Bumsig Kim

in Memoriam

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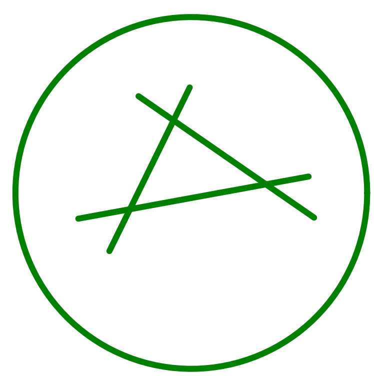
I will discuss three themes related to the work of Bumsig:

(i) Mittag-Leffler Institute $QH^*(G/B)$

(ii) Quot schemes / Quasimaps   Wall Crossing

(iii) Log geometry



(i) Bumsig's first result:

Quantum Cohomology of G/B

and quantum Toda lattices

(G, B, T) semisimple group, Borel, max torus

$H^2(G/B)$ basis $p_1, \dots, p_\ell$

rep theory of T

Quantum parameters $q_1, \dots, q_\ell$

$$QH^*(G/B) = \mathbb{Q}[p_1, \dots, p_\ell, q_1, \dots, q_\ell] / I$$

Question: What is the ideal?

Example : SL $\Rightarrow$ Complete flag variety

$\mathbb{F}_{n+1}$ flags $\mathbb{C}^1 \subset \cdots \subset \mathbb{C}^n$ in $\mathbb{C}^{n+1}$

$$H^*(\mathbb{F}_{n+1}) = \mathbb{Q}[u_0, \ldots, u_n] \Big/ \begin{matrix}\text{Symmetric} \\ \text{polys}\end{matrix}$$

$$QH^*(\mathbb{F}_{n+1}) = \mathbb{Q}[u_0, \ldots, u_n, q_1, \ldots, q_n] \Big/ \text{q-defs of symmetric polys}$$

$u_i = p_i - p_{i+1}$

$$A_n = \begin{bmatrix} u_0 & q_1 & & & & \\ -1 & u_1 & q_2 & & & \\ & -1 & u_2 & & & \\ & & & \ddots & & q_n \\ & & & & -1 & u_n \end{bmatrix}$$

Givental-Kim
Ciocan-Fontanine

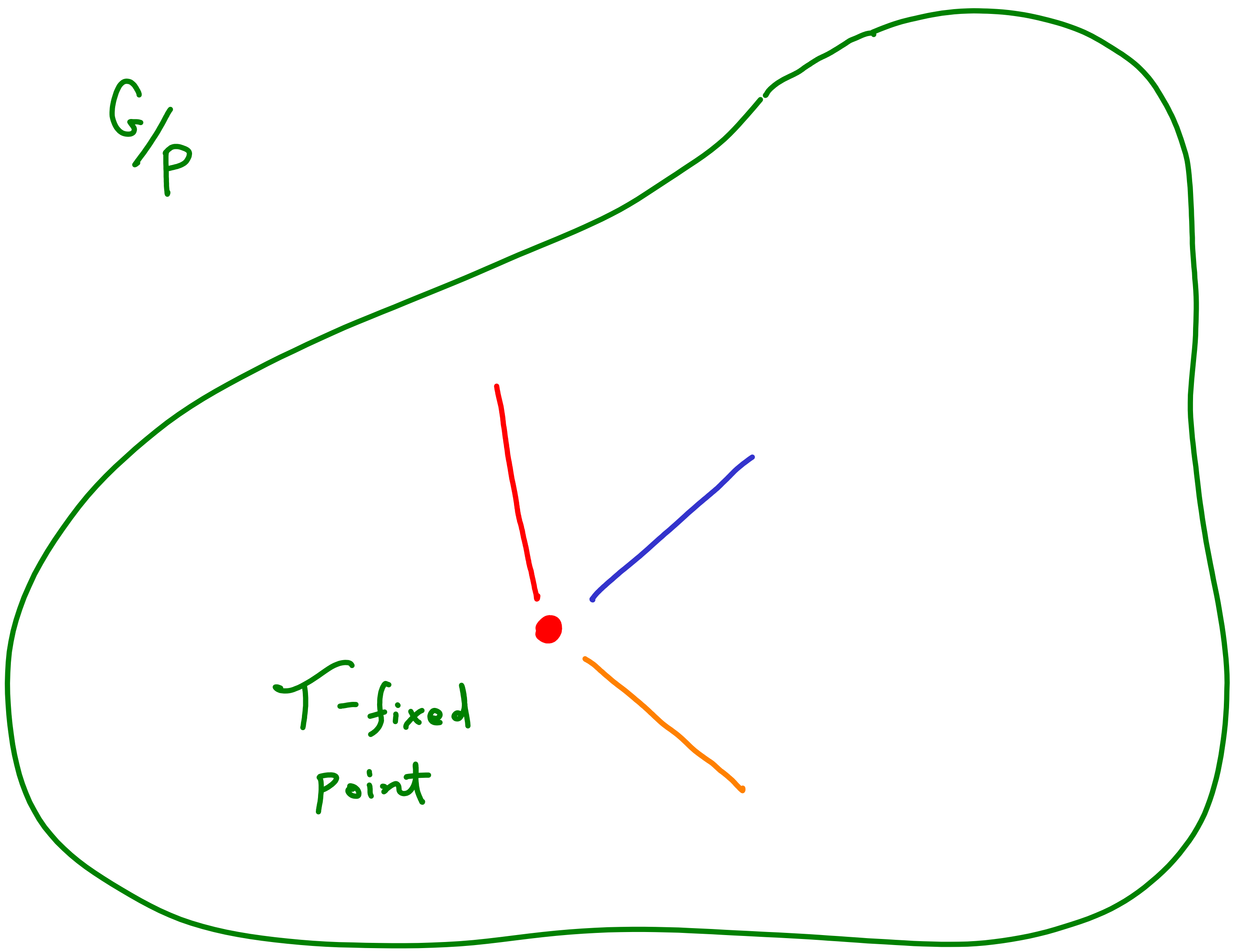
Take the coeffs of $\det(A_n + \lambda)$ for the deformation

Toda lattice interpretation

with Brusig at Mittag-Leffler,
we started discussing aspects of
the geometry of $\overline{\mathcal{M}}_{0,n}(G/P, d)$

- Paper by Hirschowitz (1988)
Rationalité des schémas de Hilbert de courbes gauches rationelles suivant Katsylo

Outcome was a paper with Brusig which proves rationality for all cases $\overline{\mathcal{M}}_{0,n}(G/P, d)$



irreducible
nonsingular
DM stack

$\overline{\mathcal{M}}_{0,n}(G/P, d)$ has an open set

which is an affine bundle over

use
Bialynicki
-Birula

$$\overline{\mathcal{M}}_{0,A} / H \subset \Sigma_A$$

We have $\mathcal{H} = \Sigma_{A_1} \times \Sigma_{A_2} \times \cdots \times \Sigma_{A_k}$

$A = \coprod_{i=1}^{k} A_i$

Then $\overline{\mathcal{M}}_{0,A} / H \subset \Sigma_A$

is rational [Katsylo, Bogomolov]

$$QH^*(G/P)$$

$\swarrow$ $\searrow$

Explicit Quantum Schubert calculus, q-deformed rules, Quantum K-theory

a lot of work

Geometry of $\overline{\mathcal{M}}_{0,n}(G/P, d)$

Oprea's study of tautological classes

Question: Is there a set of relations among tautological classes on

$\overline{\mathcal{M}}_{0,n}(G/P, d)$

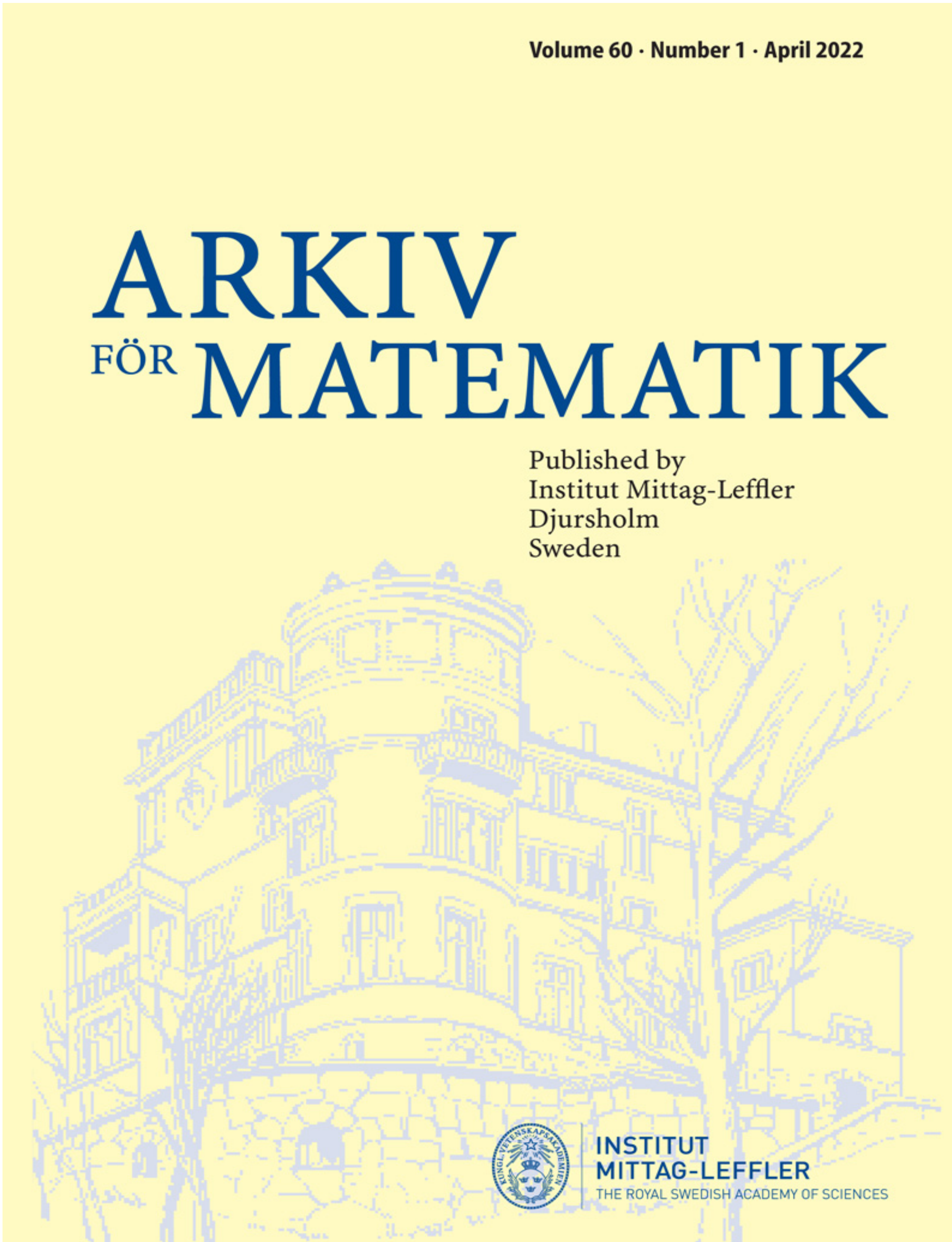
analogous to the Pixton set for $\overline{\mathcal{M}}_{g,n}$?

A step in this direction in the paper of Younghan Bae:

Tautological relations for stable maps to target varieties

Published in Arkiv för Matematik:

[Bae's article is in a different volume]



(ii) We now turn to my PhD thesis (1994):

study over $\overline{M}_g$ of the moduli of semistable sheaves on curves

$$\begin{tikzcd} \mathcal{U}_g(r) \arrow[d, "\varepsilon"'] & \supset & \mathcal{U}_C(r) \arrow[d] \\ \overline{M}_g & \ni & {[C]} \end{tikzcd}$$

Method of construction:

GIT on Quot schemes

relative $\begin{array}{c} \mathcal{C} \\ \pi \downarrow \\ \overline{\mathcal{M}}_g \end{array}$

$$\overset{\text{rk } k}{V} \otimes \mathcal{O}_C \to \overset{\text{rk } e}{E} \to 0$$

* Disadvantage: when C has nodes, $\mathcal{U}_g(r)$ is singular

Solution A

Moduli space of stable quotients

[Marian - Oprea - P 2009]

Idea: insist that E is locally free at the nodes (and markings) of C, $\omega_C \otimes (\det E)^{\lambda}$ ample $\forall \lambda > 0$

$$\overline{Q}_{g,n}(Gr(n,e), d)$$

moduli of stable quotients

<u>Theorem</u> [MOP] GW theory of

$$\overline{Q}_{g,n}(Gr(n,e), d) \quad \text{and}$$

$$\overline{M}_{g,n}(Gr(n,e), d) \quad \text{are}$$

the same in the strongest sense

CohFT

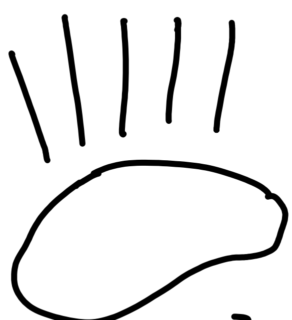
No wall crossing!

But in the hands of Y. Toda,

CF-Kim-Maulik, CF-Kim Quasimaps, Quasimap Wallcrossing

the subject was transformed

Results in the CY 3-fold case are very interesting

local CY  $\mathcal{O}(-3)$

$\mathbb{P}^2$

Wall Crossing!

Quintic $\subset \mathbb{P}^4$

Cooper-Zinger

Theorem [CF-Kim 2017]

The stable quotient series for CY3 geometries compute the B-model invariants.

Mirror Symmetry by Wall Crossing]

Holomorphic anomaly $\Rightarrow$ equation on the B-model side

Derivation with Hyenho Lho:

Stable quotients and the holomorphic anomaly equation (2018)

Lho-P

local case, quintic much harder

(iii) Bumsig was always interested in log geometry.

paper on log stable maps (2008)

paper with Kresch, Y-G Oh on unramified maps (2014)

I am sorry to have missed the opportunity to discuss with Bumsig our most recent work

Holmes - Molcho - P - Pixton - Schmitt (2022)

Solution B

(of the problem of singularities of

$\mathcal{U}_g(1)$ ← universal Picard)

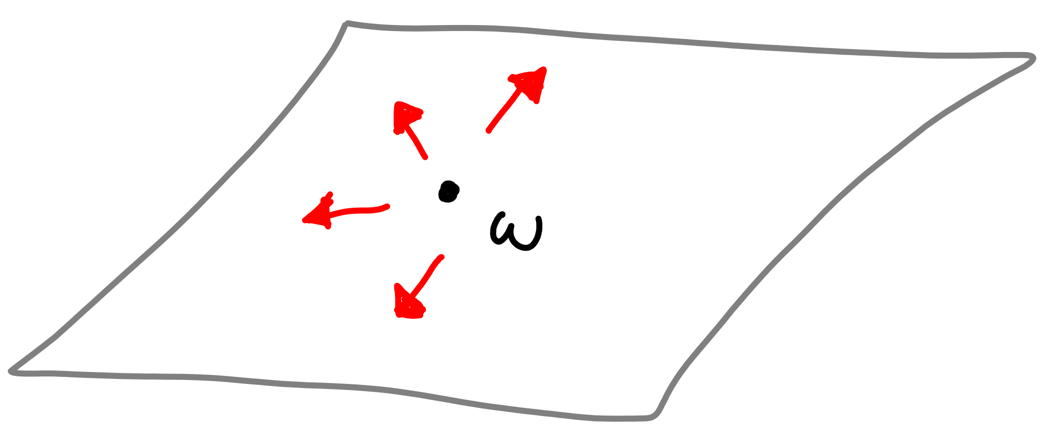
rank = 1

Caporaso (1993)

Solution: change stability conditions!

Kass-Pagani

Abreu-Pacini



Stability circa 1990   Caporaso

I will describe the idea from the perspective of my thesis:

We are interested in a proper moduli space of line bundles on nodal curves.

We have a canonical stability condition θ on stable curves parameterized by $\bar{M}_{g,n}$:

For $(C, p_1, \dots, p_n) \in \bar{M}_{g,n}$

and an irreducible component $D \subset C$

$$\theta(D) = 2g_D - 2 + \mathrm{val}_D$$ ← includes markings

Extend θ additively to subcurves $S \subset C$

Using θ, we can construct a moduli space:

$$\mathcal{P}ic^{\theta} \to \overline{M}_{g,n}$$

of θ-stable torsion free sheaves of rank 1 on stable curves by GIT.

$$\mathcal{L} \to (C, p_1, \dots, p_n) \text{ is } \theta\text{-stable}$$

we are interested in $\deg(\mathcal{L})=0$ case

$$\Updownarrow$$

for every subcurve $S \subset C$, $0 \to F_S \to \mathcal{L} \to \mathcal{L}|_S \to 0$

$$\frac{\chi(F_S)}{\theta(S)} < \frac{\chi(\mathcal{L})}{2g-2+n}$$

[Issues of non-locally free θ-stable sheaves solved by 1-step destabilization of C]

We obtain the universal Picard constructed by Caporaso, later P

Unfortunately, there are strictly semistable sheaves here.

Return to the subject almost 30 years later: Kass-Pagani, Abreu-Pacini

Idea is to study all possible stability conditions (not just θ).

We can perturb θ by finding a rule ε which assigns a rational number to every component of every

Stable curve $(C, p_1, \dots, p_n) \in \overline{M}_{g,n}$

with the additive property under smoothing

and $\varepsilon(C) = 0$.

If ε is small, $\hat{\theta} = \theta + \varepsilon$

is positive on all subcurves, and

we obtain a moduli space as before

by GIT

$$P_{ic}^{\hat{\theta}} \to \overline{M}_{g,n}$$

Abreu-Pacini construct such ε.

For generic choices $\Rightarrow$ No semistable elements!

QUESTION: Compute $\chi_{top}\left(P_{ic}^{\hat{\theta}}\right)$

Select a small and generic ε

Can be done explicitly, but there is a choice

Then we have

$$\begin{array}{c} \mathcal{P}ic^{\hat{\theta}} \\ \downarrow \\ \overline{\mathcal{M}}_{g,n} \end{array}$$

$\hat{\theta}$ determines canonically a blow-up

$$\begin{tikzcd} \overline{\mathcal{M}}_{g,n}^{\hat{\theta}} \arrow[r, "AJ"] \arrow[dr] & \mathcal{P}ic^{\hat{\theta}} \arrow[d] \\ & \overline{\mathcal{M}}_{g,n} \arrow[u, dashed, bend left, "AJ"] \end{tikzcd}$$

Choose

$A = (a_1, \ldots, a_n)$

$\Sigma a_i = 0$

on which the Abel-Jacobi map defined by A extends

By pulling back the universal family over $\mathfrak{P}ic^{\hat{\theta}}$ to $\bar{\mathcal{M}}_{g,n}^{\hat{\theta}}$, we can calculate the pull-back of the 0-section by applying the universal DR formula.

Theorem [HMPPS]

The result is the log DR cycle for the data A.

The result is about the most basic log GW geometry: maps to $\mathbb{P}^1/0 \cup \infty$.

Photos from Mittag-Leffler (96-97)

The End