundary transversely, Proposition (1) is immediate.

2.3. $\overline{M}_{0,0}(\mathbf{P}^r, d)$**.** The case $d > 0$, $n = 0$, $r \geq 2$ is now considered. Let $\overline{\pi} : \overline{S} \to C$, $\overline{\mu} : \overline{S} \to \mathbf{P}$ be a family of stable maps as above. There is canonical exact sequence

$$0 \to \overline{\pi}_*(\omega^*_{\overline{\pi}}) \to \overline{\pi}_*\overline{\mu}^*(T_{\mathbf{P}^r}) \to \lambda^*(T_{\overline{M}_{0,0}(\mathbf{P}^r,d)}) \to \bigoplus_{p\in Q} \mathbb{C}_p \to 0.$$

Hence $C \cdot \mathcal{K}_{\overline{M}} = +deg\big(\overline{\pi}_*(\omega^*_{\overline{\pi}})\big) - deg\big(\overline{\pi}_*\overline{\mu}^*(T_{\mathbf{P}^r})\big)\big) - \sum_1^{[\frac{d}{2}]} \mathcal{D}_{i,0}$. Let x_i for $1 \leq i \leq d-1$ be the number of simple base points of $\mu : S- \to \mathbf{P}^r$ of degree exactly i. If $i \neq d/2$, $C \cdot \mathcal{D}_{i,0} = x_i + x_{d-i}$. If $i = d/2$, $C \cdot \mathcal{D}_{i.0} = x_i$. Via G-R-R,

$$deg\big(\overline{\pi}_*(\omega^*_{\overline{\pi}})\big) = -\sum_1^{d-1} x_i,$$

$$deg\big(\overline{\pi}_*(\overline{\mu}^*(T_{\mathbf{P}^r}))\big) = (r+1)(d+1)k - \sum_1^{d-1} \frac{(r+1)(i^2+i)}{2} x_i.$$

Combining equations yields:

$$C \cdot \mathcal{K}_{\overline{M}} = -(r+1)(d+1)k + \sum_1^{d-1} \Big(\frac{(r+1)(i^2+i)}{2} - 2\Big) x_i.$$

Finally $C \cdot \mathcal{H}$ must be computed:

$$C \cdot \mathcal{H} = 2dk - \sum_1^{d-1} i^2 x_i.$$

These equations (plus algebra) verify Proposition (2). As before, the complete proof requires the above calculation for *any* $\mathbf{P}^1$-bundle $\pi : S \to C$. Again, the generalization to this case is trivial.

2.4. $\overline{M}_{0,n}(\mathbf{P}^r, d)$**.** Finally, the case $d > 0$, $n > 0$, $r \geq 2$ is considered. Let $\overline{\pi} : \overline{S} \to C$, $\overline{\mu} : \overline{S} \to \mathbf{P}$ be a family of stable maps as above. Let $s_1, \ldots, s_n$ be n sections of $\pi : S = \mathbf{P}^1 \times C \to C$ satisfying (1), (2), (3.ii). For $1 \leq \alpha \leq n$, let s_α be of type $(1, \sigma_\alpha)$ on $S = P^1 \times C$. There is a canonical exact sequence

$$0 \to \overline{\pi}_*(\omega^*_{\overline{\pi}}) \to \overline{\pi}_*(\omega^*_{\overline{\pi}}|_{\sum s_\alpha}) \bigoplus \overline{\pi}_*\overline{\mu}^*(T_{\mathbf{P}^r}) \to \lambda^*(T_{\overline{M}_{0,0}(\mathbf{P}^r,d)}) \to \bigoplus_{p\in Q} \mathbb{C}_p \to 0.$$

Hence $C \cdot \mathcal{K}_{\overline{M}} = +deg\big(\overline{\pi}_*(\omega^*_{\overline{\pi}})\big) - (\omega^*_{\overline{\pi}} \cdot \sum_1^n s_\alpha) - deg\big(\overline{\pi}_*\overline{\mu}^*(T_{\mathbf{P}^r})\big)\big) - \sum_{i=1}^{[\frac{d}{2}]} \sum_{j=0}^{n} \mathcal{D}_{i,j}$. Let $z_{i,j}$ for $0 \leq i \leq d$ and $0 \leq j \leq n$ be the number of special points of S that are simple base points of degree exactly i and that lie on exactly j sections. If $i \neq d/2$ or $j \neq n/2$, then

$C \cdot \mathcal{D}_{i,j} = z_{i,j} + z_{d-i,n-j}$. If $i = d/2$ and $j = n/2$, then $C \cdot \mathcal{D}_{i,j} = z_{i,j}$. Via G-R-R,

$$deg\big(\overline{\pi}_*(\omega^*_{\overline{\pi}})\big) = -\sum_{i=0}^{d}\sum_{j=0}^{n} z_{i,j},$$

$$deg\big(\overline{\pi}_*(\overline{\mu}^*(T_{\mathbf{P}^r}))\big) = (r+1)(d+1)k - \sum_{i=0}^{d}\sum_{j=0}^{n} \frac{(r+1)(i^2+i)}{2} z_{i,j}.$$

A simple calculation yields:

$$\omega^*_{\overline{\pi}} \cdot \sum_{1}^{n} s_\alpha = \sum_{1}^{n} 2\sigma_\alpha - \sum_{i=0}^{d}\sum_{j=0}^{n} j z_{i,j}.$$

The two additional intersection numbers are:

$$C \cdot \mathcal{H} = 2dk - \sum_{i=0}^{n}\sum_{j=0}^{n} i^2 z_{i,j},$$

$$C \cdot \sum_{1}^{n} \mathcal{L}_p = nk + \sum_{1}^{n} d\sigma_\alpha - \sum_{i=0}^{n}\sum_{j=0}^{n} ij z_{i,j}.$$

Now algebra yields the desired equality of intersections that establishes Proposition (3). Again the calculation must be done in case $\pi : S \to C$ is a $\mathbf{P}^1$ bundle.

3. The genus of C_d, $\tilde{C}_d$

3.1. **Singularities.** Let $C_d \subset \overline{S}(0,d)$ be the dimension 1 subvariety corresponding to curves passing through $3d-2$ general points $p_1, \ldots, p_{3d-2}$ in $\mathbf{P}^2$. Let $\hat{C}_d \subset \overline{M}_{0,0}(\mathbf{P}^2, d)$ be the dimension 1 subvariety corresponding to maps passing through $p_1, \ldots, p_{3d-2}$. The singularities of C_d, $\hat{C}_d$ will be analyzed.

Let $[\mu] \in \overline{M}_{0,0}(\mathbf{P}^2, d)$ correspond to an automorphism-free map with domain $\mathbf{P}^1$. There is a normal sequence on $\mathbf{P}^1$:

$$0 \to T_{\mathbf{P}^1} \stackrel{d\mu}{\to} \mu^*(T_{\mathbf{P}^2}) \to Norm \to 0.$$

The tangent space to $\overline{M}_{0,0}(\mathbf{P}^2, d)$ is $H^0(\mathbf{P}^1, Norm)$. If μ is an immersion, $Norm \stackrel{\sim}{=} \mathcal{O}_{\mathbf{P}^1}(3d-2)$. If μ is not an immersion $Norm$ will have torsion. A 1-*cuspidal* rational plane curve is an irreducible rational plane curve with nodal singularities except for exactly 1 cusp. If μ corresponds to a 1-cuspidal rational curve, then there is a sequence:

$$0 \to \mathbb{C}_p \to Norm \to \mathcal{O}_{\mathbf{P}^1}(3d-3) \to 0$$

where p is the point of $\mathbf{P}^1$ lying over the cusp. Since $3d-2$ distinct points of $\mathbf{P}^1$ always impose independent conditions on $H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(3d-2))$ and $H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(3d-3))$, Lemma (1) has been established:

Lemma 1. *Let $[\mu] \in \hat{C}_d$ be a point corresponding to an irreducible, nodal or 1-cuspidal rational curve with all singularities distinct from $p_1, \ldots, p_{3d-2}$. $\hat{C}_d$ is nonsingular at $[\mu]$.*

The corresponding analysis for C_d is more involved.

Lemma 2. *Let $x \in C_d$ be a point corresponding to an irreducible, nodal rational curve with nodes distinct from $p_1, \ldots, p_{3d-2}$. C_d is nonsingular at x.*

Proof. Let $X \subset \mathbf{P}^2$ be the plane curve corresponding to $x \in C_d$. $S(0,d)$ is nonsingular at x with tangent space $H^0(\tilde{X}, \mathcal{O}_{\mathbf{P}^2}(d) - N)$ where N is the divisor of points of $\tilde{X}$ lying over the nodes of X. The additional points correspond to $3d-2$ *distinct* points of $\tilde{X}$. Since $3d-2$ distinct point on $\mathbf{P}^1$ impose $3d-2$ independent linear conditions on sections of $\mathcal{O}_{\mathbf{P}^2}(d) - N \stackrel{\sim}{=} \mathcal{O}_{\mathbf{P}^1}(3d-2)$, it follows C_d is nonsingular at x. □

Actually, $\overline{M}_{0,0}(\mathbf{P}^2, d)$ and $S(0,d)$ are isomorphic on the irreducible, nodal locus. Hence Lemma (2) is unnecessary.

Lemma 3. *Let $x \in C_d$ be a point corresponding to a 1-cuspidal rational plane curve with all singularities distinct from $p_1, \ldots, p_{3d-2}$. C_d is cuspidal at x.*

Proof. The versal deformation space of the cusp $Z_0^2 + Z_1^3$ is 2 dimensional:

$$Z_0^2 + Z_1^3 + aZ_1 + b.$$

The locus in the versal deformation space corresponding to equigeneric deformations is determined by the cuspidal curve $4a^3 + 27b^2 = 0$.

Let X be the plane curve corresponding to x. Let $q \in X$ be the cusp. Let $\tilde{X}$ the normalization of X. Let $p \in \tilde{X}$ lie over q. The nodes of X, the points $p_1, \ldots, p_{3d-2}$, and the 2 dimensional subscheme supported on q and pointing in the direction of the tangent cone of X all together impose independent conditions on the linear system of degree d plane curves. First, this independence will be established.

Let A be the subspace of $H^0(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(d))$ passing through the nodes, points, and the subscheme of length 2. As before, let N denote the divisor of $\tilde{X}$ lying above the nodes. There is a natural left exact sequence obtained by pulling back sections to $\tilde{X}$:

$$0 \to \mathbb{C} \to A \to H^0(\tilde{X}, \mathcal{O}_{\mathbf{P}^2}(d) - N - p_1 - \ldots - p_{3d-2} - 3p).$$

By counting conditions,

$$dim(A) \geq \frac{(d+1)(d+2)}{2} - \frac{(d-1)(d-2)}{2} + 1 - 3d + 2 - 2 = 1$$

with equality if only if the conditions are independent. Since

$$deg_{\tilde{X}}(\mathcal{O}_{\mathbf{P}^2}(d) - N - p_1 - \ldots - p_{3d-2} - 3p) = d^2 - (d-1)(d-2) + 2 - 3d + 2 - 3 = -1,$$

$dim(A) = 1$ and the conditions are independent.

By the independence result above, the deformations of X parameterized by the linear system of plane curves through the nodes and the points $p_1, \ldots, p_{3d-2}$ surjects on the 2 dimensional versal deformation space of the cusp. The locus of equigeneric deformations of X through the points $p_1, \ldots, p_{3d-2}$ is étale locally equivalent to the cusp in the versal deformation space of the cusp. □

Lemma 4. *Let $[\mu] \in \hat{C}_d$ (resp. $x \in C_d$) be a point corresponding to an irreducible, nodal, rational curve with a node at p_1 and nodes distinct from $p_2, \ldots, p_{3d-2}$. $\hat{C}_d$ is nodal at $[\mu]$ (resp. C_d is nodal at x).*

Proof. If suffices to prove the result for $\hat{C}_d$. The divisor $D_1 \subset \overline{M}_{0,0}(\mathbf{P}^2, d)$ corresponding to curves passing through the point p_1 has two nonsingular branches with a normal crossings intersection at $[\mu]$. Let $r, s \in \mathbf{P}^1$ lie over $p_1 \in \mathbf{P}^2$. The two nonsingular branches have the following tangent spaces at X:

$$H^0(\mathbf{P}^1, Norm(-r), \quad H^0(\mathbf{P}^1, Norm(-s)).$$

The remaining $3d-3$ points impose independent conditions on each of these tangent spaces. Etale locally at $[\mu]$, $\hat{C}_d$ is the intersection of the union of linear spaces of dimensions $3d-2$ meeting along a subspace of dimension $3d-3$ with general linear space of codimension $3d-3$. Hence, $\hat{C}_d$ is nodal at $[\mu]$. $\square$

Lemma 5. *Let $x \in C_d$ be a point corresponding to the union of two irreducible, nodal, rational curves with degrees i and $d-i$ meeting transversely with nodes (including component intersection points) distinct from $p_1, \ldots, p_{3d-2}$. Also assume the components of degrees i, $d-i$ contain $3i-1$, $3(d-i)-1$ points respectively. C_d has the singularity type of the coordinate axes at the origin in $\mathbb{C}^{id-i^2}$.*

Proof. The nodes (including the intersections of the two components of X) and the points $p_1, \ldots, p_{3d-2}$ necessarily impose $(d+1)(d+2)/2$ independent conditions on $H^0(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(d))$. This independence can be established as follows. Let $\tilde{X}$ be the normalization of X (note $\tilde{X}$ is the disjoint union of two $\mathbf{P}^1$'s). Let $A \subset H^0(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(d))$ be the linear series passing through all the nodes of X. There is an exact sequence of vector spaces

$$0 \to \mathbb{C} \to A \to H^0(\tilde{X}, \mathcal{O}_{\mathbf{P}^2}(d) - N) \to 0.$$

As before, N is the divisor preimage of the nodes of X. Certainly only a 1 dimensional subspace of A corresponding to the equation of X vanishes on $\tilde{X}$. Surjectivity of the above sequence follows by a dimension count:

$$dim(A) \geq \frac{(d+1)(d+2)}{2} - \frac{(d-1)(d-2)}{2} - 1 = 3d-1,$$

$$h^0(\tilde{X}, \mathcal{O}_{\mathbf{P}^2}(d) - N) = d^2 - (d-1)(d-2) - 2 + 2 = 3d-2.$$

The points $p_1, \ldots, p_{3d-2}$ are distinct on $\tilde{X}$ and impose independent conditions on $H^0(\tilde{X}, \mathcal{O}_{\mathbf{P}^2}(d) - N)$ by the assumption of their distribution (and the fact $\tilde{X}$ is a disjoint union of $\mathbf{P}^1$'s).

At $x \in \overline{S}(0,d)$, the closed Severi variety has $id - i^2$ nonsingular branches (one for each intersection point). Let $q \in \mathbf{P}^2$ be an intersection point of the two components of X. The tangent space $T(q)$ to the branch of $\overline{S}(0,d)$ corresponding to q is simply the linear subspace $T(q) \subset H^0(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(d))$ of polynomials that vanish at all the nodes of X besides q. Let $V \subset H^0(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(d))$ be the linear subspace of polynomials that vanish at the non-intersection nodes of X and the points $p_1, \ldots, p_{3d-2}$. C_d at x is étale locally equivalent to the intersection

$$V \cap (T(q_1) \cup T(q_2) \cup \cdots \cup T(q_{id-i^2})).$$

Note $V \cong \mathbb{C}^{id-i^2}$. Since the nodes of X and the points $p_1, \ldots, p_{3d-2}$ impose independent conditions on $H^0(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(d))$, the Lemma is proven. $\square$

The last case to be consider is the analogue of Lemma (5) for $\hat{C}_d$: when $[\mu] \in \hat{C}_d$ corresponds to a map with reducible domain and image satisfying the conditions of (5). This case can be handled directly. However, it is easier to observe that at such $[\mu]$, $\overline{M}_{0,0}(\mathbf{P}^2, d)$ is locally isomorphic to the nonsingular branch in the proof of Lemma (5) determined by the attaching point of the two components. The singularity analysis then shows $[\mu] \in \hat{C}_d$ is a nonsingular point.

For general points $p_1, \ldots, p_{3d-2}$, every point $x \in C_d$, $[\mu] \in \hat{C}_d$ corresponds to exactly one of the three cases covered by Lemmas (1-5). Hence the singularities of C_d, $\hat{C}_d$ are established.

3.2. **The Arithmetic Genus.** Consider the moduli space $\overline{M}_{0,0}(\mathbf{P}^2, d)$ for $d \geq 3$ (to avoid $[0, 0, r, d] = [0, 0, 2, 2]$). For general points $p_1, \ldots, p_{3d-2}$, the intersection cycle

$$\hat{C}_d = \mathcal{H}_1 \cap \mathcal{H}_2 \cap \cdots \cap \mathcal{H}_{3d-2}$$

is a curve in $\overline{M}_{0,0}(\mathbf{P}^2, d)$. $\mathcal{H}_i$ is the divisor of maps passing through the point p_i. By the analysis in section (3.1), $\hat{C}_d$ is nonsingular except for nodes. The nodes occur precisely at the points $[\mu] \in \hat{C}_d$ corresponding to a nodal curve with a node at some p_i. Since, for general points, $\hat{C}_d \subset \overline{M}^*_{0,0}(\mathbf{P}^2, d)$, the arithmetic genus $\hat{g}_d$ of $\hat{C}_d$ can be determined by the formula for the canonical class and adjunction ($d \geq 3$):

$$2\hat{g}_d - 2 = \big(\mathcal{K}_{\overline{M}} + (3d-2)\mathcal{H}\big) \cdot \mathcal{H}^{3d-2}.$$

A computation of these intersection numbers in terms of the numbers N_d yields for all $d \geq 3$:

$$2\hat{g}_d - 2 = \frac{(2d-3)(3d+1)}{2d} N_d + \frac{1}{4d} \sum_{i=1}^{d-1} N_i N_{d-i} \Big(3i^2(d-i)^2 - 4di(d-i)\Big) \binom{3d-2}{3i-1}.$$

The natural map $\hat{C}_d \to C_d$ is a partial desingularization. The arithmetic genus of C_d differs from the arithmetic genus of $\hat{C}_d$ only by the contribution of the singularities of Lemma (3) and (5). Consider first the cusps in C_d determined by Lemma (3). The number of these cusps is exactly the number of 1-cuspidal, degree d, rational curves through $3d-2$ points in the plane. In [P2] it is shown there are

$$\frac{3d-3}{d}N_d + \frac{1}{2d}\sum_{i=1}^{d-1} N_iN_{d-i}(3i^2(d-i)^2 - 2di(d-i))\binom{3d-2}{3i-1}$$

1-cuspidal, degree d, rational curves through $3d-2$ points. Each cusp contributes 1 to the arithmetic genus of C_d. The singularities of Lemma (5) contribute

$$\frac{1}{2}\sum_{i=1}^{d-1} N_iN_{d-i}(i(d-i)-1)\binom{3d-2}{3i-1}$$

to the arithmetic genus of C_d. The formula for the arithmetic genus of C_d can be deduced by adding these contributions to the formula for $\hat{g}$:

$$2g_d - 2 = \frac{6d^2+5d-15}{2d}N_d + \frac{1}{4d}\sum_{i=1}^{d-1} N_iN_{d-i}\Big(15i^2(d-i)^2 - 8di(d-i) - 4d\Big)\binom{3d-2}{3i-1}.$$

3.3. **The Geometric Genus.** The geometric genus, $g(\tilde{C}_d)$ is simple to compute. By Bertini's Theorem, the curve $\tilde{C}_d$ determined in $\overline{M}_{0,3d-2}(\mathbf{P}^2, d)$ by $3d-2$ general points is nonsingular and contained in the automorphism-free locus $\overline{M}^*_{0,3d-2}(\mathbf{P}^2, d)$. There is sequence of natural maps exhibiting $\tilde{C}_d$ as the normalization of both $\hat{C}_d$ and C_d:

$$\tilde{C}_d \to \hat{C}_d \to C_d.$$

The genus of $\tilde{C}_d$ can be determined by the formula for the canonical class and adjunction:

$$2\tilde{g}_d - 2 = \big(\mathcal{K}_{\overline{M}} + 2\sum_{1}^{3d-2} c_1(\mathcal{L}_p)\big) \cdot c_1^2(\mathcal{L}_1)\cdots c_1^2(\mathcal{L}_{3d-2})$$

$$= \mathcal{K}_{\overline{M}} \cdot c_1^2(\mathcal{L}_1)\cdots c_1^2(\mathcal{L}_{3d-2}).$$

A computation of these intersection numbers in terms of the numbers N_d yields for all $d \geq 1$:

$$2\tilde{g}_d - 2 = -\frac{3d^2-3d+4}{2d^2}N_d + \frac{1}{4d^2}\sum_{i=1}^{d-1} N_iN_{d-i}(id-i^2)\Big((9d+4)i(d-i) - 6d^2\Big)\binom{3d-2}{3i-1}.$$

3.4. **The difference $\hat{g}_d - \tilde{g}_d$.** Let $d \geq 3$. The natural map $\tilde{C}_d \to \hat{C}_d$ is a desingularization. $\hat{C}_d$ has only nodal singularity. The difference, $\hat{g}_d - \tilde{g}_d$, equals the number of nodes of $\hat{C}_d$. Let M_d be the number of irreducible, nodal, rational degree d plane curves with a node at a fixed point and passing through $3d-3$ general point in $\mathbf{P}^2$. From the description of the nodes of $\hat{C}_d$, it follows:

$$\hat{g}_d - \tilde{g}_d = (3d-2)M_d.$$

Values for low degree d are tabulated below:

d	N_d
1	1
2	1
3	12
4	620
5	87304
6	26312976
7	14616808192
8	13525751027392

d	g_d	$\hat{g}_d$	$\tilde{g}_d$	M_d
1	0	0	0	*
2	0	0	0	*
3	55	10	3	1
4	5447	1685	725	96
5	1059729	402261	166545	18132
6	393308785	168879025	64776625	6506400
7	254586817377	119342269809	42214315809	4059366000
8	265975021514145	133411753757505	43616611944513	4081597355136

The formula for M_d (for $d \geq 3$) is:

$$M_d = \frac{d^2-1}{d^2} N_d - \frac{1}{4d^2} \sum_{i=1}^{d-1} N_i N_{d-i} (id - i^2) \Big(\frac{(6d+4)i(d-i) - 2d^2}{3d-2} \Big) \binom{3d-2}{3i-1}.$$

An alternative method of computing g_3 is the following. $\overline{S}(0,3)$ is simply the degree 12 discriminant hypersurface in the linear system of plane cubics. Therefore, C_3 is a degree 12 plane curve of arithmetic genus $11 \cdot 10/2 = 55$. In fact, C_3 has 24 cusp, 28 nodes, and geometric genus 3.

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Department of Mathematics, University of Chicago

rahul@@math.uchicago.edu