CREPANT RESOLUTION AND THE HOLOMORPHIC ANOMALY EQUATION FOR $[\mathbb{C}^3/\mathbb{Z}_3]$

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ABSTRACT. We study the orbifold Gromov-Witten theory of the quotient $[\mathbb{C}^3/\mathbb{Z}_3]$ in all genera. Our first result is a proof of the holomorphic anomaly equations in the precise form predicted by B-model physics. Our second result is an exact crepant resolution correspondence relating the Gromov-Witten theories of $[\mathbb{C}^3/\mathbb{Z}_3]$ and local $\mathbb{P}^2$. The proof of the correspondence requires an identity proven in the Appendix by T. Coates and H. Iritani.

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0. INTRODUCTION

0.1. **Overview.** Let $\mathbb{Z}_3$ be the cyclic group of order 3 with generator ω. Let $\mathbb{Z}_3$ act on $\mathbb{C}^3$ by

$$\omega \mapsto \begin{pmatrix} e^{\frac{2\pi i}{3}} & 0 & 0 \\ 0 & e^{\frac{2\pi i}{3}} & 0 \\ 0 & 0 & e^{\frac{2\pi i}{3}} \end{pmatrix} . \tag{1}$$

The central object of our paper is the orbifold (or Deligne-Mumford stack) quotient $[\mathbb{C}^3/\mathbb{Z}_3]$. The constant holomorphic 3-form of $\mathbb{C}^3$ descends to $\mathbb{C}^3/\mathbb{Z}_3$ by the specific choice of the $\mathbb{Z}_3$-action (1). We refer

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the reader to [1, 7] for an introduction to orbifolds and orbifold cohomology.

Viewed as a singular scheme, the quotient $\mathbb{C}^3/\mathbb{Z}_3$ admits a crepant resolution in the category of schemes by the total space $K\mathbb{P}^2$ of the canonical bundle of $\mathbb{P}^2$,

$$K\mathbb{P}^2 \to \mathbb{C}^3/\mathbb{Z}_3\,. \tag{2}$$

Since $[\mathbb{C}^3/\mathbb{Z}_3]$ is nonsingular as an orbifold, the morphism

$$[\mathbb{C}^3/\mathbb{Z}_3] \to \mathbb{C}^3/\mathbb{Z}_3$$

may be viewed as a crepant resolution in the category of orbifolds.

Our study of the orbifold Gromov-Witten theory of $[\mathbb{C}^3/\mathbb{Z}_3]$ in all genera yields two main results:

(i) We prove the holomorphic anomaly equations for $[\mathbb{C}^3/\mathbb{Z}_3]$ in the precise form predicted by B-model physics [2].

(ii) We prove an exact crepant resolution correspondence in all genera relating the Gromov-Witten theories of $K\mathbb{P}^2$ and $[\mathbb{C}^3/\mathbb{Z}_3]$.

For (i), our approach follows the path of the higher genus study in [24, 25]. For (ii), our correspondence is simple, explicit, and carries no unevaluated[1] analytic continuation.

0.2. **Crepant resolutions.** Following Ruan [32], Bryan-Graber [5], and, especially for $\mathbb{C}^3/\mathbb{Z}_3$, Coates-Iritani-Tseng [14], the relationship between the Gromov-Witten theories of scheme and orbifold crepant resolutions has been studied for more than a decade. In many cases where the exceptional locus of the resolution is of dimension 1, a crepant resolution correspondence is proven by matching closed form calculations of the two sides.[2] However, for the quotient $\mathbb{C}^3/\mathbb{Z}_3$, the resolution (2) has exceptional locus $\mathbb{P}^2$ of dimension 2, and closed forms are not available.

Our correspondence is proven instead by Cohomological Field Theory (CohFT) techniques [17, 33]. The crepant resolution correspondence of [29] for

$$\mathrm{Hilb}(\mathbb{C}^2, n) \to (\mathbb{C}^2)^n/\Sigma_n$$

is another recent application of the CohFT perspective. While the statements of the correspondences for $\mathbb{C}^3/\mathbb{Z}_3$ and $(\mathbb{C}^2)^n/\Sigma_n$ have no unevaluated analytic continuations, the proofs both require delicate identities governing the changes of variables. In the latter case, the

[1]The general statement of the crepant resolution correspondence (see [15, Conjecture 4.1]) asserts an equivalence up to *a choice of analytic continuation* which is not explicitly specified.

[2]See, for example, [4, 6, 8, 10, 30].

result required is related to the analytic continuation of the quantum differential equation studied in [26]. For $\mathbb{C}^3/\mathbb{Z}_3$, we require here identities proven in the Appendix by T. Coates and H. Iritani.

0.3. **Orbifold cohomology.** The diagonal $\mathsf{T} = (\mathbb{C}^*)^3$ action on $\mathbb{C}^3$ descends to the orbifold $[\mathbb{C}^3/\mathbb{Z}_3]$. The T-equivariant *Chen-Ruan orbifold cohomology* $H^*_{\mathsf{T},\mathsf{orb}}([\mathbb{C}^3/\mathbb{Z}_3])$ has three canonical elements[3],

$$\begin{aligned} 1 = \phi_0 &\in H^0_{\mathsf{T},\mathsf{orb}}([\mathbb{C}^3/\mathbb{Z}_3])\,, \\ \phi_1 &\in H^2_{\mathsf{T},\mathsf{orb}}([\mathbb{C}^3/\mathbb{Z}_3])\,, \\ \phi_2 &\in H^4_{\mathsf{T},\mathsf{orb}}([\mathbb{C}^3/\mathbb{Z}_3])\,, \end{aligned}$$

which span a basis. The classes ϕ_1 and ϕ_2 correspond (on the inertial stack) respectively to the two stabilizers ω and $\omega^2 \in \mathbb{Z}_3$ of the fixed point $0 \in \mathbb{C}^3$.

Let $\lambda_0, \lambda_1, \lambda_2$ denote the 3 weights of the representations of T on the 3 factors of $\mathbb{C}^3$. The pairing matrix for $H^*_{\mathsf{T},\mathsf{orb}}([\mathbb{C}^3/\mathbb{Z}_3])$ in the basis ϕ_0, ϕ_1, ϕ_2 is defined by residues with respect to localization by T:

$$\frac{1}{3}\begin{bmatrix} \frac{1}{\lambda_0\lambda_1\lambda_2} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}. \tag{3}$$

For the rest of the paper, the specialization

$$\lambda_k = e^{\frac{2\pi i k}{3}} \tag{4}$$

will be fixed. All homogeneous rational functions of degree 0 in the weights become constants after the specialization (4).

0.4. **Holomorphic anomaly for** $[\mathbb{C}^3/\mathbb{Z}_3]$**.** The holomorphic anomaly equation has origins in B-model physics. An interpretation of the B-model invariants in terms of stable quotients invariants and a systematic study of the holomorphic anomaly for $K\mathbb{P}^2$ was given in [24]. By the crepant resolution philosophy, parallel holomorphic anomaly equations must hold for $[\mathbb{C}^3/\mathbb{Z}_3]$. We state here the precise form of the holomorphic anomaly equations for $[\mathbb{C}^3/\mathbb{Z}_3]$ as predicted by [2].

Define the orbifold Gromov-Witten potential of $[\mathbb{C}^3/\mathbb{Z}_3]$ for $g \geq 2$ by

$$\mathcal{F}_g^{[\mathbb{C}^3/\mathbb{Z}_3]} \;=\; \sum_{d=0}^{\infty} \frac{\Theta^d}{d!} \int_{[\overline{M}^{\mathsf{orb}}_{g,d}([\mathbb{C}^3/\mathbb{Z}^3],0)]^{\mathsf{T},vir}} \prod_{i=1}^{d} \mathrm{ev}_i^*(\phi_1)\,, \tag{5}$$

where $\phi_1 \in H^2_{\mathsf{orb}}([\mathbb{C}^3/\mathbb{Z}_3])$ is the basis element of degree 2. The integral on the right side of (5), defined by residues with respect to localization

[3] Cohomology will always be taken here with $\mathbb{C}$-coefficients.

by T, is a symmetric homogeneous rational function with $\mathbb{Q}$-coefficients of degree 0 in the weights $\lambda_0, \lambda_1, \lambda_2$. Hence, after the specialization (4),

$$\mathcal{F}_g^{[\mathbb{C}^3/\mathbb{Z}_3]} \in \mathbb{Q}[[\Theta]]\,.$$

The definition of the potential in genus 0 and 1 requires insertions and will be discussed in Section 1.

From the I-function[4] for $[\mathbb{C}^3/\mathbb{Z}^3]$, we obtain

$$I_1^{[\mathbb{C}^3/\mathbb{Z}^3]}(\theta) = \sum_{n\geq 0} \frac{(-1)^{3n}\theta^{3n+1}}{(3n+1)!} \left(\frac{\Gamma(n+\frac{1}{3})}{\Gamma(\frac{1}{3})}\right)^3 . \tag{6}$$

Define the mirror map $T(\theta)$ by

$$T(\theta) = I_1^{[\mathbb{C}^3/\mathbb{Z}^3]}(\theta)\,. \tag{7}$$

In order to state the holomorphic anomaly equations, we require the following additional series in θ:

$$\begin{aligned}
L(\theta) &= -\theta(1+\frac{\theta^3}{27})^{-\frac{1}{3}} = -\theta + \frac{\theta^4}{81} - \frac{2\theta^7}{6561} + \frac{14\theta^{10}}{1594323} + \dots\,, \\
C_1(\theta) &= \theta\frac{d}{d\theta}I_1^{[\mathbb{C}^3/\mathbb{Z}^3]} = \theta - \frac{\theta^4}{162} + \frac{4\theta^7}{32805} + \dots\,, \\
A_2(\theta) &= \frac{1}{L^3}\left(3\frac{\theta\frac{d}{d\theta}C_1}{C_1} - 3 - \frac{L^3}{18}\right) = \frac{\theta^3}{4860} - \frac{41\theta^6}{472392} + \dots\,.
\end{aligned}$$

The ring $\mathbb{C}[L^{\pm 1}] = \mathbb{C}[L, L^{-1}]$ will play a basic role. Consider the free polynomial rings in the variables A_2 and C_1^{-1} over $\mathbb{C}[L^{\pm 1}]$,

$$\mathbb{C}[L^{\pm 1}][A_2]\,, \quad \mathbb{C}[L^{\pm 1}][A_2, C_1^{-1}]\,. \tag{8}$$

There are canonical maps

$$\mathbb{C}[L^{\pm 1}][A_2] \to \mathbb{C}[[\theta]]\,, \quad \mathbb{C}[L^{\pm 1}][A_2, C_1^{-1}] \to \mathbb{C}((\theta)) \tag{9}$$

given by assigning the above defined series $A_2(\theta)$ and $C_1^{-1}(\theta)$ to the variables A_2 and C_1^{-1} respectively. We may therefore consider elements of the rings (8) *either* as free polynomials in the variables A_2 and C_1^{-1} *or* as series in θ.

Let $F(\theta) \in \mathbb{C}[[\theta]]$ be a series in θ. When we write

$$F(\theta) \in \mathbb{C}[L^{\pm 1}][A_2]\,,$$

we mean there is a canonical lift $F \in \mathbb{C}[L^{\pm 1}][A_2]$ for which

$$F \mapsto F(\theta) \in \mathbb{C}[[\theta]]$$

[4]The formula for the I-function is *after* the specialization (4). The full I-function and the series I_i are defined in Section 1.

under the map (9). The symbol F *without the argument* θ is the lift. The notation

$$F(\theta) \in \mathbb{C}[L^{\pm 1}][A_2, C_1^{-1}]$$

is parallel.

Since the holomorphic anomaly equations originate in the B-model, we will consider the orbifold Gromov-Witten potential $\mathcal{F}_g^{[\mathbb{C}^3/\mathbb{Z}_3]}$ as a series in θ via the mirror map (7),

$$\Theta = T(\theta)\,.$$

The potential $\mathcal{F}_g^{[\mathbb{C}^3/\mathbb{Z}_3]}$ viewed as a series in θ will be connected to the stable quotients series of $K\mathbb{P}^2$.

Theorem 1. *The orbifold Gromov-Witten potentials of* $[\mathbb{C}^3/\mathbb{Z}^3]$ *satisfy:*

(i) $\mathcal{F}_g^{[\mathbb{C}^3/\mathbb{Z}_3]}(\theta) \in \mathbb{C}[L^{\pm 1}][A_2]$ *for* $g \geq 2$,

(ii) $\mathcal{F}_g^{[\mathbb{C}^3/\mathbb{Z}_3]}$ *is of degree at most* $3g-3$ *with respect to* A_2,

(iii) $\frac{\partial^k \mathcal{F}_g^{[\mathbb{C}^3/\mathbb{Z}_3]}}{\partial T^k}(\theta) \in \mathbb{C}[L^{\pm 1}][A_2, C_1^{-1}]$ *for* $g \geq 1$ *and* $k \geq 1$,

(iv) $\frac{\partial^k \mathcal{F}_g^{[\mathbb{C}^3/\mathbb{Z}_3]}}{\partial T^k}$ *is homogeneous of degree* k *with respect to* C_1^{-1}.

Theorem 2. *The holomorphic anomaly equations for the orbifold Gromov-Witten invariants of* $[\mathbb{C}^3/\mathbb{Z}^3]$ *hold for* $g \geq 2$*:*

$$\frac{1}{C_1^2}\frac{\partial \mathcal{F}_g^{[\mathbb{C}^3/\mathbb{Z}_3]}}{\partial A_2} = \frac{1}{2}\sum_{i=1}^{g-1}\frac{\partial \mathcal{F}_{g-i}^{[\mathbb{C}^3/\mathbb{Z}_3]}}{\partial T}\frac{\partial \mathcal{F}_i^{[\mathbb{C}^3/\mathbb{Z}_3]}}{\partial T} + \frac{1}{2}\frac{\partial^2 \mathcal{F}_{g-1}^{[\mathbb{C}^3/\mathbb{Z}_3]}}{\partial T^2}\,.$$

The derivative of the lift $\mathcal{F}_g^{[\mathbb{C}^3/\mathbb{Z}_3]}$ with respect to A_2 in the holomorphic anomaly equation of Theorem 2 is well-defined since

$$\mathcal{F}_g^{[\mathbb{C}^3/\mathbb{Z}_3]} \in \mathbb{C}[L^{\pm 1}][A_2]$$

by Theorem 1 part (i). By Theorem 1 parts (ii) and (iii),

$$\frac{\partial \mathcal{F}_{g-i}^{[\mathbb{C}^3/\mathbb{Z}_3]}}{\partial T}\,,\ \frac{\partial \mathcal{F}_i^{[\mathbb{C}^3/\mathbb{Z}_3]}}{\partial T}\,,\ \frac{\partial^2 \mathcal{F}_{g-1}^{[\mathbb{C}^3/\mathbb{Z}_3]}}{\partial T^2}\ \in\ \mathbb{C}[L^{\pm 1}][A_2, C_1^{-1}]$$

are all of degree 2 in C_1^{-1}. Hence, the holomorphic anomaly equation of Theorem 2 may be viewed as holding in $\mathbb{C}[L^{\pm 1}][A_2]$ since the factors of C_1^{-1} on the left and right sides cancel. The holomorphic anomaly equations here for $[\mathbb{C}^3/\mathbb{Z}_3]$ are exactly as presented in [2, (4.27)] via B-model physics.

Theorem 2 determines $\mathcal{F}_g^{[\mathbb{C}^3/\mathbb{Z}_3]} \in \mathbb{C}[L^{\pm 1}][A_2]$ uniquely as a polynomial in A_2 up to a constant term in $\mathbb{C}[L^{\pm 1}]$. In fact, the degree of the constant term can be bounded (as will be seen in the proof of Theorem 2). So Theorem 2 determines $\mathcal{F}_g^{[\mathbb{C}^3/\mathbb{Z}_3]}$ from the lower genus theory together with a finite amount of data.

0.5. **Crepant resolution correspondence.** We start by defining a the polynomial ring over $\mathbb{C}[L^{\pm}]$ in a new variable X,

$$\mathcal{A}^{[\mathbb{C}^3/\mathbb{Z}_3]} = \mathbb{C}[L^{\pm 1}][X]\,.$$

By setting

$$X = \frac{\theta \frac{d}{d\theta} C_1}{C_1} = \frac{1}{3} L^3 A_2 + 1 + \frac{L^3}{54}\,,$$

we obtain an isomorphism

$$\mathcal{A}^{[\mathbb{C}^3/\mathbb{Z}_3]} \cong \mathbb{C}[L^{\pm 1}][A_2]\,.$$

As a series in θ,

$$X = 1 - \frac{\theta^3}{54} + \frac{\theta^6}{1620} + \dots\,.$$

Then, by Theorem 1 part (i), we have

$$\mathcal{F}_g^{[\mathbb{C}^3/\mathbb{Z}_3]} \in \mathcal{A}^{[\mathbb{C}^3/\mathbb{Z}_3]}\,.$$

To state the crepant resolution correspondence, we require results from our study of $K\mathbb{P}^2$ in [24]. The following series in q were defined in [24, Section 0.4]:

$$\begin{aligned}
L^{K\mathbb{P}^2}(q) &= (1+27q)^{-\frac{1}{3}} = 1 - 9q + 162q^2 + \dots\,,\\
C_1^{K\mathbb{P}^2}(q) &= q\frac{d}{dq} I_1^{K\mathbb{P}^2} = 1 - 6q + 90q^2 + \dots\,,\\
X^{K\mathbb{P}^2}(q) &= \frac{q\frac{d}{dq} C_1^{K\mathbb{P}^2}}{C_1^{K\mathbb{P}^2}} = -6q + 144q^2 + \dots\,.
\end{aligned}$$

Denote the ring generated by $X^{K\mathbb{P}^2}$ over the base ring $\mathbb{C}[(L^{K\mathbb{P}^2})^{\pm 1}]$ by

$$\mathcal{A}^{K\mathbb{P}^2} = \mathbb{C}[(L^{K\mathbb{P}^2})^{\pm 1}][X^{K\mathbb{P}^2}]\,.$$

Theorem 3. ([24, Theorem 1]) *For $g \geq 2$, the stable quotients potential $\mathcal{F}_g^{K\mathbb{P}^2}$ satisfies*

$$\mathcal{F}_g^{K\mathbb{P}^2} \in \mathcal{A}^{K\mathbb{P}^2}\,.$$

Our crepant resolution correspondence is based upon a simple ring homomorphism

$$\mathsf{P} : \mathcal{A}^{K\mathbb{P}^2} \to \mathcal{A}^{[\mathbb{C}^3/\mathbb{Z}_3]} \tag{10}$$

defined by

$$\mathsf{P}(L^{K\mathbb{P}^2}) = -\frac{L}{3}, \quad \mathsf{P}(X^{K\mathbb{P}^2}) = -\frac{X}{3}.$$

Theorem 4. *For $g \geq 2$, a crepant resolution correspondence holds:*

$$\mathcal{F}_g^{[\mathbb{C}^3/\mathbb{Z}_3]} = \mathsf{P}(\mathcal{F}_g^{K\mathbb{P}^2}).$$

Theorem 4 is stated on the B-model side since we use the variables θ and q. By the mirror maps on both sides, Theorem 4 is also a direct relationship between the Gromov-Witten theories. Knowledge of one side easily determines the other. A parallel statement for genus 0 and 1 (requiring insertions for stability) is presented in Section 4. A different approach to the crepant resolution correspondence for $[\mathbb{C}^3/\mathbb{Z}_3]$ will appear in the upcoming paper [13].

Theorem 4 concerns the higher genus potentials *after* specializing the equivariant parameters by

$$\lambda_k = e^{\frac{2\pi i k}{3}}$$

on both the $[\mathbb{C}^3/\mathbb{Z}_3]$ and $K\mathbb{P}^2$ sides. In fact, because of compactness[5] of the moduli spaces, all coefficients of $\mathcal{F}_g^{[\mathbb{C}^3/\mathbb{Z}_3]}$ of positive degree in Θ are constant *before* specialization of the equivariant parameters. Similarly, all coefficients of $\mathcal{F}_g^{K\mathbb{P}^2}$ of positive degree in q are constant[6] *before* specialization. So our specialization of the equivariant parameters affects *only* the leading terms of $\mathcal{F}_g^{[\mathbb{C}^3/\mathbb{Z}_3]}$ and $\mathcal{F}_g^{K\mathbb{P}^2}$.

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[5]In the presence of orbifold markings on the domain curve, the map must factor through $B\mathbb{Z}_3$.

[6]In positive degree, the map must have image in the 0-section $\mathbb{P}^2$.

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1. Orbifold Gromov-Witten invariants of $[\mathbb{C}^3/\mathbb{Z}_3]$

Let $\phi_{a_1}, \ldots, \phi_{a_n} \in H^*_{\mathsf{T},\mathsf{orb}}([\mathbb{C}^3/\mathbb{Z}_3])$. We define the Gromov-Witten potential by

$$\mathcal{F}^{[\mathbb{C}^3/\mathbb{Z}_3]}_{g,n}(\phi_{a_1}, \ldots, \phi_{a_n}) = \sum_{d=0}^{\infty} \frac{\Theta^d}{d!} \int_{[\overline{M}^{\mathsf{orb}}_{g,n+d}([\mathbb{C}^3/\mathbb{Z}_3],0)]^{vir}} \prod_{k=1}^{n} \mathrm{ev}_i^*(\phi_{a_k}) \prod_{i=n+1}^{n+d} \mathrm{ev}_i^*(\phi_1)\,. \tag{11}$$

For the positive coefficients of Θ, the stable map factors through

$$B\mathbb{Z}_3 \subset [\mathbb{C}^3/\mathbb{Z}_3]$$

since there are orbifold markings on the domain curves. For the constant terms, the integrals on the right side of (11) are defined via T-equivariant residues. If the pair $(g, n+d)$ is not in the stable range,

$$2g - 2 + n + d > 0\,,$$

the moduli space $\overline{M}^{\mathsf{orb}}_{g,n+d}([\mathbb{C}^3/\mathbb{Z}_3], 0)$ is empty and the corresponding term in (11) vanishes. We will also use the standard double bracket notation

$$\mathcal{F}^{[\mathbb{C}^3/\mathbb{Z}_3]}_{g,n}(\phi_{a_1}, \ldots, \phi_{a_n}) = \langle\langle\, \phi_{a_1}, \ldots, \phi_{a_n} \,\rangle\rangle^{[\mathbb{C}^3/\mathbb{Z}_3]}_{g,n}\,.$$

For a beautiful introduction to the geometry of stable maps to orbifolds and the Gromov-Witten theory of $[\mathbb{C}^3/\mathbb{Z}_3]$, we refer the reader to [3, Section 1].

The small J-function of $[\mathbb{C}^3/\mathbb{Z}_3]$ is defined by

$$J^{[\mathbb{C}^3/\mathbb{Z}_3]}(\Theta) = \phi_0 + \frac{\Theta\phi_1}{z} + \sum_{i=0}^{2} \langle\langle \frac{\phi_i}{z(z-\psi)} \rangle\rangle^{[\mathbb{C}^3/\mathbb{Z}_3]}_{0,1} \phi^i\,.$$

Here, ϕ^0, ϕ^1, ϕ^2 is the basis of $H^*_{\mathsf{T},\mathsf{orb}}([\mathbb{C}^3/\mathbb{Z}_3])$ dual ϕ_0, ϕ_1, ϕ_2 with respect to the pairing (3). After the specialization (4), we have

$$\phi^0 = 3\,\phi_0\,, \quad \phi^1 = 3\,\phi_2\,, \quad \phi^2 = 3\,\phi_1\,.$$

The small I-function of $[\mathbb{C}^3/\mathbb{Z}_3]$ is defined in [10, Section 6.3] by

$$I^{[\mathbb{C}^3/\mathbb{Z}_3]}(\theta) = \sum_{i=0}^{\infty} \frac{\theta^i}{z^i i!} \prod_{\substack{0 \le k < \frac{i}{3} \\ [k] = [\frac{i}{3}]}} \left(1 - (kz)^3\right) \phi_i\,. \tag{12}$$

The elements ϕ_i occur above for all non-negative integers i via the conventions

$$\phi_0 = \phi_3 = \phi_6 = \ldots = \phi_{3k} = \ldots\,, \qquad \phi_1 = \phi_4 = \phi_7 = \ldots = \phi_{3k+1} = \ldots\,,$$

$$\phi_2 = \phi_5 = \phi_8 = \ldots = \phi_{3k+2} = \ldots\,.$$

There are no positive powers of z on the side of (12). Moreover, the coefficient of z^{-i} always has basis vector ϕ_i. Hence, we can define the functions $I_i(\theta)$ by

$$I^{[\mathbb{C}^3/\mathbb{Z}_3]}(\theta) = \sum_{i=0}^{\infty} \frac{I_i(\theta)}{z^i}\phi_i\,.$$

In particular, I_1 is given (6).

The I-function satisfies following Picard-Fuchs equation:

$$\left[\frac{(z\frac{\theta\partial}{\partial\theta})^3}{27} + 1 - \theta^{-3}\left(z\frac{\theta\partial}{\partial\theta}\right)\left(z\frac{\theta\partial}{\partial\theta} - z\right)\left(z\frac{\theta\partial}{\partial\theta} - 2z\right)\right] I^{[\mathbb{C}^3/\mathbb{Z}_3]}(\theta) = 0\,.$$

Theorem 5. (Coates-Corti-Iritani-Tseng [10, Section 6.3]) *After the change of variables*

$$\Theta(\theta) = I_1(\theta)\,,$$

the following mirror result holds:

$$J^{[\mathbb{C}^3/\mathbb{Z}_3]}(\Theta(\theta)) = I^{[\mathbb{C}^3/\mathbb{Z}_3]}(\theta)\,.$$

2. Semisimple Frobenius manifolds

2.1. **Frobenius manifolds.** We briefly review here Givental's formula for the higher genus theory associated to a semisimple Frobenius manifold. We refer the reader to [17, 23, 27, 28, 33] for more leisurely treatments.

Definition 6. *A Frobenius manifold* $(\mathbf{M}, \mathbf{g}, \bullet, A, \mathbf{1})$ *satisfies the following conditions:*

(i) $\mathbf{g}$ *is Riemmanian metric on* $\mathbf{M}$*,*
(ii) $\bullet$ *is commutative and associative product on* $T\mathbf{M}$*,*
(iii) A *is a symmetric tensor,*

$$A : T\mathbf{M} \otimes T\mathbf{M} \otimes T\mathbf{M} \to \mathcal{O}_{\mathbf{M}}\,,$$

(iv) $\mathbf{g}(X \bullet Y, Z) = A(X, Y, Z)$*,*
(v) $\mathbf{1}$ *is a* $\mathbf{g}$*-flat unit vector field.*

For us, $\mathbf{M}$ will be a complex manifold of dimension m. The metric $\mathbf{g}$ will be symmetric and non-degenerate, but the positivity condition of a Riemmanian metric will be dropped (and is not necessary for the theory).

2.2. **Flat coordinates.** Let p be a point of $\mathbf{M}$. As $\mathbf{g}$ is flat, we can find flat coordinates $(t^0, t^1, \ldots, t^{m-1})$ in a neighborhood of p. Let

$$\phi_i = \frac{\partial}{\partial t^i}$$

denote the corresponding flat vector fields. By convention,

$$\mathbf{1} = \phi_0 \,.$$

2.3. **Semisimple points and canonical coordinates.** A point

$$p \in \mathbf{M}$$

is *semisimple* if the tangent algebra $(T_p\mathbf{M}, \bullet, \mathbf{1})$ is semisimple. For a semisimple point p, we can find *canonical coordinates*

$$(u^0, u^1, \ldots, u^{m-1})$$

in a neighborhood of p for which the corresponding vector fields

$$e_i = \frac{\partial}{\partial u^i}$$

are orthogonal idempotents:

$$e_i \bullet e_j = \delta_{ij} e_i$$

and $\mathbf{g}(e_i, e_j) = 0$ for $i \neq j$.

A *normalized canonical basis* $\{\widetilde{e}_i\}$ is constructed by

$$\widetilde{e}_i = \mathbf{g}(e_i, e_i)^{-\frac{1}{2}} e_i \,.$$

The normalized coordinates require choices of square roots (but the final formulas are independent of these choices).

Let $\mathbf{\Psi}$ be the transition matrix from the basis $\{\phi_i\}$ to the basis $\{\widetilde{e}_\alpha\}$. By the orthonormality of $\widetilde{e}_\alpha$, the elements of $\mathbf{\Psi}$ are

$$\mathbf{\Psi}_{\alpha i} = g(\widetilde{e}_\alpha, \phi_i) \,.$$

2.4. **Fundamental solutions and the R-matrix.** We define

$$\mathbf{R}(z) = \sum_{k=0}^{\infty} \mathbf{R}_k z^k$$

by following flatness equation:

$$zd\mathbf{\Psi}^{-1}\mathbf{R} + z\mathbf{\Psi}^{-1}d\mathbf{R} + \mathbf{\Psi}^{-1}\mathbf{R}d\mathbf{U} - \mathbf{\Psi}^{-1}d\mathbf{U}\mathbf{R}, \tag{13}$$

where $\mathbf{U}$ is the diagonal matrix with coefficients

$$\mathbf{U} = \mathrm{Diag}(u^0, u^1, \ldots, u^{m-1})\,.$$

The $\mathbf{R}$-matrix $\mathbf{R}(z)$ is uniquely determined by (13) and the *symplectic condition*,

$$\mathbf{R}(z) \cdot \mathbf{R}^t(-z) = \mathrm{Id}\,, \tag{14}$$

up to right multiplication by a constant matrix

$$\exp\left(\sum_{k\geq 1} \mathbf{a}_{2k-1} z^{2k-1}\right)\,,$$

where the matrices $\mathbf{a}_{2k-1}$ are diagonal with constant coefficients

$$\mathbf{a}_{2k-1} = \mathrm{Diag}[a^0_{0,2k-1}, a^1_{1,2k-1}, \ldots, a^{m-1}_{m-1,2k-1}]\,.$$

The $\mathbf{R}$-matrix determines an endomorphism

$$\mathbf{R}(z) \in \mathrm{End}(T_p\mathbf{M})[[z]]$$

defined in the basis $\{\widetilde{e}_i\}$. Given a vector $v \in T_p\mathbf{M}$,

$$\mathbf{R}(z)v \in T_p\mathbf{M}[[z]]\,.$$

2.5. **Higher genus potentials.**

2.5.1. *Topological field theory.* Let $p \in \mathbf{M}$ be a semisimple point, and let $\Omega_{g,n}$ be the *Topological Field Theory* on $T_p\mathbf{M}$ defined by

$$\mathbf{g}(u \bullet v, w) = \Omega_{0,3}(u, v, w).$$

The CohFT axioms easily yield:

$$\Omega_{g,n}(\widetilde{e}_{i_1}, \widetilde{e}_{i_2}, \ldots, \widetilde{e}_{i_n}) = \begin{cases} \sum_{j=0}^{m-1} \mathbf{g}(e_j, e_j)^{1-g} & \text{if } n = 0\,, \\ \mathbf{g}(e_{i_1}, e_{i_1})^{-\frac{2g-2+n}{2}} & \text{if } i_1 = i_2 = \cdots = i_n\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

In Section 2.5, for the higher genus potential, we will use the basis $\{\widetilde{e}_i\}$ of $T_p\mathbf{M}$ in all formulas.

2.5.2. *Potentials.* Let $\mathsf{G}_{g,n}$ be the finite set of stable graphs of genus g with n legs. Givental's higher genus (cycle valued) potential functions at $p \in \mathbf{M}$ are defined by the following formula

$$\mathcal{F}_{g,n}(v_1, v_2, \ldots, v_n) = \sum_{\Gamma \in \mathsf{G}_{g,n}} \frac{1}{\mathrm{Aut}(\Gamma)} \mathrm{Cont}_\Gamma$$

for $v_i \in T_p\mathbf{M}$. The contributions Cont_Γ are determined by

$$\mathrm{Cont}_\Gamma = \xi_{\Gamma *}\left(\prod_{v \in \mathrm{Vert}(\Gamma)} \sum_{k=0}^{\infty} \frac{1}{k!} \pi_* \Omega_{g(v),n(v)+k} \right)$$

where ξ_Γ is the standard map of the stratum indexed by Γ,

$$\xi_\Gamma : \overline{M}_\Gamma \to \overline{M}_{g,n}\,,$$

π is the forgetful map at the vertex dropping the last k markings,

$$\pi : \overline{M}_{g(v),n(v)+k} \to \overline{M}_{g(v),n(v)}\,,$$

and the insertions in the arguments of $\prod_v \Omega_{g(v),n(v)+k}$ are specified by the following rules:

- For the insertion corresponding to the i^{th} original marking, place $\mathbf{R}^{-1}(\psi_i)v_i$.
- For each pair of insertions corresponding to an edge, place the bivector
$$\sum_{ij} \left[\frac{\mathbf{g}^{-1} - \mathbf{R}^{-1}(\psi)\mathbf{g}^{-1}(\mathbf{R}^{-1}(\psi'))^t}{\psi + \psi'} \right]_{ij} \widetilde{e}_i \otimes \widetilde{e}_j \in V^{\otimes 2}[[\psi, \psi']]\,,$$
well-defined by symplectic property $\mathbf{R}$.

  Here, $\mathbf{g}$ and $\mathbf{g}^{-1}$ denote the matrices obtained from the metric in the *normalized canonical basis.* In fact, both are the identity matrix.
- For each additional insertion at a vertex, place
$$T(\psi) = \psi \left(\mathrm{Id} - \mathbf{R}^{-1}(\psi) \right) \phi_0\,.$$

2.6. **Givental-Teleman classification.** Let Λ be a semisimple CohFT with unit and state space $(V, \mathbf{g}, \mathbf{1})$. The genus 0 part of Λ determines a Frobenius manifold structure on the complex vector space V for which $0 \in V$ is a semisimple point.

The Givental-Teleman classification [17, 33] states: *there exists a unique* $\mathbf{R}$*-matrix for the Frobenius manifold* V *for which Givental's potential (as defined in Section 2.5.2) equals the CohFT evaluation*

$$\Lambda_{g,n}(v_1, v_2, \ldots, v_n) \in H^*(\overline{M}_{g,n})$$

for all g *and* n *in the stable range.*

3. Genus 0 theory for $[\mathbb{C}^3/\mathbb{Z}_3]$

3.1. **Summary.** We review the genus 0 orbifold Gromov-Witten theory of $[\mathbb{C}^3/\mathbb{Z}_3]$. We follow the notations and conventions of [23]. The main difficult result that we will use in the genus 0 theory is the mirror transformation of Theorem 5 proven by Coates-Corti-Iritani-Tseng [10]. Similar computations appeared in [20] for the study of genus one FJRW invariants associated to the quintic threefold.

3.2. **Frobenius structure.** The orbifold Gromov-Witten theory determines an Frobenius manifold structure[7] on $H^*_{\mathsf{T},\mathsf{orb}}([\mathbb{C}^3/\mathbb{Z}_3])$ viewed with flat basis ϕ_0, ϕ_1, ϕ_2 and with specialization (4). The inner product and the quantum product are as follows.

- *Inner product.* In the flat basis

$$\{\phi_0, \phi_1, \phi_2\}\,, \tag{15}$$

  the inner product $\mathbf{g}$, given by

$$\mathbf{g} = \frac{1}{3}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \tag{16}$$

  has already appeared in (3).
- *Potential.* The full genus 0 Gromov-Witten potential is a function of the coordinates $\{t_0, t_1, t_2\}$ in the flat basis (15) and of the additional variable Θ,

$$\mathcal{F}_0^{[\mathbb{C}^3/\mathbb{Z}_3]}(t,\Theta) \;=\; \sum_{n=0}^{\infty}\sum_{d=0}^{\infty}\int_{[\overline{M}^{\mathsf{orb}}_{g,n+d}([\mathbb{C}^3/\mathbb{Z}^3],0)]^{\mathsf{T},vir}} \frac{1}{n!d!}\prod_{i=1}^{n}\mathrm{ev}_i^*(\gamma)\prod_{i=n+1}^{n+d}\mathrm{ev}_i^*(\Theta\phi_1)\,,$$

  where $\gamma = \sum_{i=0}^{2} t_i\phi_i$. The potential satisfies

$$\frac{\partial}{\partial t_1}\mathcal{F}_0^{[\mathbb{C}^3/\mathbb{Z}_3]} = \frac{\partial}{\partial \Theta}\mathcal{F}_0^{[\mathbb{C}^3/\mathbb{Z}_3]}\,.$$

[7] The Frobenius manifold here is over the ring $\mathbb{C}[[\Theta]]$.

- *Quantum product.* The 6 products at $0 \in H^*_{\mathsf{T},\mathsf{orb}}([\mathbb{C}^3/\mathbb{Z}_3])$ are

$$\begin{aligned}
\phi_0 \bullet \phi_0 &= \phi_0 \,, \\
\phi_0 \bullet \phi_1 &= \phi_1 \,, \\
\phi_0 \bullet \phi_2 &= \phi_2 \,, \\
\phi_1 \bullet \phi_1 &= -\frac{L^3}{C_1^3}\phi_2 \,, \\
\phi_1 \bullet \phi_2 &= \phi_0 \,, \\
\phi_2 \bullet \phi_2 &= -\frac{C_1^3}{L^3}\phi_1 \,.
\end{aligned}$$

Since the quantum product is at $0 \in H^*_{\mathsf{T},\mathsf{orb}}([\mathbb{C}^3/\mathbb{Z}_3])$, only the variable Θ appears in the functions on the right side of the above formulas. In fact, both L and C are defined in Section 0.4 in terms of θ, so the Θ dependence appears only after inverting the mirror map $\Theta(\theta)$. We will give a proof of the quantum product in the following subsection.

3.3. **Calculation of the quantum product.** To compute the quantum product of $[\mathbb{C}^3/\mathbb{Z}_3]$, we require the 3-point functions in genus 0.

Lemma 7. *The nonvanishing* 3*-point function in genus 0 are:*

$$\langle\langle \phi_0, \phi_0, \phi_0 \rangle\rangle_{0,3}^{[\mathbb{C}^3/\mathbb{Z}_3]} = \frac{1}{3} \,, \qquad \langle\langle \phi_0, \phi_1, \phi_2 \rangle\rangle_{0,3}^{[\mathbb{C}^3/\mathbb{Z}_3]} = \frac{1}{3} \,,$$

$$\langle\langle \phi_1, \phi_1, \phi_1 \rangle\rangle_{0,3}^{[\mathbb{C}^3/\mathbb{Z}_3]} = -\frac{1}{3}\frac{L^3}{C_1^3} \,, \qquad \langle\langle \phi_2, \phi_2, \phi_2 \rangle\rangle_{0,3}^{[\mathbb{C}^3/\mathbb{Z}_3]} = -\frac{1}{3}\frac{C_1^3}{L^3} \,.$$

For other choices of insertions, the 3*-point functions in genus 0 vanish.*

Proof. By [10], the I-function $I^{[\mathbb{C}^3/\mathbb{Z}_3]}$ lies on the Lagrangian[8] cone $\mathcal{L}^{[\mathbb{C}^3/\mathbb{Z}_3]}$ encoding the genus 0 Gromov-Witten theory of $[\mathbb{C}^3/\mathbb{Z}^3]$. By standard properties of the Lagrangian cone, we obtain the following results, see for example [22, 31]:

$$\begin{aligned}
\mathbb{S}^{[\mathbb{C}^3/\mathbb{Z}_3]}(\Theta(\theta), z)(\phi_0) &= I^{[\mathbb{C}^3/\mathbb{Z}_3]} \,, \\
\mathbb{S}^{[\mathbb{C}^3/\mathbb{Z}_3]}(\Theta(\theta), z)(\phi_1) &= \frac{z\mathsf{D}\mathbb{S}^{[\mathbb{C}^3/\mathbb{Z}_3]}(\phi_0)}{C_1} \,, \\
\mathbb{S}^{[\mathbb{C}^3/\mathbb{Z}_3]}(\Theta(\theta), z)(\phi_2) &= \frac{z\mathsf{D}\mathbb{S}^{[\mathbb{C}^3/\mathbb{Z}_3]}(\phi_1)}{C_2} \,.
\end{aligned} \tag{17}$$

Here, the $\mathbb{S}$-operator for $[\mathbb{C}^3/\mathbb{Z}_3]$ is defined as usual by

$$\mathbb{S}^{[\mathbb{C}^3/\mathbb{Z}_3]}(\Theta, z)(\gamma) = \sum_i \phi^i \langle\langle \frac{\phi_i}{z-\psi}, \gamma \rangle\rangle_{0,2}^{[\mathbb{C}^3/\mathbb{Z}_3]} \,, \quad \text{for}\, \gamma \in H^*_{\mathsf{T},\mathsf{orb}}([\mathbb{C}^3/\mathbb{Z}_3]) \,,$$

[8] See [12, 18] for the definition of the Lagrangian cone.

The differential operator $\mathsf{D} = \theta \frac{d}{d\theta}$ acts on $\mathbb{S}$ via variable change $\Theta(\theta)$. The functions

$$C_0 = 1\,, \quad C_1 = \mathsf{D} I_1\,, \quad C_2 = \mathsf{D}\left(\frac{\mathsf{D} I_2}{C_1}\right)\,.$$

appear on the right side of (17).

Using the methods of [35, Theorem 2], we obtain

$$C_1^2 C_2 = -L^3\,.$$

Observe that the I-function has following expansion,

$$I^{[\mathbb{C}^3/\mathbb{Z}_3]} = \phi_0 + \frac{I_1\phi_1}{z} + \frac{I_2\phi_2}{z^2} + \mathsf{O}(\frac{1}{z^3})\,.$$

Then, equation (17) immediately yields

$$\langle\langle \phi_0, \phi_1, \phi_2 \rangle\rangle_{0,3}^{[\mathbb{C}^3/\mathbb{Z}_3]} = \frac{1}{3}\,,$$
$$\langle\langle \phi_1, \phi_1, \phi_1 \rangle\rangle_{0,3}^{[\mathbb{C}^3/\mathbb{Z}_3]} = \frac{1}{3}\frac{C_2}{C_1} = -\frac{1}{3}\frac{L^3}{C_1^3}\,.$$

By definition of the Frobenius structure,

$$\mathbf{g}(X \bullet Y, Z) = \langle\langle X, Y, Z \rangle\rangle_{0,3}^{[\mathbb{C}^3/\mathbb{Z}_3]}\,, \quad \text{for } X, Y, Z \in H^*_{\mathsf{T},\mathrm{orb}}([\mathbb{C}^3/\mathbb{Z}_3])\,.$$

The remaining two evaluations follow from the associativity of the quantum product. $\square$

3.4. **Canonical coordinates.** After normalizing the basis $\{\phi_0, \phi_1, \phi_2\}$ by

$$\widetilde{\phi}_0 = \phi_0\,, \quad \widetilde{\phi}_1 = -\frac{C_1}{L}\phi_1\,, \quad \widetilde{\phi}_2 = -\frac{L}{C_1}\phi_2\,, \tag{18}$$

we obtain the relation

$$\widetilde{\phi}_i \bullet \widetilde{\phi}_j = \widetilde{\phi}_{i+j}\,.$$

The quantum product at $0 \in H^*_{\mathsf{T},\mathsf{orb}}([\mathbb{C}^3/\mathbb{Z}_3])$ can then be checked to be semisimple with an idempotent basis,

$$e_\alpha \bullet e_\beta = \delta_{\alpha\beta} e_\alpha\,,$$

given by the formula

$$e_\alpha = \frac{1}{3}\sum_{i=0}^{3} \zeta^{-\alpha i}\widetilde{\phi}_i \quad \text{for} \quad \alpha = 0, 1, 2, \tag{19}$$

where $\zeta = e^{\frac{2\pi i}{3}}$ is a third root of unity. The normalized idempotents are

$$\widetilde{e}_\alpha = \frac{e_\alpha}{\sqrt{\mathbf{g}(e_\alpha, e_\alpha)}} = 3e_\alpha\,. \tag{20}$$

Equations (18)-(20) take place at the point $0 \in H^*_{\mathsf{T},\mathsf{orb}}([\mathbb{C}^3/\mathbb{Z}_3])$ of the Frobenius manifold and hence only depend upon the variable Θ.

Let $\{u^\alpha\}$ be the canonical coordinates associated to the above idempotent basis with constants fixed by

$$u^\alpha(t_i = 0, \Theta = 0) = 0\,. \tag{21}$$

Since $e_\alpha = \frac{\partial}{\partial u^\alpha}$, we have

$$\sum_{\alpha=1}^{3} e_\alpha \frac{du^\alpha}{dt_1} = \phi_1\,. \tag{22}$$

The standard convention for equations such as (22) is that the derivative $\frac{\partial}{\partial t_1}$ on the left side is taken *before* all the t_i are set to 0.

Lemma 8. *We have*

$$\frac{du^\alpha}{dt} = \zeta^\alpha\left(-\frac{L}{C_1}\right)$$

Proof. The result is a consequence of equations (19) and (22). □

Before restriction to the point $0 \in H^*_{\mathsf{T},\mathsf{orb}}([\mathbb{C}^3/\mathbb{Z}_3])$, the genus 0 potential $\mathcal{F}_0^{[\mathbb{C}^3/\mathbb{Z}_3]}$, the components of the idempotents in flat coordinates, and the canonical coordinates $\{u^\alpha\}$ all are functions of the variables t_0, t_1, t_2, Θ which are annihilated by the operator[9]

$$\frac{\partial}{\partial t_1} - \frac{\partial}{\partial \Theta}\,. \tag{23}$$

By the argument of [29, Section 3], the $\mathbf{R}$-matrix of the associated CohFT is also annihilated by (23).

[9]The proof is elementary starting with the annihilation of the potential $\mathcal{F}_0^{[\mathbb{C}^3/\mathbb{Z}_3]}$. The components of du^α in the basis $\{dt_i\}$ are eigenvalues of matrices with coefficients all annihilated by (23), so also annihilated by (23). Hence,

$$d\left(\frac{\partial u^\alpha}{\partial t_1} - \frac{\partial u^\alpha}{\partial \Theta}\right) = 0$$

and hence $\frac{\partial u^\alpha}{\partial t_1} - \frac{\partial u^\alpha}{\partial \Theta}$ must be a function $f^\alpha(\Theta)$ only of Θ. Then, we can find *unique canonical coordinates* (by shifting by the integral $\int f^\alpha(\Theta) d\Theta$ which satisfy (21) and are annihilated by (23).

Since the operator (23) annihilates u^α, we can rewrite Lemma 8 at the point $0 \in H^*_{\mathsf{T},\mathsf{orb}}([\mathbb{C}^3/\mathbb{Z}_3])$ using

$$\frac{du^\alpha}{dt} = \frac{du^\alpha}{d\Theta} = \frac{du^\alpha}{d\theta}\frac{d\theta}{d\Theta}\,.$$

We then obtain the equation

$$\frac{du^\alpha}{d\theta} = \zeta^\alpha(-L)\frac{1}{\theta}\,.$$

3.5. **Transition matrix.** The transition matrix $\mathbf{\Psi}$ from flat coordinate to normalized canonical basis is given by

$$\mathbf{\Psi}_{\alpha i} = g(\widetilde{e}_\alpha, \phi_i).$$

At the point $0 \in H^*_{\mathsf{T},\mathsf{orb}}([\mathbb{C}^3/\mathbb{Z}_3])$, we can calculate using (19):

$$\mathbf{\Psi} = \frac{1}{3}\begin{bmatrix} 1 & -\frac{L}{C_1} & -\frac{C_1}{L} \\ 1 & -\zeta\frac{L}{C_1} & -\zeta^2\frac{C_1}{L} \\ 1 & -\zeta^2\frac{L}{C_1} & -\zeta\frac{C_1}{L} \end{bmatrix}.$$

Viewed a functions of t_0, t_1, t_2, Θ, the coefficients of Θ are annihilated by (23).

3.6. **Fundamental solution matrix.** Consider the coefficient of z^k in the flatness equation (13). We will study the solutions along the line $\{t_0 = 0, t_2 = 0\}$, so we consider only the flatness equation with respect to the t_1 directional derivative in (13). Using the annihilation of all functions by (23) and the change of variable relation

$$\frac{\partial}{\partial\Theta} = \frac{\theta}{C_1}\frac{\partial}{\partial\theta}\,,$$

we obtain,

$$\mathbf{\Psi}\left(\frac{\theta\partial}{\partial\theta}\mathbf{\Psi}^{-1}\right)\mathbf{R}_{k-1} + \frac{\theta\partial}{\partial\theta}\mathbf{R}_{k-1} + \mathbf{R}_k\frac{\theta\partial}{\partial\theta}\mathbf{U} - \left(\frac{\theta\partial}{\partial\theta}\mathbf{U}\right)\mathbf{R}_k = 0\,, \tag{24}$$

or equivalently, in most useful form,

$$\frac{\theta\partial}{\partial\theta}\left(\mathbf{\Psi}^{-1}\mathbf{R}_{k-1}\right) + \left(\mathbf{\Psi}^{-1}\mathbf{R}_k\right)\frac{\theta\partial}{\partial\theta}\mathbf{U} - \mathbf{\Psi}^{-1}\left(\frac{\theta\partial}{\partial\theta}\mathbf{U}\right)\mathbf{\Psi}\left(\mathbf{\Psi}^{-1}\mathbf{R}_k\right) = 0\,.$$

We then restrict to the point $0 \in H^*_{\mathsf{T},\mathsf{orb}}([\mathbb{C}^3/\mathbb{Z}_3])$, so (24) and the second form, become equations purely of the variable θ.

From Lemma 8, we obtain

$$\frac{\theta\partial}{\partial\theta}\mathbf{U} = \begin{bmatrix} -L & 0 & 0 \\ 0 & -\zeta L & 0 \\ 0 & 0 & -\zeta^2 L \end{bmatrix}.$$

We also have

$$\mathbf{\Psi}^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ -\frac{C_1}{L} & -\zeta^2\frac{C_1}{L} & -\zeta\frac{C_1}{L} \\ -\frac{L}{C_1} & -\zeta\frac{L}{C_1} & -\zeta^2\frac{L}{C_1} \end{bmatrix}.$$

Let P^k_{ij} denote the (i,j) coefficient of the matrix $\mathbf{\Psi}^{-1}\mathbf{R}_k$ restricted to $0 \in H^*_{\mathsf{T},\mathsf{orb}}([\mathbb{C}^3/\mathbb{Z}_3])$. From the second form of (24), we obtain the following equations for $j = 0, 1, 2$:

$$\begin{aligned} \frac{\theta\partial}{\partial\theta}P^{k-1}_{0j} &= C_1 P^k_{2j} + LP^k_{0j}\zeta^j, \\ \frac{\theta\partial}{\partial\theta}P^{k-1}_{1j} &= C_1 P^k_{0j} + LP^k_{1j}\zeta^j, \\ \frac{\theta\partial}{\partial\theta}P^{k-1}_{2j} &= -\frac{L^3}{C_1^2}P^k_{1j} + LP^k_{2j}\zeta^j\,. \end{aligned} \tag{25}$$

3.7. **Generators and relations.** As before, let $\mathsf{D} = \theta\frac{\partial}{\partial\theta}$.

Lemma 9. *We have the following relation between L and $X = \frac{\mathsf{D}C_1}{C_1}$:*

$$\begin{aligned} \mathsf{D}L &= L\left(\frac{L^3}{27} + 1\right)\,, \\ X^2 - 3\frac{\mathsf{D}L}{L}X &+ 2\frac{\mathsf{D}L}{L} + \mathsf{D}X = 0\,. \end{aligned}$$

Proof. The first relation follows from the definition of L. The second relation follows from case $k = 2$ of (25). $\square$

By above result, we view the differential ring

$$\mathbb{C}[L^{\pm 1}][X, \mathsf{D}X, \mathsf{D}\mathsf{D}X, \ldots]$$

as simply the polynomial ring $\mathbb{C}[L^{\pm 1}][X]$.

The following normalizations will be convenient for us:

$$\begin{aligned} \widetilde{P}^k_{0j} &= P^k_{0j}\zeta^{kj} \\ \widetilde{P}^k_{1j} &= -\frac{L}{C_1\zeta^{2j}}P^k_{1j}\zeta^{kj} \\ \widetilde{P}^k_{2j} &= -\frac{C_1}{L\zeta^j}P^k_{2j}\zeta^{kj}\,, \quad k \geq 0, \quad j = 0, 1, 2. \end{aligned} \tag{26}$$

From (25), we can calculate $\widetilde{P}^k_{ij}$ explicitly with initial conditions

$$\tilde{P}^k_{ij}|_{\theta=0} = 0\,, \quad k \geq 1. \tag{27}$$

For example, for $j = 0, 1, 2$, we have

$$\begin{aligned}
\widetilde{P}^0_{0j} &= 1,\\
\widetilde{P}^1_{0j} &= \frac{L^2}{162},\\
\widetilde{P}^2_{0j} &= \frac{L}{81} + \frac{25}{52488}L^4,\\
\widetilde{P}^3_{0j} &= \frac{7}{4374}L^3 + \frac{1225}{25509168}L^6\,.
\end{aligned}$$

Using (25) and Lemma 9, we obtain the Lemma 10 below. Lemma 11 follows from an argument parallel to [35, Section 1].

Lemma 10. *For $j = 0, 1, 2$, we have:*

$$\begin{aligned}
\widetilde{P}^{k+1}_{2j} &= \widetilde{P}^{k+1}_{0j} - \frac{\mathsf{D}\widetilde{P}^k_{0j}}{L},\\
\widetilde{P}^{k+1}_{1j} &= \widetilde{P}^{k+1}_{2j} - \frac{\mathsf{D}\widetilde{P}^k_{2j}}{L} - \left(\frac{\mathsf{D}L}{L^2} - \frac{X}{L}\right)\widetilde{P}^k_{2j},\\
\widetilde{P}^{k+1}_{0j} &= \widetilde{P}^{k+1}_{1j} - \frac{\mathsf{D}\widetilde{P}^k_{1j}}{L} + \left(\frac{\mathsf{D}L}{L^2} - \frac{X}{L}\right)\widetilde{P}^k_{1j}\,.
\end{aligned} \tag{28}$$

Lemma 11. *We have $\widetilde{P}^k_{0j} \in \mathbb{C}[L^{\pm 1}]$.*

The following result is a direct consequence of Lemmas 10 and 11.

Lemma 12. *For all $k \geq 0$ and $j = 0, 1, 2$, we have*

$$\begin{aligned}
\widetilde{P}^k_{2j} &\in \mathbb{C}[L^{\pm 1}],\\
\widetilde{P}^k_{1j} &= \widetilde{Q}^k_{1j} + \frac{\widetilde{P}^{k-1}_{2j}}{L}X,
\end{aligned}$$

with $\widetilde{Q}^k_{1j} \in \mathbb{C}[L^{\pm 1}]$.

4. The holomorphic anomaly equations

4.1. **R-matrix.** Let $\widetilde{\mathsf{R}}^{[\mathbb{C}^3/\mathbb{Z}^3]}$ be the matrix whose z^k coefficient is the solution $\mathbf{R}_k$ of (24) with initial conditions[10]

$$\left[\widetilde{\mathsf{R}}^{[\mathbb{C}^3/\mathbb{Z}^3]}(z)\right]\Big|_{\theta=0} = \mathrm{Id}\,. \tag{29}$$

Define a new diagonal matrix

$$\mathsf{B}(z) = \mathrm{Diag}\left[B_0^{[\mathbb{C}^3/\mathbb{Z}_3]}(z), B_1^{[\mathbb{C}^3/\mathbb{Z}_3]}(z), B_2^{[\mathbb{C}^3/\mathbb{Z}_3]}(z)\right]$$

[10] The symplectic condition (14) is not imposed on (and not satisfied by) $\widetilde{\mathsf{R}}^{[\mathbb{C}^3/\mathbb{Z}^3]}$.

where for i=0,1,2,

$$B_i^{[\mathbb{C}^3/\mathbb{Z}_3]}(z) = \mathrm{Exp}\left(3\sum_{k=1}^{\infty}(-1)^{k+1}\frac{B_{3k+1}\left(\frac{\mathsf{Inv}(i)}{3}\right)}{3k+1}\frac{z^{3k}}{3k}\right).$$

Here, the involution $\mathsf{Inv} : \{0,1,2\} \to \{0,1,2\}$ is defined by

$$\mathsf{Inv}(0) = 0\,,\ \ \mathsf{Inv}(1) = 2\,,\ \ \mathsf{Inv}(2) = 1\,.$$

The Bernoulli polynomials $B_m(x)$ are defined by

$$\frac{te^{tx}}{e^t - 1} = \sum_{m\geq 0}\frac{B_m(x)t^m}{m!}\,.$$

For example,

$$B_0(x) = 1\,,\ B_1(x) = x - \frac{1}{2}\,,\ B_2(x) = x^2 - x + \frac{1}{6}\,.$$

Especially, $B_k(0)$ is the Bernoulli numbers.

Via the orbifold quantum Riemann-Roch theorem in [34, Section 4.2], we obtain the following result.

Proposition 13. *The true* $\mathbf{R}$*-matrix* $\mathsf{R}^{[\mathbb{C}^3/\mathbb{Z}_3]}$ *for the Gromov-Witten theory of* $[\mathbb{C}^3/\mathbb{Z}_3]$ *has the following form after restriction* $\theta = 0$*:*

$$\left[\mathsf{R}^{[\mathbb{C}^3/\mathbb{Z}_3]}(z)\right]_{ij}\Big|_{\theta=0} = (\mathbf{\Psi}|_{\theta=0})\cdot\mathsf{B}(z)\cdot(\mathbf{\Psi}|_{\theta=0})^{-1}\,.$$

Corollary 14. *The true* $\mathbf{R}$*-matrix* $\mathsf{R}^{[\mathbb{C}^3/\mathbb{Z}_3]}$ *for the Gromov-Witten theory of* $[\mathbb{C}^3/\mathbb{Z}_3]$ *in the normalized canonical basis is given by*

$$\mathsf{R}^{[\mathbb{C}^3/\mathbb{Z}_3]}(z) = (\mathbf{\Psi}|_{\theta=0})\cdot\mathsf{B}(z)\cdot(\mathbf{\Psi}|_{\theta=0})^{-1}\cdot\left[\widetilde{\mathsf{R}}^{[\mathbb{C}^3/\mathbb{Z}^3]}(z)\right]\,.$$

Proof. The coefficients of the matrices $\mathsf{R}^{[\mathbb{C}^3/\mathbb{Z}_3]}(z)$ and $\widetilde{\mathsf{R}}^{[\mathbb{C}^3/\mathbb{Z}^3]}(z)$ satisfy the same system of differential equations (25). Therefore, the solutions differ by a constant (with respect to θ) matrix which can be determined using (29) and Lemma 13. □

4.2. **Decorated graphs.** Let the genus g and the number of markings n be in the stable range

$$2g - 2 + n > 0\,.$$

A decorated graph $\Gamma \in \mathsf{G}_{g,n}^{\mathrm{Dec}}(3)$ consists of the data $(\mathsf{V},\mathsf{E},\mathsf{N},\gamma,\nu)$ where

(i) V is the vertex set,
(ii) E is the edge set (including possible self-edges),
(iii) $\mathsf{N} : \{1,2,\ldots,n\} \to \mathsf{V}$ is the marking assignment,

(iv) $\mathsf{g} : \mathsf{V} \to \mathbb{Z}_{\geq 0}$ is a genus assignment satisfying

$$g = \sum_{v \in \mathsf{V}} \mathsf{g}(v) + h^1(\Gamma)$$

and for which $(\mathsf{V}, \mathsf{E}, \mathsf{N}, \gamma)$ is stable graph,

(v) $\mathsf{p} : \mathsf{V} \to \{0, 1, 2\}$ is an assignment to each vertex $v \in \mathsf{V}$.

4.3. **Decomposition theorem.** By the formula for the higher genus potential of Section 2.5.2, we can decompose $\mathcal{F}_g^{[\mathbb{C}^3/\mathbb{Z}_3]}$ into contributions of decorated graphs of genus g. Furthermore, we can write the contribution corresponding to a graph $\Gamma \in \mathsf{G}_g^{\mathrm{Dec}}(3)$ in terms of vertex and edge contributions,

$$\mathcal{F}_g^{[\mathbb{C}_3/\mathbb{Z}_3]} = \sum_{\Gamma \in \mathsf{G}_g^{\mathrm{Dec}}(3)} \mathrm{Cont}_\Gamma \, .$$

Proposition 15. *We have*

$$\mathrm{Cont}_\Gamma = \frac{1}{\mathrm{Aut}(\Gamma)} \sum_{\mathsf{A} \in \mathbb{Z}_{>0}^{\mathsf{F}}} \prod_{v \in \mathsf{V}} \mathrm{Cont}_\Gamma^{\mathsf{A}}(v) \prod_{e \in \mathsf{E}} \mathrm{Cont}_\Gamma^{\mathsf{A}}(e) \, ,$$

where the vertex[11] *and edge contributions with incident flag* A*-values* $(a_1, a_2, ..., a_n)$ *and* (b_1, b_2) *respectively are:*

- $$\mathrm{Cont}_\Gamma^{\mathsf{A}}(v) = \left[\sum_{k \geq 0} \frac{\mathbf{g}(\widetilde{e}_{p(v)}, \widetilde{e}_{p(v)})^{1-g}}{k!} \cdot \int_{\overline{M}_{g,n+k}} \psi_1^{a_1} \ldots \psi_n^{a_n} T(\psi_{n+1}) \ldots T(\psi_{n+k}) \right] \Bigg|_{t_0 = t_1 = 0, t_{j \geq 2} = Q_{j-1\, p(v)}} ,$$

where $Q_{kp(v)}$ *is the coefficient of* z^k *in* $[(-1)^{k+1} (\mathsf{R}^{[\mathbb{C}^3/\mathbb{Z}_3]}(z))^t \cdot \mathbf{\Psi}]_{0p(v)}$,

- $$\mathrm{Cont}_\Gamma^{\mathsf{A}}(e) = (-1)^{b_1+b_2} \, 3 \cdot \left[\frac{N_{0p(v_1)}(z) N_{0p(v_2)}(w) + N_{1p(v_1)}(z) N_{2p(v_2)}(w) + N_{2p(v_1)}(z) N_{1p(v_1)}(w)}{z+w} - \frac{1}{z+w} \right]_{z^{b_1-1} w^{b_2-1}} ,$$

where $N_{ij}(z)$ *is the* (i, j) *component of* $(\mathsf{R}^{[\mathbb{C}^3/\mathbb{Z}_3]}(-z))^t \cdot \mathbf{\Psi}$.

[11] Strictly, we should have $g(\widetilde{e}_{p(v)}, \widetilde{e}_{p(v)})^{1-g-\frac{n}{2}}$ in the vertex contribution, but we shift here $g(\widetilde{e}_{p(v)}, \widetilde{e}_{p(v)})^{-\frac{n}{2}}$ to the n incident edge contributions to be consistent with our formula for $K\mathbb{P}^2$ in [24].

4.4. **Legs.** To compute the potentials $\mathcal{F}_{g,n}^{[\mathbb{C}^3/\mathbb{Z}_3]}(\phi_{a_1},\ldots,\phi_{a_n})$, the contributions of stable graphs with markings are required,

$$\mathcal{F}_{g,n}^{[\mathbb{C}_3/\mathbb{Z}_3]}(\phi_{a_1,\ldots,\phi_{a_n}}) = \sum_{\Gamma\in\mathsf{G}_{g,n}^{\mathrm{Dec}}(3)} \mathrm{Cont}_\Gamma(\phi_{a_1},\ldots,\phi_{a_n})\,.$$

Proposition 16. *We have*

$$\mathrm{Cont}_\Gamma(\phi_{k_1},\ldots,\phi_{k_n}) = \frac{1}{|\mathrm{Aut}(\Gamma)|}\sum_{\mathsf{A}\in\mathbb{Z}_{>0}^{\mathsf{F}}}\prod_{v\in\mathsf{V}}\mathrm{Cont}_\Gamma^{\mathsf{A}}(v)\prod_{e\in\mathsf{E}}\mathrm{Cont}_\Gamma^{\mathsf{A}}(e)\prod_{l\in\mathsf{L}}\mathrm{Cont}_\Gamma^{\mathsf{A}}(l)\,,$$

where the leg contribution $\mathrm{Cont}_\Gamma^{\mathsf{A}}(l)$ *is given by* $z^{\mathsf{A}(l)-1}$ *coefficient of*

$$[(-1)^{\mathsf{A}(l)-1}(\mathsf{R}^{[\mathbb{C}^3/\mathbb{Z}_3]}(z))^t\cdot\mathbf{\Psi}]_{\mathsf{Inv}(k_l)p(l)}\,.$$

The vertex and edge contributions are same as in Proposition 15.

4.5. **Vertex, edge, and legs analysis.** We analyze here the vertex and edge contributions of Proposition 15.

Lemma 17. *We have* $\mathrm{Cont}_\Gamma^{\mathsf{A}}(v)\in\mathbb{C}[L^{\pm1}]$.

Proof. The result is a direct consequence of from Proposition 15 and with Lemma 11. $\square$

Let $e\in\mathsf{E}$ be an edge connecting the vertices $v_1,v_2\in\mathsf{V}$. Let the A-values of the respective half-edges be (k,l).

Lemma 18. *We have* $\mathrm{Cont}_\Gamma^{\mathsf{A}}(e)\in\mathbb{C}[L^{\pm1},X]$ *and*

- *the degree of* $\mathrm{Cont}_\Gamma^{\mathsf{A}}(e)$ *with respect to* X *is 1,*
- *the coefficient of* X *in* $\mathrm{Cont}_\Gamma^{\mathsf{A}}(e)$ *is*

$$(-1)^{k+l}\frac{3\widetilde{P}_{2p(v_1)}^{k-1}\widetilde{P}_{2p(v_2)}^{l-1}}{L\lambda_{p(v_1)}^{k-1}\lambda_{p(v_2)}^{l-2}}\,.$$

Proof. The claims follow from Proposition 15 together with Lemmas 11 and 12. $\square$

Similarly, using the contribution formula of Proposition 16, we obtain the following result.

Lemma 19. *The leg contributions satisfy:*

- *when the insertion at the marking* l *is* ϕ_0,

$$\mathrm{Cont}_\Gamma^{\mathsf{A}}(l)\in\mathbb{C}(\lambda_0,\lambda_1,\lambda_2)[L^{\pm1}]\,,$$

- *when the insertion at the marking* l *is* ϕ_1,

$$C_1\cdot\mathrm{Cont}_\Gamma^{\mathsf{A}}(l)\in\mathbb{C}(\lambda_0,\lambda_1,\lambda_2)[L^{\pm1}]\,,$$

- *when the insertion at the marking l is ϕ_2,*

$$\frac{1}{C_1}\cdot \mathrm{Cont}^{\mathsf{A}}_{\Gamma}(l) \in \mathbb{C}(\lambda_0,\lambda_1,\lambda_2)[L^{\pm 1}, X]\,.$$

4.6. **Proof of Theorem 1.** By definition, we have

$$A_2(\theta) = \frac{1}{L^3}\left(3X - 3 - \frac{L^3}{18}\right)\,. \tag{30}$$

Hence, claim (i) of Theorem 1,

$$\mathcal{F}_g^{[\mathbb{C}^3/\mathbb{Z}_3]}(\theta) \in \mathbb{C}[L^{\pm 1}][A_2]\,,$$

follows from Proposition 15 and Lemma 17-18. Claim (ii),

$$\mathcal{F}_g^{[\mathbb{C}^3/\mathbb{Z}_3]} \text{ has at most degree } 3g-3 \text{ with respect to } A_2\,,$$

holds since a stable graph of genus g has at most $3g-3$ edges. Since

$$\frac{\partial}{\partial T} = \frac{\theta}{C_1}\frac{\partial}{\partial \theta}\,,$$

claim (iii),

$$\frac{\partial^k \mathcal{F}_g^{[\mathbb{C}^3/\mathbb{Z}_3]}}{\partial T^k}(\theta) \in \mathbb{C}[L^{\pm 1}][A_2][C_1^{-1}]\,, \tag{31}$$

follows since the ring

$$\mathbb{C}[L^{\pm 1}][A_2] = \mathbb{C}[L^{\pm 1}][X]$$

is closed under the action of the differential operator

$$\mathsf{D} = \theta\frac{\partial}{\partial\theta}$$

by Lemma 9. The degree of C_1^{-1} in (31) is 1 which yields claim (iv). $\square$

The same argument can also be applied to potentials with insertions to immediately yield the parallel result for part (i).

Theorem 1′. *After the change of variables given by the inverse of mirror map,*

$$\mathcal{F}_{g,n}^{[\mathbb{C}^3/\mathbb{Z}_3]}(\phi_{a_1},\ldots,\phi_{a_n})(\theta) \in \mathbb{C}[L^{\pm 1}][A_2, C_1^{\pm}]$$

for $2g-2+n>0$.

4.7. **Proof of Theorem 2.** Let $\Gamma \in \mathsf{G}^{\mathrm{Dec}}_{g,0}(3)$ be a decorated graph. Let us fix an edge $f \in \mathsf{E}(\Gamma)$:

- if Γ is connected after deleting f, denote the resulting graph by
$$\Gamma^0_f \in \mathsf{G}^{\mathrm{Dec}}_{g-1,2}(3),$$
- •• if Γ is disconnected after deleting f, denote the resulting two graphs by
$$\Gamma^1_f \in \mathsf{G}^{\mathrm{Dec}}_{g_1,1}(3) \quad \text{and} \quad \Gamma^2_f \in \mathsf{G}^{\mathrm{Dec}}_{g_2,1}(3)$$
where $g = g_1 + g_2$.

There is no canonical order for the 2 new markings. We will always sum over the 2 labellings. So more precisely, the graph Γ^0_f in case • should be viewed as sum of 2 graphs
$$\Gamma^0_{f,(1,2)} + \Gamma^0_{f,(2,1)}\,.$$
Similarly, in case ••, we will sum over the ordering of g_1 and g_2. As usually, the summation will be later compensated by a factor of $\frac{1}{2}$ in the formulas.

By Proposition 15, we have the following formula for the contribution of the graph Γ to the Gromov-Witten theory of $[\mathbb{C}^3/\mathbb{Z}_3]$,
$$\mathrm{Cont}_\Gamma = \frac{1}{\mathrm{Aut}(\Gamma)} \sum_{\mathsf{A} \in \mathbb{Z}^{\mathsf{F}}_{>0}} \prod_{v \in \mathsf{V}} \mathrm{Cont}^{\mathsf{A}}_\Gamma(v) \prod_{e \in \mathsf{E}} \mathrm{Cont}^{\mathsf{A}}_\Gamma(e)\,.$$

Let f connect the vertices $v_1, v_2 \in \mathsf{V}(\Gamma)$. Let the A-values of the respective half-edges be (k,l). By Lemma 18, we have
$$\frac{\partial \mathrm{Cont}^{\mathsf{A}}_\Gamma(f)}{\partial X} = (-1)^{k+l} \frac{3 \widetilde{P}^{k-1}_{2p(v_1)} \widetilde{P}^{l-1}_{2p(v_2)}}{L \lambda^{k-2}_{p(v_1)} \lambda^{l-2}_{p(v_2)}}\,. \tag{32}$$

- If Γ is connected after deleting f, we have

$$\frac{1}{|\mathrm{Aut}(\Gamma)|} \sum_{\mathsf{A} \in \mathbb{Z}^{\mathsf{F}} \geq 0} \left(\frac{L^3}{3C_1^2}\right) \frac{\partial \mathrm{Cont}^{\mathsf{A}_\Gamma(f)}}{\partial X} \prod_{v \in \mathsf{V}} \mathrm{Cont}^{\mathsf{A}_\Gamma(v)} \prod_{e \in \mathsf{E}} \mathrm{Cont}^{\mathsf{A}}_\Gamma(e)$$
$$= \frac{1}{2} \mathrm{Cont}_{\Gamma^0_f}(\phi_1, \phi_1)\,.$$

The derivation is simply by applying (32) on the left and Proposition 16 on the right.

- If Γ is disconnected after deleting f, we obtain

$$\frac{1}{|\mathrm{Aut}(\Gamma)|}\sum_{\mathsf{A}\in\mathbb{Z}^{\mathsf{F}}_{\geq 0}}\left(\frac{L^3}{3C_1^2}\right)\frac{\partial \mathrm{Cont}^{\mathsf{A}_\Gamma(f)}}{\partial X}\prod_{v\in\mathsf{V}}\mathrm{Cont}^{\mathsf{A}_\Gamma(v)}\prod_{e\in\mathsf{E}}\mathrm{Cont}^{\mathsf{A}}_\Gamma(e)$$
$$=\frac{1}{2}\mathrm{Cont}_{\Gamma^1_f}(\phi_1)\mathrm{Cont}_{\Gamma^2_f}(\phi_1)$$

by the same method. By combining the above two equations for all the edges of all the graphs $\Gamma\in\mathsf{G}^{\mathrm{Dec}}_g(3)$ and using the vanishing

$$\frac{\partial \mathrm{Cont}^{\mathsf{A}}_\Gamma(v)}{\partial X}=0$$

of Lemma (17), we obtain

$$\left(\frac{L^3}{3C_1^2}\right)\frac{\partial}{\partial X}\langle\langle\rangle\rangle^{[\mathbb{C}^3/\mathbb{Z}_3]}_g = \frac{1}{2}\sum_{i=1}^{g-1}\langle\langle\phi_1\rangle\rangle^{[\mathbb{C}^3/\mathbb{Z}_3]}_{g-i,1}\langle\langle\phi_1\rangle\rangle^{[\mathbb{C}^3/\mathbb{Z}_3]}_{i,1}+\frac{1}{2}\langle\langle\phi_1,\phi_1\rangle\rangle^{[\mathbb{C}^3/\mathbb{Z}_3]}_{g-1,2}. \tag{33}$$

We have followed here the notation of Section 0.4. The equality (33) holds in the ring $\mathbb{C}[L^{\pm1}][A_2,C_1^{-1}]$.

Since $A_2=\frac{1}{L^3}\left(3X-3-\frac{L^3}{18}\right)$ and $\langle\langle\ \rangle\rangle^{[\mathbb{C}^3/\mathbb{Z}_3]}_g=\mathcal{F}^{[\mathbb{C}^3/\mathbb{Z}_3]}_g$, the left side of (33) is, by the chain rule,

$$\frac{1}{C_2}\frac{\partial \mathcal{F}^{[\mathbb{C}^3/\mathbb{Z}_3]}_g}{\partial A_2}\in\mathbb{C}[L^{\pm1}][A_2,C_1^{-1}]\,.$$

On the right side of (33), we have

$$\langle\langle\phi_1\rangle\rangle^{[\mathbb{C}^3/\mathbb{Z}_3]}_{g-i,1}=\frac{\partial \mathcal{F}^{[\mathbb{C}^3/\mathbb{Z}_3]}_{g-i}}{\partial T}\in\mathbb{C}[[\theta]]\,.$$

Similarly, we obtain

$$\begin{aligned}\langle\langle\phi_1\rangle\rangle^{[\mathbb{C}^3/\mathbb{Z}_3]}_{i,1} &= \frac{\partial \mathcal{F}^{[\mathbb{C}^3/\mathbb{Z}_3]}_{i}}{\partial T}\in\mathbb{C}[[\theta]]\,,\\ \langle\langle\phi_1,\phi_1\rangle\rangle^{[\mathbb{C}^3/\mathbb{Z}_3]}_{g-1,2} &= \frac{\partial^2 \mathcal{F}^{[\mathbb{C}^3/\mathbb{Z}_3]}_{g-1}}{\partial T^2}\in\mathbb{C}[[\theta]]\,.\end{aligned}$$

Together, the above equations transform (33) into exactly the holomorphic anomaly equation of Theorem 2,

$$\frac{1}{C_1^2}\frac{\partial \mathcal{F}^{[\mathbb{C}^3/\mathbb{Z}_3]}_g}{\partial A_2}=\frac{1}{2}\sum_{i=1}^{g-1}\frac{\partial \mathcal{F}^{[\mathbb{C}^3/\mathbb{Z}_3]}_{g-i}}{\partial T}\frac{\partial \mathcal{F}^{[\mathbb{C}^3/\mathbb{Z}_3]}_{i}}{\partial T}+\frac{1}{2}\frac{\partial^2 \mathcal{F}^{[\mathbb{C}^3/\mathbb{Z}_3]}_{g-1}}{\partial T^2}\,,$$

as an equality in $\mathbb{C}[[\theta]]$.

The series L and A_2 are expected to be algebraically independent. Since we do not have a proof of the independence, to lift holomorphic anomaly equation to the equality

$$\frac{1}{C_1^2}\frac{\partial \mathcal{F}_g^{[\mathbb{C}^3/\mathbb{Z}_3]}}{\partial A_2} = \frac{1}{2}\sum_{i=1}^{g-1} \frac{\partial \mathcal{F}_{g-i}^{[\mathbb{C}^3/\mathbb{Z}_3]}}{\partial T}\frac{\partial \mathcal{F}_{i}^{[\mathbb{C}^3/\mathbb{Z}_3]}}{\partial T} + \frac{1}{2}\frac{\partial^2 \mathcal{F}_{g-1}^{[\mathbb{C}^3/\mathbb{Z}_3]}}{\partial T^2}$$

in the ring $\mathbb{C}[L^{\pm 1}][A_2, C_1^{-1}]$, we must prove the equalities

$$\langle\langle \phi_1 \rangle\rangle_{g-i,1}^{[\mathbb{C}^3/\mathbb{Z}_3]} = \frac{\partial \mathcal{F}_{g-i}^{[\mathbb{C}^3/\mathbb{Z}_3]}}{\partial T}\,, \quad \langle\langle \phi_1 \rangle\rangle_{i,1}^{[\mathbb{C}^3/\mathbb{Z}_3]} = \frac{\partial \mathcal{F}_{i}^{[\mathbb{C}^3/\mathbb{Z}_3]}}{\partial T} \in \mathbb{C}[[\theta]]\,,$$

$$\langle\langle \phi_1, \phi_1 \rangle\rangle_{g-1,2}^{[\mathbb{C}^3/\mathbb{Z}_3]} = \frac{\partial^2 \mathcal{F}_{g-1}^{[\mathbb{C}^3/\mathbb{Z}_3]}}{\partial T^2} \in \mathbb{C}[[\theta]]$$

hold in the ring $\mathbb{C}[L^{\pm 1}][A_2, C_1^{-1}]$. The lifting follow from the argument in Section 7.3 in [22].

We do not study the genus 1 unpointed series $\mathcal{F}^{[\mathbb{C}^3/\mathbb{Z}_3]}$ in the paper, so we take

$$\langle\langle \phi_1 \rangle\rangle_{g-i,1}^{[\mathbb{C}^3/\mathbb{Z}_3]} = \frac{\partial \mathcal{F}_{g-i}^{[\mathbb{C}^3/\mathbb{Z}_3]}}{\partial T}\,,$$

$$\langle\langle \phi_1 \rangle\rangle_{i,1}^{[\mathbb{C}^3/\mathbb{Z}_3]} = \frac{\partial \mathcal{F}_{i}^{[\mathbb{C}^3/\mathbb{Z}_3]}}{\partial T} \in \mathbb{C}[[\theta]]$$

as definitions of the right side in the genus 1 case. There is no difficulty in calculating these series explicitly using Proposition 16,

$$\frac{\partial \mathcal{F}_{1}^{[\mathbb{C}^3/\mathbb{Z}_3]}}{\partial T} = \frac{1}{18C_1}L^3 A_2\,,$$

$$\frac{\partial^2 \mathcal{F}_{1}^{[\mathbb{C}^3/\mathbb{Z}_3]}}{\partial T^2} = \frac{1}{C_1}\mathsf{D}\left(\frac{1}{18C_1}L^3 A_2\right)\,.$$

4.8. **Bounding the degree.** The degrees in L of the terms of

$$\mathcal{F}_g^{[\mathbb{C}^3/\mathbb{Z}_3]} \in \mathbb{C}[L^{\pm 1}][A_2]$$

for $[\mathbb{C}^3/\mathbb{Z}_3]$ always fall in the range

$$[9-9g, 6g-6]\,. \tag{34}$$

In particular, the constant (in A_2) term of $\mathcal{F}_g^{[\mathbb{C}^3/\mathbb{Z}_3]}$ missed by the holomorphic anomaly equation for $[\mathbb{C}^3/\mathbb{Z}_3]$ is a Laurent polynomial in L with degrees in the range (34). The bound (34) is a consequence of Proposition 15, the vertex and edge analysis of Section 4.5, and the following result.

Lemma 20. *The degrees in L of $\widetilde{P}^k_{ij}$ fall in the range*

$$[-i, 2k]\,.$$

Proof. The proof for the functions $\widetilde{P}^k_{0j}$ follows from the arguments of [35]. The proof for $\widetilde{P}^k_{1j}$ and $\widetilde{P}^k_{2j}$ follows from Lemma 12. $\square$

For $\mathcal{F}_2^{[\mathbb{C}^3/\mathbb{Z}_3]}$ (resp. $\mathcal{F}_3^{[\mathbb{C}^3/\mathbb{Z}_3]}$), the L degrees can be seen to vary between 0 and 6 (resp. 0 and 12) in the formula in Section 6 when rewritten in terms of A_2 using (30). The sharper range

$$[0, 6g-6]$$

proposed in [2] for the L degrees of $\mathcal{F}_g^{[\mathbb{C}^3/\mathbb{Z}_3]}$ is found in examples. How to derive the sharper bound from properties of the functions $\widetilde{P}^k_{ij}$ is an interesting question.

5. Crepant resolution correspondence

5.1. **R-matrix of $K\mathbb{P}^2$.** For Gromov-Witten theories in the torus equivariant setting, Givental proved a reconstruction result in the semisimple case using the localization of the virtual class [19]. We have applied the method to the stable quotient theory of local $\mathbb{P}^2$ in [24]. The results are summarized here.

Let H be the hyperplane class in $H^*_{\mathsf{T}}(\mathbb{P}^2)$, and let

$$\{1, H, H^2\} \in H^*_{\mathsf{T}}(\mathbb{P}^2) \tag{35}$$

be a basis. The inner product $\mathbf{g}^{K\mathbb{P}^2}$ in the basis (35) is given by:

$$\mathbf{g}^{K\mathbb{P}^2} = -\frac{1}{3}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}. \tag{36}$$

For $\gamma \in H^*_{\mathsf{T}}(K\mathbb{P}^2)$, we define a q-series $\overline{\mathbb{S}}_i$ using quasimap invariants of $K\mathbb{P}^2$ by

$$\overline{\mathbb{S}}_i(\gamma) = e_i^{K\mathbb{P}^2}\left\langle \frac{\phi_i^{K\mathbb{P}^2}}{z-\psi}, \gamma \right\rangle_{0,2}^{K\mathbb{P}^2}.$$

We follow here the notation[12] of [24, Section 3] where

$$e_i^{K\mathbb{P}^2} = -3(1-\lambda_i)(1-\lambda_i^2) = -9$$

[12] In particular, the formulas hold after the specialization (4) of equivariant parameters.

is the equivariant Euler class of the tangent space of $K\mathbb{P}^2$ at the fixed point p_i, and $\phi_i^{K\mathbb{P}^2}$ is the canonical basis element

$$\phi_i^{K\mathbb{P}^2} = \frac{-3\lambda_i \prod_{j\neq i}(H-\lambda_j)}{e_i^{K\mathbb{P}^2}}\,.$$

The following asymptotic form[13] of the series $\overline{\mathbb{S}}_j(H^i)$ plays a crucial role [24, Section 3.4]:

$$\overline{\mathbb{S}}_j(1) = e^{\frac{\mu\zeta_j}{z}}\left(\widetilde{P}_{00}^{0,K\mathbb{P}^2} + \widetilde{P}_{00}^{1,K\mathbb{P}^2}(\frac{z}{\zeta^j}) + \widetilde{P}_{00}^{2,K\mathbb{P}^2}(\frac{z}{\zeta^j})^2 + \dots\right),$$

(37)

$$\overline{\mathbb{S}}_j(H) = e^{\frac{\mu\zeta_j}{z}}\frac{L^{K\mathbb{P}^2}\zeta^j}{C_1^{K\mathbb{P}^2}}\left(\widetilde{P}_{20}^{0,K\mathbb{P}^2} + \widetilde{P}_{20}^{1,K\mathbb{P}^2}(\frac{z}{\zeta^j}) + \widetilde{P}_{20}^{2,K\mathbb{P}^2}(\frac{z}{\zeta^j})^2 + \dots\right),$$

$$\overline{\mathbb{S}}_j(H^2) = e^{\frac{\mu\zeta_j}{z}}\frac{(L^{K\mathbb{P}^2})^2\zeta^{2j}}{C_1^{K\mathbb{P}^2}C_2^{K\mathbb{P}^2}}\left(\widetilde{P}_{10}^{0,K\mathbb{P}^2} + \widetilde{P}_{10}^{1,K\mathbb{P}^2}(\frac{z}{\zeta^j}) + \widetilde{P}_{10}^{2,K\mathbb{P}^2}(\frac{z}{\zeta^j})^2 + \dots\right),$$

for $0 \le j \le 2$. Here, $\mu(q) = \int_0^q (L^{K\mathbb{P}^2}(x)-1)\frac{dx}{x}$. Define

$$X^{K\mathbb{P}^2} = \frac{\mathsf{D}^{K\mathbb{P}^2}C_1^{K\mathbb{P}^2}}{C_1^{K\mathbb{P}^2}}\,,$$

where $\mathsf{D}^{K\mathbb{P}^2} = q\frac{d}{dq}$. The series $\widetilde{P}_{ij}^{k,K\mathbb{P}^2}$ satisfy following system of equations for $j=0,1,2$:

$$\widetilde{P}_{2j}^{k+1,K\mathbb{P}^2} = \widetilde{P}_{0j}^{k+1,K\mathbb{P}^2} + \frac{\mathsf{D}^{K\mathbb{P}^2}\widetilde{P}_{0j}^{k,K\mathbb{P}^2}}{L^{K\mathbb{P}^2}}\,,$$

(38)

$$\widetilde{P}_{1j}^{k+1,K\mathbb{P}^2} = \widetilde{P}_{2j}^{k+1,K\mathbb{P}^2} + \frac{\mathsf{D}^{K\mathbb{P}^2}\widetilde{P}_{2j}^{k,K\mathbb{P}^2}}{L^{K\mathbb{P}^2}} + \left(\frac{\mathsf{D}^{K\mathbb{P}^2}L^{K\mathbb{P}^2}}{(L^{K\mathbb{P}^2})^2} - \frac{X^{K\mathbb{P}^2}}{L^{K\mathbb{P}^2}}\right)\widetilde{P}_{2j}^{k,K\mathbb{P}^2}\,,$$

$$\widetilde{P}_{0j}^{k+1,K\mathbb{P}^2} = \widetilde{P}_{1j}^{k+1,K\mathbb{P}^2} + \frac{\mathsf{D}^{K\mathbb{P}^2}\widetilde{P}_{1j}^{k,K\mathbb{P}^2}}{L^{K\mathbb{P}^2}} - \left(\frac{\mathsf{D}^{K\mathbb{P}^2}L^{K\mathbb{P}^2}}{(L^{K\mathbb{P}^2})^2} - \frac{X^{K\mathbb{P}^2}}{L^{K\mathbb{P}^2}}\right)\widetilde{P}_{1j}^{k,K\mathbb{P}^2}\,,$$

with the initial conditions

$$\widetilde{P}_{ij}^{0,K\mathbb{P}^2}|_{q=0} = 1\,,$$
$$\widetilde{P}_{ij}^{k,K\mathbb{P}^2}|_{q=0} = 0 \quad \text{for } k \ge 1\,.$$

[13]The notation $\widetilde{P}_{i0}^{k,K\mathbb{P}^2}$ differs slightly from R_{ik} in [24]. More precisely, the correspondence of the index i between $\widetilde{P}_{i0}^{k,K\mathbb{P}^2}$ and R_{ik} in [24] is $\{0,1,2\} \to \{0,2,1\}$. The difference will not have any effect in the higher genus formula.

Denote by $\mathsf{Q}(z)$ the matrix with the coefficient of z^k in (i,j) component $P_{ij}^{k,K\mathbb{P}2}$ are defined by the following equations for $k \geq 0$ and $j = 0, 1, 2$:

$$\begin{aligned}
\widetilde{P}_{00}^{k,K\mathbb{P}^2} &= P_{0j}^{k,K\mathbb{P}2}\zeta^{kj} \\
\widetilde{P}_{10}^{k,K\mathbb{P}^2} &= \frac{L}{C_1\zeta^{2j}}P_{1j}^{k,K\mathbb{P}^2}\zeta^{kj} \\
\widetilde{P}_{20}^{k,K\mathbb{P}2} &= \frac{C_1}{L\zeta^j}P_{2j}^{k,K\mathbb{P}2}\zeta^{kj}\,.
\end{aligned}$$

Define a new matrix $\widetilde{\mathsf{R}}^{K\mathbb{P}^2}(z)$ by

$$\widetilde{\mathsf{R}}^{K\mathbb{P}^2}(z) = \mathbf{\Psi} \cdot \mathsf{Q}(z),$$

where

$$\mathbf{\Psi}^{K\mathbb{P}^2} = \frac{-i}{3}\begin{bmatrix} 1 & \frac{L^{K\mathbb{P}^2}}{C_1^{K\mathbb{P}^2}} & \frac{C_1^{K\mathbb{P}^2}}{L^{K\mathbb{P}^2}} \\ 1 & \zeta\frac{L^{K\mathbb{P}^2}}{C_1^{K\mathbb{P}^2}} & \zeta^2\frac{C_1^{K\mathbb{P}^2}}{L^{K\mathbb{P}^2}} \\ 1 & \zeta^2\frac{L^{K\mathbb{P}^2}}{C_1^{K\mathbb{P}^2}} & \zeta\frac{C_1^{K\mathbb{P}^2}}{L^{K\mathbb{P}^2}} \end{bmatrix}.$$

Define a new diagonal matrix

$$\mathsf{B}^{K\mathbb{P}^2}(z) = \mathrm{Diag}\left[B_0^{K\mathbb{P}^2}(z), B_1^{K\mathbb{P}^2}(z), B_2^{K\mathbb{P}^2}(z)\right]$$

where

$$B_j^{K\mathbb{P}^2}(z) = \mathrm{Exp}\left(-\sum_{k=1}^{\infty}\frac{N_{2k-1,j}}{2k-1}\frac{B_{2k}(0)}{2k}\left(\frac{z}{\zeta^i}\right)^{2k-1}\right)$$

and $N_{k,j} = (-\frac{1}{3\zeta^j})^k + \sum_{l=1}^2(\frac{1}{\zeta^j-\zeta^{j+l}})^k$.

The $\mathbf{R}$-matrix $\mathsf{R}^{K\mathbb{P}^2}$ in the normalized canonical basis for $K\mathbb{P}^2$ is given by the following result, see [17, 24].

Proposition 21. *We have*

$$\left[\mathsf{R}^{K\mathbb{P}^2}(z)\right]_{ij} = \left[\widetilde{\mathsf{R}}^{K\mathbb{P}^2}(z)\right]_{ij}\cdot\mathsf{B}^{K\mathbb{P}^2}(z)\,.$$

5.2. **Proof of Theorem 4.** The $\mathbf{R}$-matrix approach to Theorem 4 will establish a more general results for potentials with insertions. Let

$$\mathcal{F}_{g,n}^{K\mathbb{P}^2}(H^{a_1},\ldots,H^{a_n}) = \sum_{d=0}^{\infty}q^d\int_{[\overline{M}_{g,n}(K\mathbb{P}^2,d)]^{vir}}\prod_{k=1}^{n}\mathrm{ev}_i^*(H^{a_k})\,,$$

be the Gromov-Witten potential for $K\mathbb{P}^2$. Define the map

$$\iota : (H^*_{\mathsf{T},\mathrm{orb}}([\mathbb{C}^3/\mathbb{Z}_3]),\mathbf{g}) \to (H^*_{\mathsf{T}}(K\mathbb{P}^2),\mathbf{g}^{K\mathbb{P}^2})$$

by the rule

$$\iota(\phi_0) = 1\,, \;\; \iota(\phi_1) = H\,, \;\; \iota(\phi_2) = H^2\,.$$

By Theorem 1′ and [24, Theorem 1] respectively, we have

$$\begin{aligned} \mathcal{F}_{g,n}^{[\mathbb{C}^3/\mathbb{Z}_3]}(\phi_{a_1},\ldots,\phi_{a_n}) &\in \mathbb{C}[L^{\pm 1}][X, C_1^{\pm 1}]\,, \\ \mathcal{F}_{g,n}^{K\mathbb{P}^2}(H^{a_1},\ldots,H^{a_n}) &\in \mathbb{C}[(L^{K\mathbb{P}^2})^{\pm 1}][X^{K\mathbb{P}^2}, (C_1^{K\mathbb{P}^2})^{\pm 1}]\,. \end{aligned}$$

The following result specializes to Theorem 4 in case there are no insertions.

Theorem 4′. *For g and n in the stable range, the crepant resolution correspondence*

$$\mathcal{F}_{g,n}^{[\mathbb{C}^3/\mathbb{Z}_3]}(\phi_{i_1},\ldots,\phi_{i_n}) = (-1)^{2g-2+n}\cdot \mathsf{P}\left(\mathcal{F}_{g,n}^{K\mathbb{P}^2}(\iota(\phi_{i_1}),\ldots,\iota(\phi_{i_n}))\right)$$

holds with the ring homomorphism

$$\mathsf{P} : \mathbb{C}[(L^{K\mathbb{P}^2})^{\pm 1}][X^{K\mathbb{P}^2}, (C_1^{K\mathbb{P}^2})^{\pm 1}] \to \mathbb{C}[L^{\pm 1}][X, C_1^{\pm 1}]$$

defined by

$$\mathsf{P}(L^{K\mathbb{P}^2}) = -\frac{L}{3}\,, \;\;\; \mathsf{P}(X^{K\mathbb{P}^2}) = -\frac{X}{3}\,, \;\;\; \mathsf{P}(C_1^{K\mathbb{P}^2}) = \frac{1}{3}C_1\,.$$

Proof. The first step is to prove that the map ι matches the CohFT structures in genus 0 (up to sign). Certainly, ι preserves pairings up to a sign,

$$\mathbf{g} = -\mathbf{g}^{K\mathbb{P}^2}\,. \tag{39}$$

Using the result of [24, Section 5], we easily obtain the following genus 0 results for $K\mathbb{P}^2$:

$$\langle 1,1,1\rangle_{0,3}^{K\mathbb{P}^2} = -\frac{1}{3}\,, \qquad \langle 1,H,H^2\rangle_{0,3}^{K\mathbb{P}^2} = -\frac{1}{3}\,,$$

$$\langle H,H,H\rangle_{0,3}^{K\mathbb{P}^2} = -\frac{1}{3}\frac{(L^{K\mathbb{P}^2})^3}{(C_1^{K\mathbb{P}^2})^3}\,, \quad \langle H^2,H^2,H^2\rangle_{0,3}^{K\mathbb{P}^2} = -\frac{1}{3}\frac{(C_1^{K\mathbb{P}^2})^3}{(L^{K\mathbb{P}^2})^3}\,.$$

For other choices of insertions, the 3-point functions in genus 0 vanish. The genus 0 invariants for $K\mathbb{P}^2$ match Lemma 7 via the ring homomorphism P and the map ι up to the sign $(-1)^{2g-2+n}$.

The second step is to match the $\mathbf{R}$-matrices of the two CohFTs. The two system of equations (28) and (38) are equivalent via the transformation P defined by (10) up to another sign change. The effect of the latter sign change cancels the former sign change (39) in the higher genus formula. More precisely, the sign changes will contribute the global factors $(-1)^{1-g}$ and $(-1)^{3g-3+n}$ respectively in the genus g, n-marked Gromov-Witten potential function.

Therefore, to prove Theorem 4, we must only match the constant terms of the $\mathbf{R}$-matrices. We apply the result of [35] to (38), to conclude

$$\widetilde{P}_{i0}^{k,K\mathbb{P}^2} \in \mathbb{C}[L^{K\mathbb{P}^2}] \text{ for } i = 0, 2\,,$$
$$\widetilde{P}_{10}^{k,K\mathbb{P}^2} \in \mathbb{C}[(L^{K\mathbb{P}^2})^{\pm 1}, X^{K\mathbb{P}^2}]\,.$$

Denote by a_{ik} the constant term in the Laurent series of $\widetilde{P}_{i0}^{k,K\mathbb{P}^2}$ in $L^{K\mathbb{P}^2}$. From (38), we can prove a_{ik} is independent of $X^{K\mathbb{P}^2}$. Therefore, we have

$$a_{ik} \in \mathbb{Q} \text{ for } i = 0, 1, 2\,.$$

The last step in the proof of Theorem 4 is the following Lemma proven in the Appendix by T. Coates and H. Iritani.

Lemma 22. *The equality of power series in z,*

$$\mathrm{Exp}\left(-\sum_{k=1}^{\infty} \frac{N_{2k-1,0}}{2k-1} \frac{B_{2k}(0)}{2k} z^{2k-1} \right) \sum_{k=0}^{\infty} a_{\mathsf{Inv}(i)k} z^k = \mathrm{Exp}\left(3 \sum_{k=1}^{\infty} (-1)^{k+1} \frac{B_{3k+1}(i/3)}{3k+1} \frac{z^{3k}}{3k} \right),$$

holds for $i = 0, 1, 2$.

The left side of equality of Lemma 22 for $i = 0$ computes the constant term with respect to $L^{K\mathbb{P}^2}$ of the coefficients of the first row of the $\mathbf{\Psi}^{-1}\mathbf{R}$-matrix of $K\mathbb{P}^2$ by Proposition 21. For $i = 1$ and $i = 2$, the left side of Lemma 22 computes the constant terms in $L^{K\mathbb{P}^2}$ of the coefficients of the second row and the third rows (after multiplication[14] by $\frac{L^{K\mathbb{P}^2}}{C_1^{K\mathbb{P}^2}}$ and $\frac{C_1^{K\mathbb{P}^2}}{L^{K\mathbb{P}^2}}$ respectively).

Similarly, the right side of Lemma 22 for $i = 0$ computes the constant term with respect to L of the coefficients of the first row of the $\mathbf{\Psi}^{-1}\mathbf{R}$-matrix of $[\mathbb{C}^3/\mathbb{Z}_3]$ by Proposition 13. For $i = 1$ and $i = 2$, the right side of Lemma 22 computes the constant terms in L of the coefficients of the second row and the third rows (after multiplication[15] by $-\frac{L}{C_1}$ and -$\frac{C_1}{L}$ respectively).

Since the constant terms match by Lemma 22, the $\mathbf{R}$-matrix (or, equivalently, the $\mathbf{\Psi}^{-1}\mathbf{R}$-matrix) of $[\mathbb{C}^3/\mathbb{Z}_3]$ exactly equals the $\mathbf{R}$-matrix

[14]Both $\frac{L^{K\mathbb{P}^2}}{C_1^{K\mathbb{P}^2}}$ and $\frac{C_1^{K\mathbb{P}^2}}{L^{K\mathbb{P}^2}}$ have constant term in q equal to 1.

[15]Both $-\frac{L}{C_1}$ and $-\frac{C_1}{L}$ have constant term in θ equal to 1.

(or, equivalently, the $\mathbf{\Psi}^{-1}\mathbf{R}$-matrix) of $K\mathbb{P}^2$ via the transformation P. $\square$

6. Calculations in low genus

We present here the formula for the potential function in genus 2 and 3 for $[\mathbb{C}^3/\mathbb{Z}_3]$ and $K\mathbb{P}^2$ obtained via the $\mathbf{R}$-matrix method of Section 4. In genus 2, we have

$$
\begin{aligned}
\mathcal{F}_2^{[\mathbb{C}^3/\mathbb{Z}^3]} &= \frac{-291600 - 25893L^3 - 784L^6 - 8L^9}{466560L^3} \\
&\quad + \left(\frac{1}{9} + \frac{15}{8L^3} + \frac{13L^3}{7776}\right) X \\
&\quad + \left(-\frac{1}{18} - \frac{15}{8L^3}\right) X^2 + \frac{5X^3}{8L^3}\,,
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{F}_2^{K\mathbb{P}^2} &= \frac{400 - 959\tilde{L}^3 + 784\tilde{L}^6 - 216\tilde{L}^9}{17280\tilde{L}^3} \\
&\quad + \left(-\frac{1}{3} + \frac{5}{24\tilde{L}^3} + \frac{13\tilde{L}^3}{96}\right) \tilde{X} \\
&\quad + \left(-\frac{1}{2} + \frac{5}{8\tilde{L}^3}\right) \tilde{X}^2 + \frac{5}{8\tilde{L}^3}\tilde{X}^3.
\end{aligned}
$$

To simplify the formulas, we have used the notation $\tilde{L} = L^{K\mathbb{P}^2}$ and $\tilde{X} = X^{K\mathbb{P}^2}$ for $K\mathbb{P}^2$.

The formula for $\mathcal{F}_3^{[\mathbb{C}^3/\mathbb{Z}^3]}$ is much more complicated:

$$\frac{26784626400 + 7043364720L^3 + 767774781L^6 + 44032896L^9 + 1398288L^{12} + 23328L^{15} + 160L^{18}}{9523422720L^6}$$

$$+ \frac{(-318864600 - 66331710L^3 - 5521446L^6 - 228393L^9 - 4681L^{12} - 38L^{15})X}{18895680L^6}$$

$$+ \frac{(531441000 + 83980800L^3 + 4996566L^6 + 132147L^9 + 1307L^{12})X^2}{12597120L^6}$$

$$+ \frac{(-47239200 - 5318784L^3 - 200772L^6 - 2539L^9)X^3}{839808L^6}$$

$$+ \left(\frac{35}{648} + \frac{675}{16L^6} + \frac{289}{96L^3}\right) X^4 - \frac{5(324 + 11L^3)X^5}{96L^6} + \frac{45X^6}{16L^6}.$$

For $\mathcal{F}_3^{K\mathbb{P}^2}$, we have:

$$\frac{16800 - 119280\tilde{L}^3 + 351063\tilde{L}^6 - 543616\tilde{L}^9 + 466096\tilde{L}^{12} - 209952\tilde{L}^{15} + 38880\tilde{L}^{18}}{4354560\tilde{L}^6}$$

$$+ \frac{(600 - 3370\tilde{L}^3 + 7574\tilde{L}^6 - 8459\tilde{L}^9 + 4681\tilde{L}^{12} - 1026\tilde{L}^{15})\tilde{X}}{8640\tilde{L}^6}$$

$$+ \frac{(3000 - 12800\tilde{L}^3 + 20562\tilde{L}^6 - 14683\tilde{L}^9 + 3921\tilde{L}^{12})\tilde{X}^2}{5760\tilde{L}^6}$$

$$+ \frac{(2400 - 7296\tilde{L}^3 + 7436\tilde{L}^6 - 2539\tilde{L}^9)\tilde{X}^3}{1152\tilde{L}^6}$$

$$+ \left(\frac{35}{8} + \frac{75}{16\tilde{L}^6} - \frac{289}{32\tilde{L}^3}\right) \tilde{X}^4 - \frac{15(-12 + 11\tilde{L}^3)\tilde{X}^5}{32\tilde{L}^6} + \frac{45\tilde{X}^6}{16\tilde{L}^6}.$$

As stated in Theorem 4, the above potentials match after the ring homomorphism P,

$$\mathsf{P}(L^{K\mathbb{P}^2}) = -\frac{L}{3}, \quad \mathsf{P}(X^{K\mathbb{P}^2}) = -\frac{X}{3}.$$

In [3], the constant terms of $\mathcal{F}_2^{[\mathbb{C}^3/\mathbb{Z}^3]}$ and $\mathcal{F}_3^{[\mathbb{C}^3/\mathbb{Z}^3]}$ were computed directly by studying the geometry of the moduli space of curves. Our calculations agree with their results.

Appendix A. The $\mathbf{R}$-matrix identity

by Tom Coates and Hiroshi Iritani

A.1. **Overview.** We will prove Lemma 22 by analyzing the oscillatory integrals occurring in Givental's equivariant mirror [16]. We briefly recall the so-called *saddle point method* for finding their asymptotic behaviour, see [11, Section 6.2]. Let $f(t)$, $g(t)$ be holomorphic functions on $\mathbb{C}^n$ and consider the oscillatory integral

$$\int_\Gamma e^{f(t)/z} g(t) dt^1 \dots dt^n ,$$

where the real n-dimensional cycle $\Gamma \subset \mathbb{C}^n$ is chosen so that the integral converges. Let t_0 be a non-degenerate critical point of f and choose Γ to be the stable manifold of the Morse function

$$t \to \mathscr{R}(f(t))$$

associated with t_0 (the union of downward gradient trajectories converging to t_0). Here we assume $z < 0$ and study the asymptotic behaviour of the integral as z approaches zero from the negative real axis. The asymptotic behaviour as $z \to 0$ is determined only by the integrand around the critical point t_0. We expand the integrand $e^{f(t)/z} g(t)$ in Taylor series at t_0 and perform termwise integration with respect to the Gaussian measure

$$e^{\frac{1}{2z}\sum_{i,j} h_{i,j}(t^i - t_0^i)(t^j - t_0^j)} dt^1 \dots dt^n$$

where $h_{i,j} = \partial_i \partial_j f(t_0)$ is the Hessian matrix and $\partial = \frac{\partial}{\partial t^i}$. We then obtain

$$\int_\Gamma e^{f(t)/z} g(t) dt^1 \dots dt^n \sim (-2\pi z)^{n/2} e^{f(t_0)/z} \sum_{k=0}^{\infty} c_k z^k \ \text{ as } z \to 0$$

with

$$\sum_{k=0}^{\infty} c_k z^k = \frac{1}{\sqrt{\det(h_{i,j})}} \left[e^{-\frac{z}{2}\sum_{i,j} h^{i,j}\partial_i\partial_j} e^{f_{\geq 3}/z} g(t) \right]_{t=t_0} \tag{40}$$

where $f_{\geq 3}(t) = f(t) - f(t_0) - \frac{1}{2}\sum_{i,j} h_{i,j}(t - t_0^i)(t^j - t_0^j)$ and $(h^{i,j})$ are the coefficients of the matrix inverse to $(h_{i,j})$.

H.I. thanks Atsushi Kanazawa for inviting Hyenho Lho to Kyoto and for providing an occasion to discuss the identity in the Appendix and the Crepant Resolution Conjecture.

Definition 23. *For a non-degenerate critical point* t_0 *of* $f(t)$*, we define the* formal asymptotic expansion

$$\mathrm{Asym}_{t_0}(e^{f(t)/z}g(t)dt) \in \mathbb{C}[[z]]$$

to be the right-hand side of (40)*. Since the definition only involves the Taylor expansion at* t_0*, this is well-defined for germs* $f(t)$*,* $g(t)$ *at* t_0*.*

A.2. **Givental's equivariant mirror for** $K\mathbb{P}^2$**.** The equivariant mirror for local $\mathbb{P}^2$ was introduced by Givental, which is given by the (multivalued) Landau-Ginzburg potential

$$F = w_0 + w_1 + w_2 + w_3 + \sum_{i=0}^{2} \lambda_i \mathrm{log} w_i$$

defined on the family of affine varieties

$$Y_q = \{(w_0, w_1, w_2, w_3) \in \mathbb{C}^4 : w_0 w_1 w_2 = q w_3^3\}\,.$$

The associated oscillatory integral is of the form

$$\mathcal{I} = \int_{\Gamma \subset Y_q} e^{F/z} g(w) \omega \tag{41}$$

where ω is the (meromorphic) volume form on Y_q:

$$\omega = \frac{d\log w_0 \wedge d\log w_1 \wedge d\log w_2 \wedge d\log w_3}{d\log q}\,.$$

Using the coordinate system (w_0, w_1, w_2) on Y_q, we have

$$\mathcal{I} = \int_{\Gamma \subset (\mathbb{C}^*)^3} e^{(w_0+w_1+w_2+q^{-1/3}(w_0w_1w_2)^{1/3}+\sum_{i=0}^{2}\lambda_i \log w_i)/z} g(w) \frac{1}{3}\frac{dw_0 dw_1 dw_2}{w_0 w_1 w_2}\,.$$

A.3. **Formal asymptotic expansion.** The proof of Lemma 22 is based on the computation of the formal asymptotic expansion of the integral $\mathcal{I}$ for $g(w) = 1, w_3, w_3^2$.

We will use the specialization

$$\lambda_i = \zeta^i\,,$$

where ζ is the primitive third root of unity. With this specialization, the critical points of F_λ are easy to calculate:

$$w_i = \begin{cases} L^{K\mathbb{P}^2} - \zeta^i & \text{for } 0 \le i \le 2\,, \\ -3L^{K\mathbb{P}^2} & \text{for } i = 3\,, \end{cases} \tag{42}$$

where $L^{K\mathbb{P}^2} = (1+27q)^{-\frac{1}{3}}$ as before. The three choices for the branch of $L^{K\mathbb{P}^2}$ give rise to three critical points. For the sake of clarity, let us assume $q > 0$ and choose the critical point corresponding to a real positive $L^{K\mathbb{P}^2}$ in the following discussion. The critical value is given by

$$F_\lambda(\mathrm{cr}) = \sum_{i=0}^{2} \zeta^i \log\left(L^{K\mathbb{P}^2} - \zeta^i\right)$$

where cr means the critical point (42). It can be decomposed as

$$F_\lambda(\mathrm{cr}) = \log(-9q) + \zeta\log(1-\zeta) + \zeta^2\log(1-\zeta^2) + \mu$$

with

$$\mu = \int_0^q (L^{K\mathbb{P}^2} - 1)\frac{dq}{q} \in q\mathbb{C}[[q]] \,.$$

The Hessian of F_λ at the critical point with respect to logarithmic coordinates $(\log w_0, \log w_1, \log w_2)$ is given by

$$\det\left(\frac{\partial^2 F_\lambda(\mathrm{cr})}{\partial\log w_i \partial\log w_j}\right)_{0\le i,j\le 2} = -1\,.$$

The I-function of $K\mathbb{P}^2$ was defined in [24] to be $H^*_{\mathsf{T}}(\mathbb{P}^2)$-valued power series:

$$I^{K\mathbb{P}^2}(q,z) = \sum_{d=0}^{\infty} q^d \frac{\prod_{k=0}^{3d-1}(-3H - kz)}{\prod_{i=0}^{2}\prod_{k=1}^{d}(H - \lambda_i + kz)}\,.$$

In [11, Proposition 6.9], a relationship between the formal asymptotic expansion of the mirror oscillatory integral (41) and the equivariant I-function was established for toric Deligne-Mumford stacks. Applying the result to $K\mathbb{P}^2$, we obtain:

Proposition 24. *We have*

$$\begin{aligned}
e^{\mu/z}\mathrm{Asym}_{\mathrm{cr}}(e^{\frac{F_\lambda}{z}}\omega) =& I(q,z)|_{p_0}\cdot \frac{1}{3\sqrt{-1}}\mathrm{Exp}\left(-\sum_{k=1}^{\infty}\frac{B_{k+1}(0)}{k(k+1)}N_{k,0}z^k\right)\,,\\
e^{\mu/z}\mathrm{Asym}_{\mathrm{cr}}(e^{\frac{F_\lambda}{z}}w_3\omega) =& (-3z\mathsf{D}^{K\mathbb{P}^2} - 3H)I(q,z)|_{p_0}\\
&\cdot \frac{1}{3\sqrt{-1}}\mathrm{Exp}\left(-\sum_{k=1}^{\infty}\frac{B_{k+1}(0)}{k(k+1)}N_{k,0}z^k\right)\,,\\
e^{\mu/z}\mathrm{Asym}_{\mathrm{cr}}(e^{\frac{F_\lambda}{z}}w_3^2\omega) =& (3z\mathsf{D}^{K\mathbb{P}^2} + 3H + z)(3z\mathsf{D}^{K\mathbb{P}^2} + 3H)I(q,z)|_{p_0}\\
&\cdot \frac{1}{3\sqrt{-1}}\mathrm{Exp}\left(-\sum_{k=1}^{\infty}\frac{B_{k+1}(0)}{k(k+1)}N_{k,0}z^k\right)\,,
\end{aligned}$$

where the I-function in the right-hand side should be expanded in Laurent series at $z = 0$.

Here $\mathsf{D}^{K\mathbb{P}^2} = q\frac{d}{dq}$. The definition of $\widetilde{P}_{i0}^{k,K\mathbb{P}^2}$ immediately yields:

Corollary 25. *We have*

$$3\sqrt{-1}\mathrm{Asym}_{\mathrm{cr}}(e^{\frac{F_\lambda}{z}}\omega) = \left(\sum_{k=0}^{\infty} \widetilde{P}_{00}^{k,K\mathbb{P}^2} z^k\right) \mathrm{Exp}\left(-\sum_{k=1}^{\infty} \frac{B_{k+1}(0)}{k(k+1)} N_{k,0} z^k\right) ,$$

$$\frac{-\sqrt{-1}}{L}\mathrm{Asym}_{\mathrm{cr}}(e^{\frac{F_\lambda}{z}}w_3\omega) = \left(\sum_{k=0}^{\infty} \widetilde{P}_{20}^{k,K\mathbb{P}^2} z^k\right) \mathrm{Exp}\left(-\sum_{k=1}^{\infty} \frac{B_{k+1}(0)}{k(k+1)} N_{k,0} z^k\right) ,$$

$$\frac{\sqrt{-1}}{3L^2}\mathrm{Asym}_{\mathrm{cr}}(e^{\frac{F_\lambda}{z}}w_3^2\omega) = \left(\sum_{k=0}^{\infty} \widetilde{P}_{10}^{k,K\mathbb{P}^2} z^k\right) \mathrm{Exp}\left(-\sum_{k=1}^{\infty} \frac{B_{k+1}(0)}{k(k+1)} N_{k,0} z^k\right) .$$

Proof. In [24], the evaluation of $\overline{\mathbb{S}}_j(H^i)$ was obtained from $I^{K\mathbb{P}^2}$ via Birkhoff factorization:

$$\begin{aligned} \overline{\mathbb{S}}_j(1) &= I^{K\mathbb{P}^2}|_{p_j} , \\ \overline{\mathbb{S}}_j(H) &= \frac{(H + z\mathsf{D}^{K\mathbb{P}^2})\overline{\mathbb{S}}_j(1)}{C_1^{K\mathbb{P}^2}} , \\ \overline{\mathbb{S}}_j(H^2) &= \frac{(H + z\mathsf{D}^{K\mathbb{P}^2})\overline{\mathbb{S}}_j(H)}{C_2^{K\mathbb{P}^2}} . \end{aligned} \tag{43}$$

First two equations in the Corollary follow immediately from Proposition 24 using (37) and (43). The last equation in the Corollary requires further explanation.

$$\begin{aligned} C_2^{K\mathbb{P}^2}\overline{\mathbb{S}}_j(H^2) &= (H + z\mathsf{D}^{K\mathbb{P}^2})\overline{\mathbb{S}}(H) \\ &= (H + z\mathsf{D}^{K\mathbb{P}^2})(C_1^{K\mathbb{P}^2})^{-1}(H + z\mathsf{D}^{K\mathbb{P}^2})I^{K\mathbb{P}^2}|_{p_j} . \end{aligned} \tag{44}$$

In particular,

$$C_2^{K\mathbb{P}^2}\overline{\mathbb{S}}_j(H^2) = -z\frac{\mathsf{D}^{K\mathbb{P}^2}C_1^{K\mathbb{P}^2}}{(C_1^{K\mathbb{P}^2})^2}(H+z\mathsf{D}^{K\mathbb{P}^2})I^{K\mathbb{P}^2} + \frac{1}{C_1^{K\mathbb{P}^2}}(H+z\mathsf{D}^{K\mathbb{P}^2})^2 I^{K\mathbb{P}^2}\Big|_{p_j} .$$

The analytic continuation of the hypergeometric series

$$C_1^{K\mathbb{P}^2} = \sum_{d=0}^{\infty} \frac{(3d)!}{(d!)^3}(-q)^d$$

gives (see Appendix A of [21])

$$C_1^{K\mathbb{P}^2} = \frac{1}{3}\sum_{d=0}^{\infty} \frac{(-1)^d}{d!}\frac{\Gamma(\frac{1}{3}+\frac{n}{3})}{\Gamma(\frac{2}{3}-\frac{n}{3})^2} q^{-\frac{1}{3}-\frac{n}{3}} = \frac{1}{3}\frac{\Gamma(\frac{1}{3})}{\Gamma(\frac{2}{3})^2} q^{-\frac{1}{3}} + \mathcal{O}(q^{-\frac{1}{3}}) \quad \text{as } q \to \infty .$$

Since we are interested in the asymptotic expansion of $C_2^{K\mathbb{P}^2}\overline{\mathbb{S}}_j(H^2)$ at $q = \infty$, we can replace

$$\frac{\mathsf{D}^{K\mathbb{P}^2} C_1^{K\mathbb{P}2}}{C_1^{K\mathbb{P}2}} \;\text{ with }\; -\frac{1}{3}$$

and consider instead of (44) the function

$$\frac{1}{C_1^{K\mathbb{P}^2}}(H + z\mathsf{D}^{K\mathbb{P}^2} + \frac{z}{3})(H + z\mathsf{D}^{K\mathbb{P}^2}) I^{K\mathbb{P}^2}\Big|_{p_j} .$$

The last equation of Corollary then follows from Proposition 24 using (37). $\square$

A.4. **Proof of Lemma 22.** We will obtain the identities of Lemma 22 by computing the analytic continuation of

$$\mathrm{Asym}_{\mathrm{cr}}(e^{\frac{F_\lambda}{z}} g\omega) \;\text{ as } q \to \infty .$$

Note that $L = 0$ at $q = \infty$. The Landau-Ginzburg potential F_λ near $q = \infty$ is mirror to $[\mathbb{C}^3/\mathbb{Z}_3]$. Thus, $\mathrm{Asym}_{\mathrm{cr}}(e^{\frac{F_\lambda}{z}} g\omega)$ near $q = \infty$ can be computed in terms of the equivariant I-function of $[\mathbb{C}^3/\mathbb{Z}^3]$, again by [11, Proposition 6.9].

It is instructive to evaluate directly the oscillatory integral at $q = \infty$. The key ingredient is the Stirling approximation for the Γ-function:

$$\log\Gamma(h+x) \sim \left(x + h - \frac{1}{2}\right)\log x - x + \frac{1}{2}\log(2\pi) + \sum_{k=1}^{\infty} \frac{(-1)^{k+1} B_{k+1}(h)}{k(k+1)x^k} .$$

We have

$$\begin{aligned}
\int_\Gamma e^{F_\lambda/z}\omega\Big|_{q=\infty} &= \int_\Gamma e^{\sum_{i=0}^2 (w_i + \lambda_i \log w_i)/z} \frac{1}{3} \frac{dw_0 dw_1 dw_2}{w_0 w_1 w_2} \\
&= \frac{1}{3}\prod_{i=0}^{2} \Gamma\left(\frac{\lambda_i}{z}\right)(-z)^{\lambda_i/z} .
\end{aligned}$$

Using the Stirling approximation, we obtain

$$\begin{aligned}
\mathrm{Asym}_{\mathrm{cr}}(e^{F_\lambda/z}\omega)\Big|_{q=\infty} &= \frac{1}{3\sqrt{-\lambda_0\lambda_1\lambda_2}} \mathrm{Exp}\left(\sum_{i=0}^{2}\sum_{k=1}^{\infty} \frac{(-1)^{k+1} B_k(0)}{k(k+1)\lambda_i^k} z^k\right) \\
&= \frac{1}{3\sqrt{-1}} \mathrm{Exp}\left(3\sum_{k=1}^{\infty} \frac{(-1)^{3k+1} B_{3k}(0)}{3k(3k+1)} z^{3k}\right) .
\end{aligned}$$

This, together with Corollary 25, immediately gives the first identity in Lemma 22. We also have

$$\int_\Gamma e^{F_\lambda/z}\frac{w_3}{L}\omega\big|_{q=0}=\int_\Gamma e^{\sum_{i=0}^2(w_i+\lambda_i\log w_i)/z}(w_0w_1w_2)^{1/3}\frac{dw_0dw_1dw_2}{w_0w_1w_2}$$
$$=-z\prod_{i=0}^2\Gamma\left(\frac{1}{3}+\frac{\lambda_i}{z}\right)(-z)^{\lambda_i/z}.$$

Again by the Stirling approximation, we obtain

$$\mathrm{Asym}_{\mathrm{cr}}\left(e^{F_\lambda/z}\frac{w_3}{L}\omega\right)\big|_{q=\infty}=\frac{-(\lambda_0\lambda_1\lambda_2)^{1/3}}{\sqrt{-\lambda_0\lambda_1\lambda_2}}\mathrm{Exp}\left(\sum_{i=0}^2\sum_{k=1}^\infty\frac{(-1)^{k+1}B_{k+1}(\frac{1}{3})}{k(k+1)\lambda_i^k}z^k\right)$$
$$=\frac{-1}{\sqrt{-1}}\mathrm{Exp}\left(3\sum_{k=1}^\infty\frac{(-1)^{3k+1}B_{3k+1}(\frac{1}{3})}{3k(k+1)}z^{3k}\right)$$

which, together with Corollary 25, gives the second identity in Lemma 22. Similarly, we also have

$$\mathrm{Asym}_{\mathrm{cr}}\left(e^{F_\lambda/z}\frac{w_3^2}{L^2}\omega\right)\big|_{q=\infty}=\frac{3}{\sqrt{-1}}\mathrm{Exp}\left(3\sum_{k=1}^\infty\frac{(-1)^{3k+1}B_{3k+1}(\frac{2}{3})}{3k(k+1)}z^{3k}\right)$$

which, together with Corollary 25, gives the third identity in Lemma 22. $\square$

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