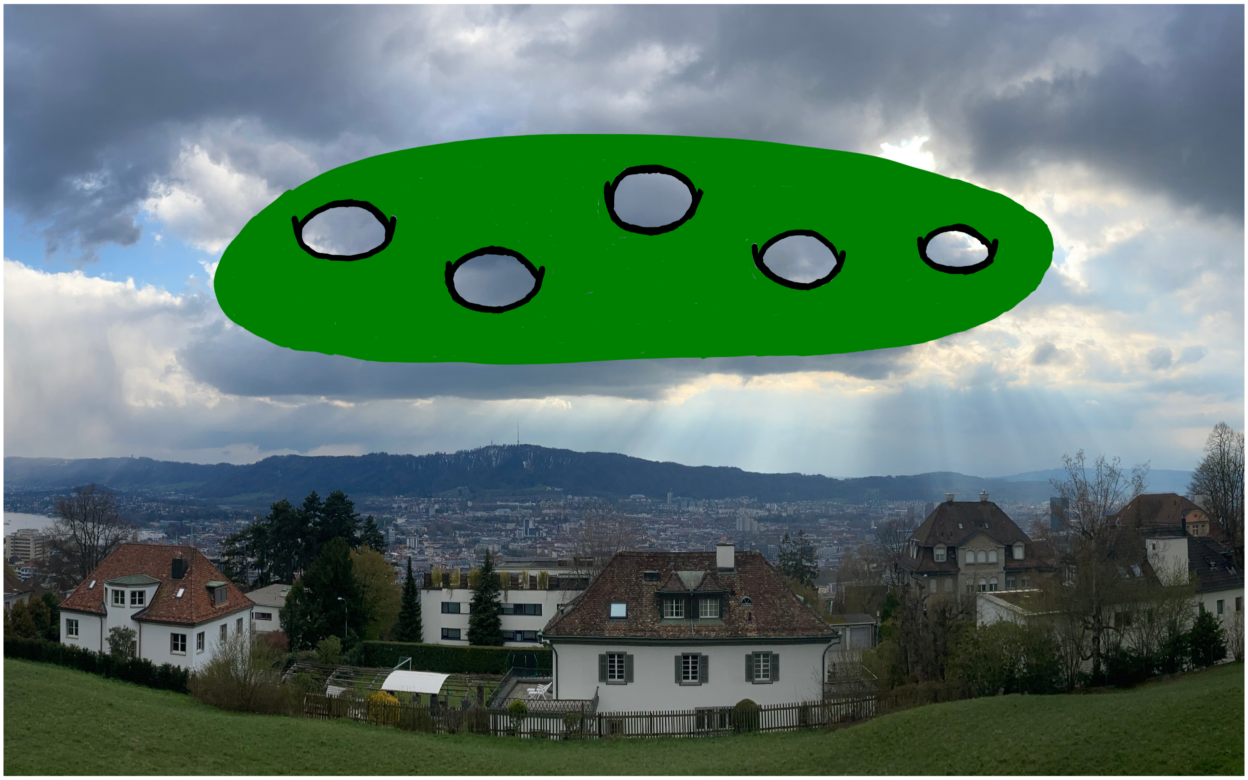
Tevelev degrees and Hurwitz moduli spaces



Algebraic Geometry Seminar

HU Berlin, 26 May 2021

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I. Start with the moduli space of stable maps to rubber:

$$\overline{M}_{g,\mu}(\mathbb{P}^1)^{\sim}$$

denotes $\mathbb{C}^*$ scaling

domain genus

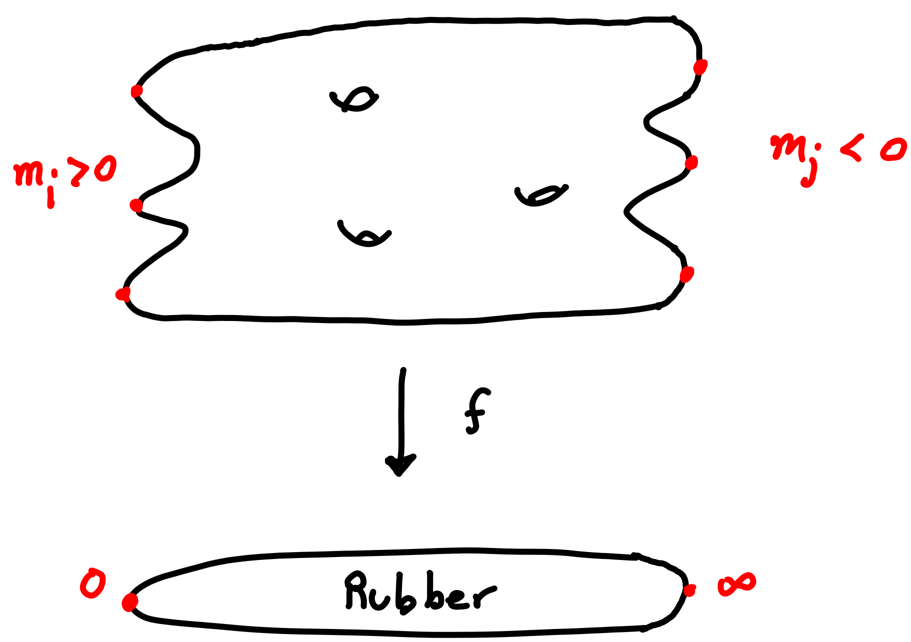
vector

$\mu = (m_1, \ldots, m_\ell)$

$m_i \in \mathbb{Z}$

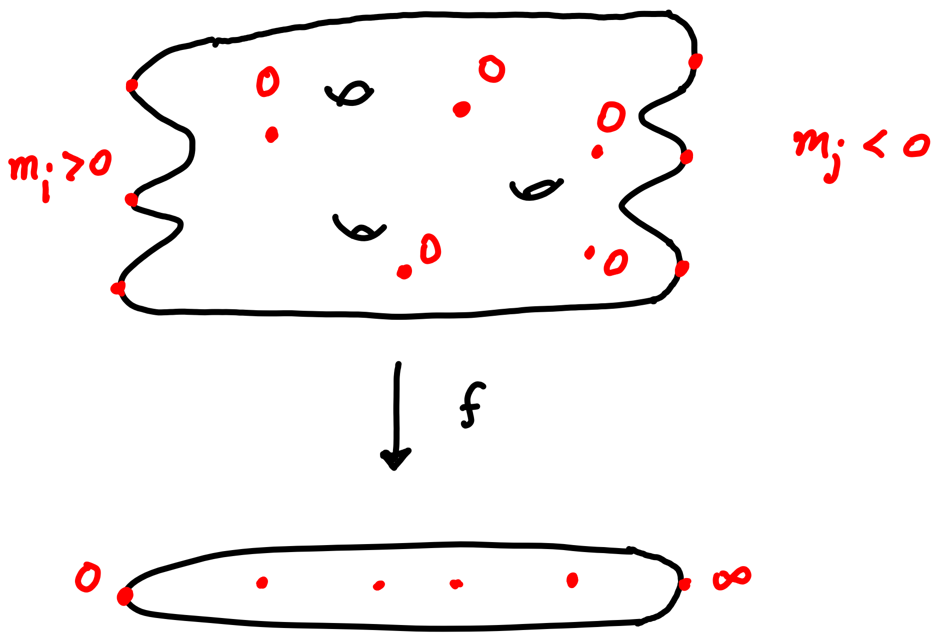
$\sum_{i=1}^{\ell} m_i = 0$

moduli space of maps



$\mathbb{C}^*$ scaling equivalence

We can include parts $m = 0$



$\mathbb{C}^*$ scaling equivalence

For clear notation:

\# of 0-parts

$$\overline{\mathcal{M}}_{g,\mu,n}(\mathbb{P}^1)^\sim$$

domain genus

vector

$\mu = (m_1, \dots, m_\ell, \underbrace{0, \dots, 0}_{n})$

$m_i \neq 0$

We obtain a correspondence

$$\begin{tikzcd}[column sep=small] & \overline{\mathcal{M}}_{g,\mu,n}(\mathbb{P}^1)^{\sim} \arrow[dl, "\text{domain}"'] \arrow[dr, "\text{target}"] & \\ \overline{\mathcal{M}}_{g,\ell+n} & & ? \end{tikzcd}$$

$\boxed{\text{What is the target moduli space?}}$

Answer: $\overline{\mathcal{M}}_{0,2|n}$ Losev-Manin space, arxiv/math/0001003

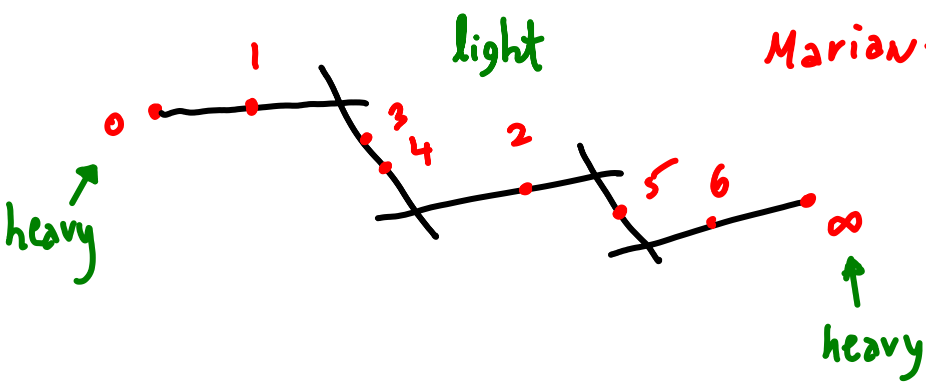
the Hassett space with

2 heavy

n light

Notation from Stable Quotients

Marian-Oprea-P, arxiv:0904.2992



moduli of pointed chains

Claim : Chow / H^* of $\overline{M}_{0,2|n}$ generated by strata ← chain type

[almost trivial]

The Betti Numbers of $\overline{M}_{0,2|n}$ (by Losev-Manin) :

$$1+\sum_{n=1}^{\infty}\sum_{i=0}^{n-1} y^n q^i \frac{h^{2i}(\overline{M}_{0,2|n})}{n!} = \frac{q-1}{q-e^{(q-1)y}}$$

in [Marian-Oprea-P], we calculate

$$\sum q^i h^{2i}\left(\overline{M}_{0,2|n}/\Sigma_n\right) = (1+q)^{n-1}$$

Strata freely generate

Correspondence:

$$\begin{array}{ccc} & \left[\overline{M}_{g,\mu,n}(\mathbb{P}^1)^{\sim}\right]^{vir} & \\ \swarrow \pi_d & & \searrow \pi_t \\ \overline{M}_{g,\ell+n} & & \overline{M}_{0,2|n} \end{array}$$

dim $3g-3+\ell+n$ dim $n-1$

$$\left[\overline{\mathcal{M}}_{g,\mu,n}(\mathbb{P}^1)^{\sim}\right]^{vir}$$ algebraic cycle class of dimension $2g-3+\ell+n$

Question: Compute

$$(\pi_d \times \pi_t)_* \left[\overline{\mathcal{M}}_{g,\mu,n}(\mathbb{P}^1)^{\sim}\right]^{vir}$$

$$\cap$$

$$H^*\left(\overline{\mathcal{M}}_{g,\ell+n} \times \overline{\mathcal{M}}_{0,2+n}\right)$$

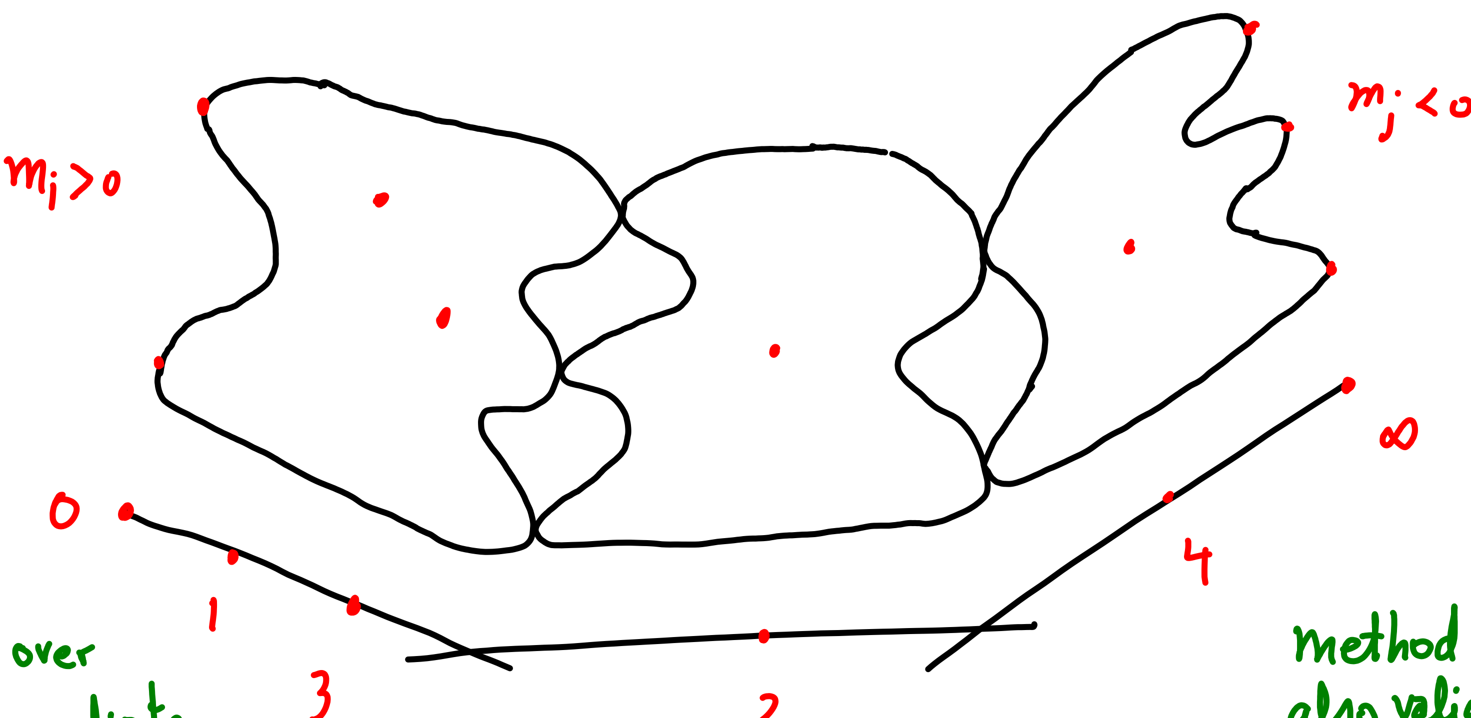
Complete solution via Pixton's formula for the DR cycle.

Sketch: We must calculate

$$\pi_{d\,\star}\left(\left[\overline{\mathcal{M}}_{g,\mu,n}\left(\mathbb{P}^1\right)^{\sim}\right]^{vir}\cdot\pi_t^{\star}\left(\begin{array}{c}\text{Chain}\\ \text{Stratum}\end{array}\right)\right)$$

$=$

$\boxed{\text{To each link, apply Pixton}}$



Sum over intermediate partitions

method also valid in CHOW

Question: Can we use the correspondence for relations in $\overline{\mathcal{M}}_{g,\ell+n}$?

II. Hurwitz Correspondences

My first encounter was in 1997

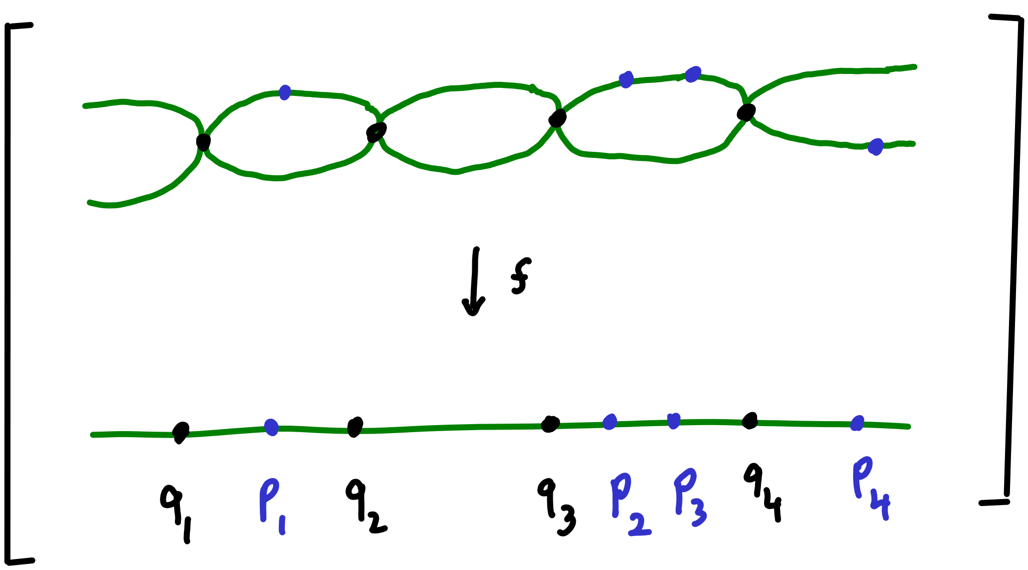
ad covers [Harris-Mumford]

branch points

markings

$$\begin{tikzcd} & \overline{\mathcal{H}}_{1,2,4,4} \arrow[dl, "\pi_d"'] \arrow[dr, "\pi_t"] & \\ \overline{\mathcal{M}}_{1,8} & & \overline{\mathcal{M}}_{0,8} \end{tikzcd}$$

genus degree



$\in \overline{\mathcal{H}}_{1,2,4,4}$

Idea in 1997 : alg-geom/9705016

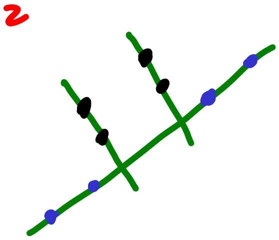
$$
\begin{tikzcd}
 & \overline{H}_{1,2,4,4} \arrow[dl, "\pi_d"'] \arrow[dr, "\pi_t"] & \\
\overline{M}_{1,8} \arrow[d, "\varepsilon_d"'] & & \overline{M}_{0,8} \arrow[d, "\varepsilon_t"] \\
\overline{M}_{1,4} & & \overline{M}_{0,4}
\end{tikzcd}
$$

marking points

marking points

Take $\zeta \in M_{0,4}$,

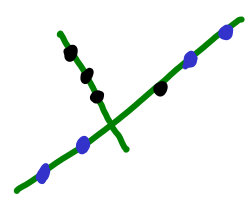
A_{22}



$$\varepsilon_{d*} \, \pi_{d*} \, \pi_t^{-1}\left(\varepsilon_t^{-1}(\zeta) \cap A_{22} \right)$$

proportional to

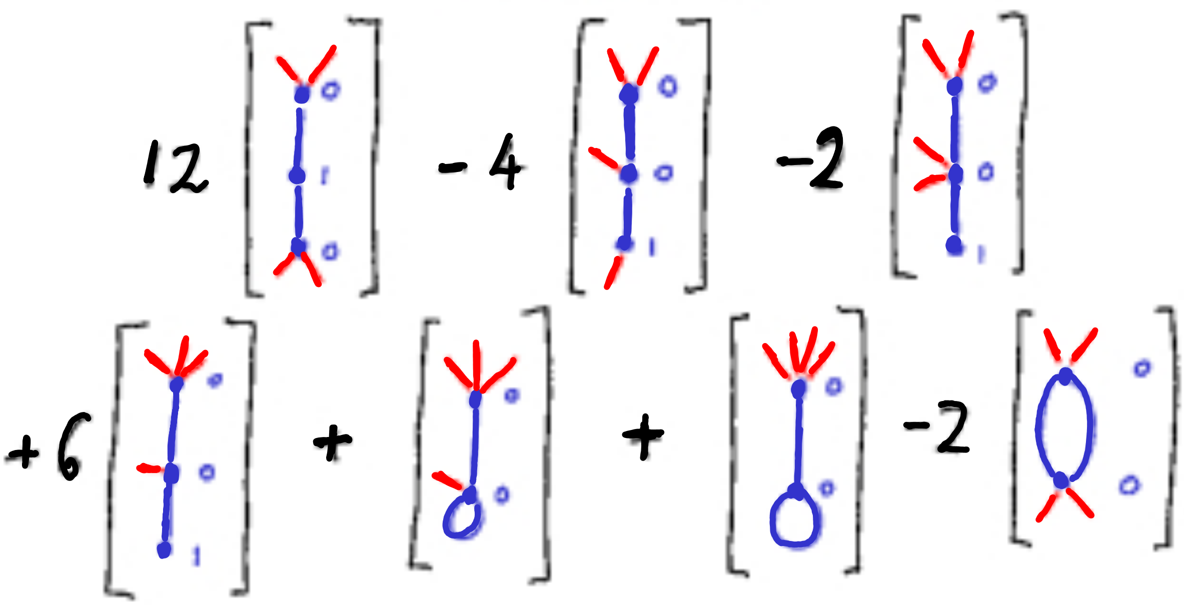
A_{31}



$$\varepsilon_{d*} \, \pi_{d*} \, \pi_t^{-1}\left(\varepsilon_t^{-1}(\zeta) \cap A_{31} \right)$$

Outcome is an algebraic construction of

Getzler's relation:



$$= 0 \in H^4(\overline{\mathcal{M}}_{1,4})$$

also $A^2(\overline{\mathcal{M}}_{1,4})$

Speculation/Question (Later with Faber 2006)

$$\begin{tikzcd} & \overline{H}_{g,d,b,n} \arrow[dl, "\pi_d"'] \arrow[dr, "\pi_t"] & \\ \overline{M}_{g,b+n} & & \overline{M}_{0,b+n} \end{tikzcd}$$

$$\boxed{\begin{array}{l}\text{Are all relations among tautological classes}\\ \text{in higher genus generated by}\\ \qquad \pi_{d*}\,\pi_t^*\big(\text{Relations in genus } 0\big)\ ?\end{array}}$$

The question only makes sense if Hurwitz loci are tautological.

III. Classes of Hurwitz loci

Theorem (Faber–P 2006)

$$(\pi_d \times \pi_t)_* \left[\overline{H}_{g,d,b,n} \right] \in R^*(\overline{M}_{g,b+n}) \otimes R^*(\overline{M}_{0,b+n})$$

as an algebraic cycle class.

Sketch of proof:

We prove a more general claim:

- Consider arbitrary ramification profiles (not just simple ramification)

most interesting direction

- do not specify all ramifications

- include all cotangent line classes

The idea is simple:

Suppose we are interested

$$ {}^{GW}\mathcal{H}^{\sim}_{g,\mu^1,\mu^2,\dots,\mu^k} $$

mixed space with k moving relative points and stable map geometry away

$$ \mu^k = (m_1, m_2, \dots) $$

Then Consider

stable map marking

$$ {}^{GW}\mathcal{H}^{\mathbb{P}^1}_{g,l,\mu^1,\mu^2,\dots,\mu^{k-1}} \qquad +3 \text{ vir dim} $$

Calculate the following relation by Virtual localization

$$ \pi_{d*}\left(ev_1^{2+\delta}(\infty) \prod_{i=1}^{l} \psi_i^{m_i-1} ev_i(0) \left[{}^{GW}\mathcal{H}^{\mathbb{P}^1}_{g,l,\mu^1,\mu^2,\dots,\mu^{k-1}} \right]^{vir} \right) $$

forces vanishing

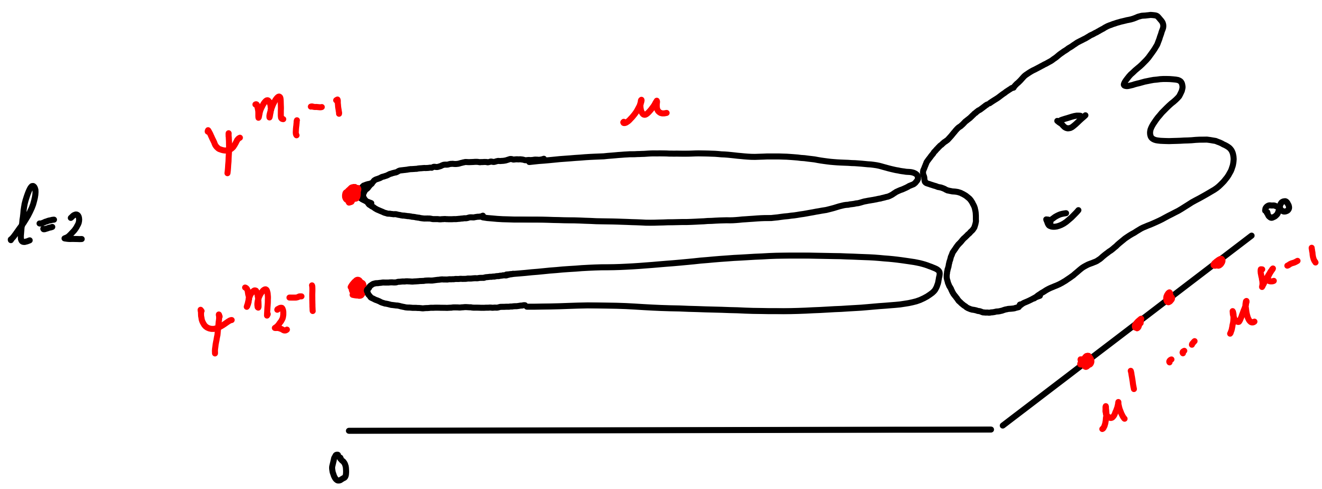
arrange the dim

$$ = 0 \in A(\overline{\mathcal{M}}_{g,\text{all}}) $$

Massive induction...

Principal terms

genus g



Challenge: Prove the Hurwitz loci yield tautological classes without Gromov-Witten theory stable maps, virtual classes, localization.

Side remark: A lot of the above geometry used in the JPPZ study of the DR cycle.

IV. Return to the Hurwitz Correspondence

genus g

$$\begin{array}{c} C \\ \pi \big\downarrow \\ \mathbb{P}^1 \end{array}$$

degree d

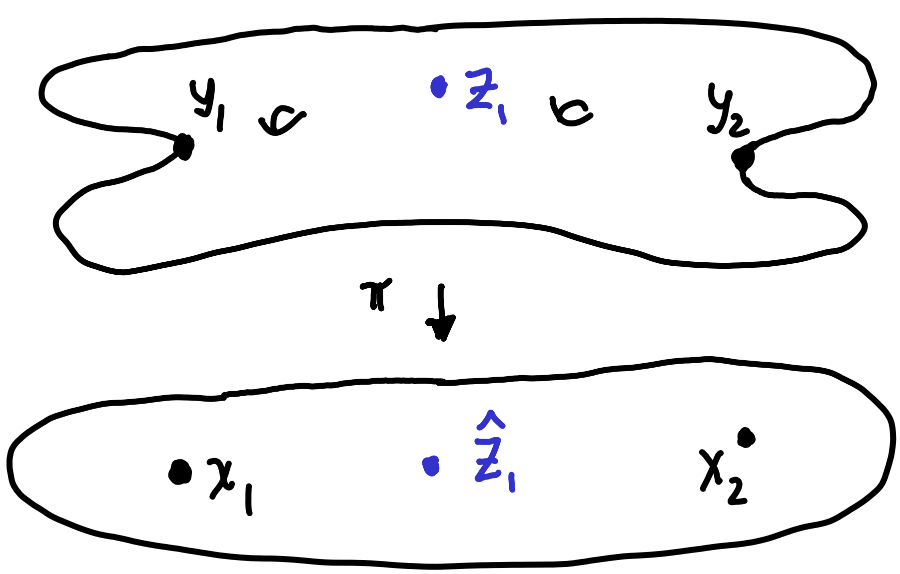
Hurwitz cover with C connected

ramification points $\pi(y_i) = x_i$

$$\begin{array}{l} (C, \; y_1, \ldots, y_b, \; z_1, \ldots, z_n) \\ \pi \big\downarrow \\ (\mathbb{P}^1, \; x_1, \ldots, x_b, \; \hat{z}_1, \ldots, \hat{z}_n) \end{array}$$

extra points satisfying $\pi(z_i) = \hat{z}_i$

branch points



Admissible covers

Harris-Mumford

1980s

Hurwitz Correspondence:

$$\begin{tikzcd}[column sep=small] & \overline{\mathcal{H}}_{g,d,b,n} \arrow[dl, "\pi_d"'] \arrow[dr, "\pi_t"] & \\ \overline{\mathcal{M}}_{g,b+n} & & \overline{\mathcal{M}}_{0,b+n} \end{tikzcd}$$

markings

Simple branch points

Connected Hurwitz number

- $$\mathcal{H}^{0}_{g,d} = \frac{\deg(\pi_t)}{d^n}$$

[disregard the markings]

Hurwitz Numbers are a small part of the information of the correspondence

- For a glimpse of the correspondence from another angle :

$$\mathcal{E} : \overline{\mathcal{H}}_{g,d,b,n} \to \overline{\mathcal{M}}_{g,n} \times \overline{\mathcal{M}}_{0,n}$$

(π_d, π_t) followed by forgetting the b ramification and b branch points

disregard the branch points

$$\dim \overline{\mathcal{H}}_{g,d,b,n} = 2g+2d-5+n$$

$$\dim \overline{\mathcal{M}}_{g,n} + \dim \overline{\mathcal{M}}_{0,n} = 3g-6+2n$$

dimensions usually different

Tevelev considered the case 

$d = g+1, \quad n = g+3$

where both dimensions are $5g$

Theorem (Tevelev, 2020) for $d = g+1$, $n = g+3$

$$\frac{\deg(\mathcal{E})}{(4g)!} = 2^g \qquad [4g = b]$$

Motivated by study of Scattering amplitudes by

N. Arkani-Hamed, Bourjaily, Cachazo, Postnikov, Trnka   arXiv:1412.8475

See Tevelev's paper arXiv: 2007.03831

A basic goal of the construction is to define a probability measure on $\mathcal{M}_{0,g+3}$ via a canonical map

$$\Upsilon : \mathrm{Pic}^{g+1}(C_g, p_1, \dots, p_{g+3}) \dashrightarrow \mathcal{M}_{0,g+3}$$

the translation invariant measure here

The first question about the construction concerns the degree which is $\frac{\deg(\Upsilon)}{(4g)!} = 2^g$

Most general case where image and range of

$$\varepsilon : \overline{\mathcal{H}}_{g,d,b,n} \to \overline{\mathcal{M}}_{g,n} \times \overline{\mathcal{M}}_{0,n}$$

have the same dimension is

$$d = g+1+\ell, \quad n = g+3+2\ell \qquad \begin{cases} g \geq 0 \\ \ell \in \mathbb{Z} \end{cases}$$

We define $\displaystyle \mathrm{Tev}_{g,\ell} = \frac{\deg(\varepsilon)}{b!}$

Tevelev's result is $\mathrm{Tev}_{g,0} = 2^g$

In March, with Alessio Cela and Johannes Schmitt we have computed $T_{g,\ell}$ in all cases.

Theorem (Cela, P, Schmitt, 2021):

arxiV:2103.14055

$$\mathrm{Tev}_{g,\ell} = 0 \quad \text{unless} \quad g \geq -2\ell$$

$$\mathrm{Tev}_{g,\ell} = 2^g - 2\sum_{i=0}^{-\ell-2}\binom{g}{i} + (-\ell-2)\binom{g}{-\ell-1} + \ell\binom{g}{-\ell}$$

Proof uses excess intersection calculations in the boundary of $\overline{\mathcal{H}}_{g,d,b,n}$

other approaches:

Cavalieri-Markwig-Ranganathan

Farkas-Lian

Historical Note: For $h \geq 0$,

1885

$$\mathrm{Tev}_{2h,-h} = \text{Castelnuovo's Count (1889) of linear series } g^1_{h+1} \text{ on a genus } 2h \text{ curve}$$

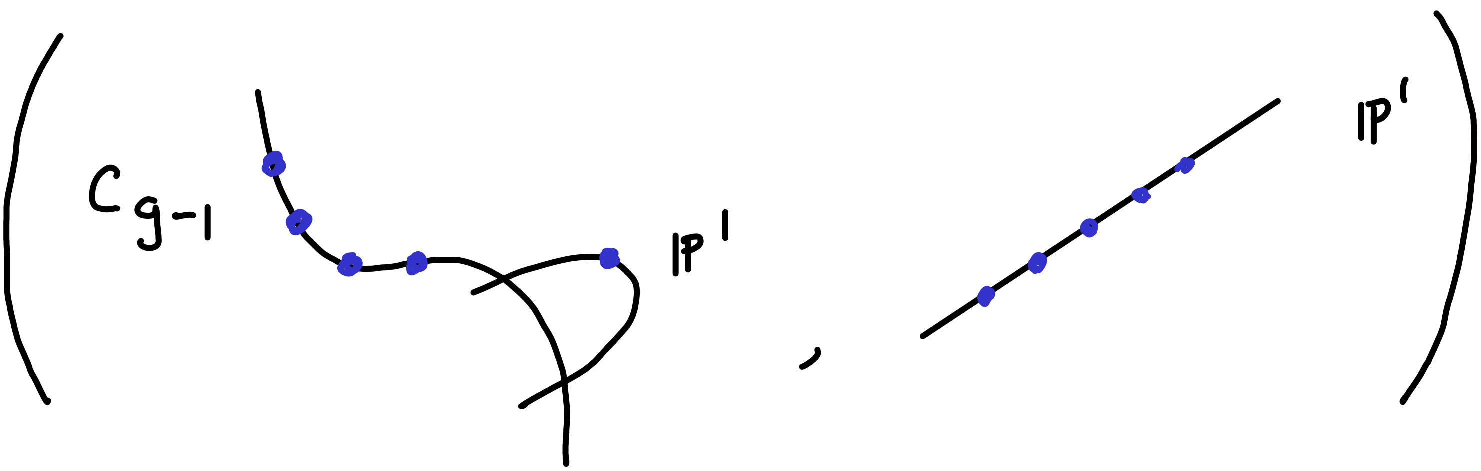
$$= \mathrm{Catalan}(h)$$

Strategy is to calculate the excess intersection determined by the fiber

$$\varepsilon : \overline{\mathcal{H}}_{g,d,b,n} \to \overline{\mathcal{M}}_{g,n} \times \overline{\mathcal{M}}_{0,n}$$

over the point

$$\left(\quad , \quad \right)$$



The result is a genus recursion which solves the problem (and also other problems with more refined Hurwitz data).

V. General targets X

Hurwitz moduli spaces concern the target $\mathbb{P}^1$

For the general case, let X be a nonsingular projective variety.

Curve class $\beta \in H_2(X, \mathbb{Z})$

Stable maps

$$\begin{tikzcd} & \overline{M}_{g,n}(X,\beta) \arrow[dl, "\pi_d"'] \arrow[dr, "\pi_t"] & \\ \overline{M}_{g,n} & & X^n \end{tikzcd}$$

For the correspondence, we push-forward the virtual fundamental class

$$\left[\overline{M}_{g,n}(X,\beta) \right]^{vir}$$

The calculation of the push-forward in cohomology

$$(\pi_d \times \pi_t)_* \left[\overline{M}_{g,n}(X,\beta) \right]^{vir} \in H^*\left(\overline{M}_{g,n} \times X^n\right)$$

is exactly the question of computing the $\mathrm{CohFT}(X)$.

The Tevelev question here:

$$\boxed{\text{Compute } \deg^{vir}(\pi_d \times \pi_t)}$$

degree with respect to vir class

when the dimensions match,

$$\text{virdim } \overline{M}_{g,n}(X,\beta) \underset{\substack{\uparrow \\ \text{dim constraint}}}{=} \dim \overline{M}_{g,n} + n \cdot \dim X$$

There is a perfect answer in GW theory.

Theorem: When the dimensions match,

$$\deg^{vir}(\pi_d \times \pi_t) = \mathrm{Coeff}_{q^\beta \cdot p}\left(p^{*n} * \Delta^{*g} \right)$$

quantum product in $QH^*(X)$

Sketch: We have

$$p, \Delta \in QH^*(X) \cong H^*(X) \otimes_{\mathbb{Q}} Nov$$

$\mathbb{Q}$-vector

point class

$p \in H^{2\dim X}(X)$

diagonal class

$$\Delta = \sum_{ij} g^{ij}\, T_i * T_j$$

quantum $*$

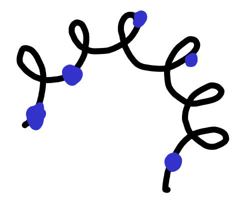
By usual applications of splitting axioms :

does not require convexity

$$\deg^{\mathrm{vir}}\left(\pi_d \times \pi_t\right)$$

$$= \mathrm{Coeff}_{q^\beta \cdot p}\left(p^{\star n} \star \Delta^{\star g}\right)$$

use the Curve



How to think about Δ ?

$$\Delta = \sum_{ij} g^{ij}\, T_i \star T_j$$

$$= \sum_{ij} g^{ij}\, T_i \cdot T_j \; + \; \text{quantum corrections}$$

$$= \chi_{top}(X) \cdot p \; + \; \cdots$$

deformed Euler characteristic

Example (i): $X = \mathbb{P}^1$

$$\Delta = P \star \mathrm{Id} + \mathrm{Id} \star P = 2P$$

$QH^*(\mathbb{P}^1) \Rightarrow$ $P \star P = q^1 \cdot \mathrm{Id}$

$P \star P \star P = q^1 \cdot P$

dimension match : $2g + 2d - 2 + n = 3g - 3 + n + n$

rewrite as : $n + g = 2d + 1$ odd

$$\mathrm{Coeff}_{q^d \cdot P}\left(P^{\star n} \star \Delta^{\star g}\right) = 2^g$$

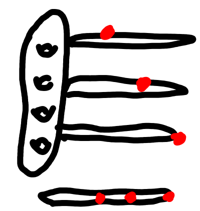
for $n + g = 2d + 1$

appears already in CohFT study of $\mathbb{P}^1$ by Janda

Teleman's original formula is true in <u>all cases</u> in GW theory!

Exercise: why is the answer of

Cela-P-Schmitt different?

[Hint: virtual contributions ]

Example (ii): $X = \mathbb{P}^N$

dimension: $(N+1)d + (3-N)(g-1) + n = 3g-3+n+Nn$

implies: $Nn + Ng = (N+1)d + N$

We must calculate:

$$\mathrm{Coeff}_{q^d \cdot p}\left(p^{*n} * \Delta^{*g}\right)$$

for $Nn + Ng = (N+1)d + N$

For $\mathbb{P}^N$, quantum corrections vanish here

$$\Delta = \sum_{ij} g^{ij}\, T_i \star T_j = \sum_{ij} g^{ij}\, T_i \cdot T_j + \dots$$

$$= \chi_{top}(\mathbb{P}^N)\cdot p = (N+1)\cdot p$$

$$\mathrm{Coeff}_{q^d \cdot p}\left(p^{\star n} \star \Delta^{\star g}\right) =$$

$$\mathrm{Coeff}_{q^d \cdot p}\left(p^{\star (n+g)}\right)\cdot (N+1)^g$$

Answer is always $(N+1)^g$

for $Nn + Ng = (N+1)d + N$

- Farkas-Lian analyse the enumerative (non-virtual) count: result is $(N+1)^g$ in a certain range.
  arxiv:2105.09340
- Bertram-Daskalopoulos-Wentworth
  arxiv:9306005 [used Quot scheme]

Example (iii): Grassmannians, flag varieties, G/P, etc.

$$\deg^{vir}(\pi_d \times \pi_t) =$$

$$\mathrm{Coeff}_{q^\beta \cdot p}\left(p^{*(n+g)}\right) \cdot \left(\chi_{top}(X)\right)^g$$

$+$ quantum correction terms from Δ

$QH^*(X)$ known very well, so the formula can be expanded explicitly.

Should equal enumerative count in certain asymptotic ranges [where virtual contributions vanish]

Argument by A. Buch $\Rightarrow$

$\mathrm{Coeff}_{q^d \cdot p}\left(p^{*(n+g)}\right)$ is always $\begin{cases} 1 \\ \text{or} \\ 0 \end{cases}$

for all Grassmannians

when the dim constraint holds.

Answer depends modular condition

Also true for the cominuscule case

[P is a Seidel element]

Sub Example : $Gr(2,4)$

$$\triangle = \underbrace{6}_{\chi_{top}(G(2,4))} \; p + 2 \underbrace{q}_{\text{quantum correction}} \, \mathrm{Id}$$

Buch's Calculator

Can we compute $\mathrm{Coeff}_{q^d \cdot p}\left(p^{*n} * \triangle^{*g}\right)$?

Use relation $p^2 = q^2$, two cases:

n even

$$\mathrm{Coeff}_{q^d \cdot p}\left(p^{*n} * \Delta^{*g}\right) = \sum_{\substack{0 \le i \le g \\ i \text{ odd}}} \binom{g}{i} 6^i 2^{g-i}$$

n odd

$$\mathrm{Coeff}_{q^d \cdot p}\left(p^{*n} * \Delta^{*g}\right) = \sum_{\substack{0 \le i \le g \\ i \text{ even}}} \binom{g}{i} 6^i 2^{g-i}$$

Theorem: For $Gr(2,4)$ and g, n, d satisfying the dimension constraint $d+1 = n+g$,

$$\deg^{vir} = \sum_{0 \le i \le g,\ i \text{ odd}} \binom{g}{i} 6^i 2^{g-i} \qquad \text{for } n \text{ even}$$

$$\deg^{vir} = \sum_{0 \le i \le g,\ i \text{ even}} \binom{g}{i} 6^i 2^{g-i} \qquad \text{for } n \text{ odd}$$

Open directions: Virtual analysis / other Grassmannians

Buch explains that the quadric

$$Q^N \subset \mathbb{P}^{N+1} \quad \text{for } N \geq 3$$

is the simplest generalization of the $Gr(2,4)$ case.

$$\triangle = \chi_N \cdot p + \delta_N q \cdot Id$$

$$\chi_N = \chi_{top}(Q^N) = \begin{cases} N+2 & N \text{ even} \\ N+1 & N \text{ odd} \end{cases} \qquad \delta_N = \begin{cases} N-2 & N \text{ even} \\ N-1 & N \text{ odd} \end{cases}$$

$$\boxed{\begin{array}{l} \text{Theorem (Buch): For } Q^{N \geq 3} \text{ and } g, n, d \\ \text{satisfying dimension constraint } d+1 = n+g, \\ \deg^{vir} = \displaystyle\sum_{0 \leq i \leq g,\ i \text{ odd}} \binom{g}{i} \chi_N^{\,i} \cdot \delta_N^{\,g-i} \quad \text{for } n \text{ even} \\ \deg^{vir} = \displaystyle\sum_{0 \leq i \leq g,\ i \text{ even}} \binom{g}{i} \chi_N^{\,i} \cdot \delta_N^{\,g-i} \quad \text{for } n \text{ odd} \end{array}}$$

Question: Find the range where these degrees are enumerative [Genus 0 always enumerative for G/P]

Exercise: What is the answer for $Q^2 = \mathbb{P}^1 \times \mathbb{P}^1$? [Answer 4^g]

Product rule for $X \times Y$:

Suppose we have $g, n, \beta_X \in H_2(X,\mathbb{Z}), \beta_Y \in H_2(Y,\mathbb{Z})$

Satisfying

- $\text{virdim}\, \overline{M}_{g,n}(X, \beta_X) = \dim \overline{M}_{g,n} + n \cdot \dim X$
- $\text{virdim}\, \overline{M}_{g,n}(Y, \beta_Y) = \dim \overline{M}_{g,n} + n \cdot \dim Y$

Then we also have

- $\text{virdim}\, \overline{M}_{g,n}(X \times Y, \beta_X + \beta_Y) = \dim \overline{M}_{g,n} + n \cdot \dim(X \times Y)$

AND $\boxed{\deg^{vir}_{X \times Y} = \deg^{vir}_X \cdot \deg^{vir}_Y}$ Geometrically expected

Proof uses product formula in GW theory by

K. Behrend

$\underline{\overline{\text{VI}}}$ Descendent integrals

$$
\begin{tikzcd}
 & \overline{\mathcal{H}}_{g,d,b,n} \arrow[dl, "\pi_d"'] \arrow[dr, "\pi_t"] & \\
\overline{\mathcal{M}}_{g,b+n} & & \overline{\mathcal{M}}_{0,b+n}
\end{tikzcd}
$$

b: simple branch points

n: markings

We can study numerical aspects of the correspondence via the integrals:

$$\int_{\overline{\mathcal{H}}_{g,d,b,n}} \pi_d^* \left(\prod_{i=1}^{b+n} \psi_i^{\alpha_i} \right) \cdot \pi_t^* \left(\prod_{i=1}^{b+n} \psi_i^{\alpha_i} \right)$$

Can we compute?

Answer: Yes!

Sketch:

- Cotangent line classes on

$$\overline{\mathcal{H}}_{g,d,b,n} \quad \text{and} \quad \overline{\mathcal{M}}_{0,b+n}$$

differ only by ramification factors

- Cotangent line classes on

$$\overline{\mathcal{H}}_{g,d,b,n} \quad \text{and} \quad \overline{\mathcal{M}}_{g,b+n}$$

differ by boundary rules

painful

See Lian for a modern treatment

Using these two observations, we can reduce first to

$$\int_{\overline{\mathcal{H}}_{g,d,b,n}} \prod_{i=1}^{b+n} \psi_i^{\alpha_i}$$

$$\|$$

$$\left(\frac{d^n}{2^b}\right) \cdot H^{\circ}_{g,d} \cdot \int_{\overline{\mathcal{M}}_{0,b+n}} \prod_{i=1}^{b+n} \psi_i^{\alpha_i}$$

easy

so Hurwitz numbers and genus 0 descendents determine everything

Such a path has been known forever, but is awaiting application

see Costello

VII Other moduli directions

$$\begin{tikzcd}
 & \left[\overline{\mathcal{M}}_{g,n}(\pi,\beta)\right]^{red} \arrow[dl] \arrow[dr] & \\
\overline{\mathcal{M}}_{g,n} & & \mathcal{Z}^n \arrow[d, dash] \\
 & & \mathcal{M}_h^{K3}
\end{tikzcd}$$

where $\pi : \mathcal{Z} \to \mathcal{M}_h^{K3}$ universal family of K3 surfaces

Correspondence is much harder to study than the Hurwitz correspondence.

Most questions open

[P - Q. Yin low genus results for study of $R^*(\mathcal{M}_h^{K3})$

The End

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