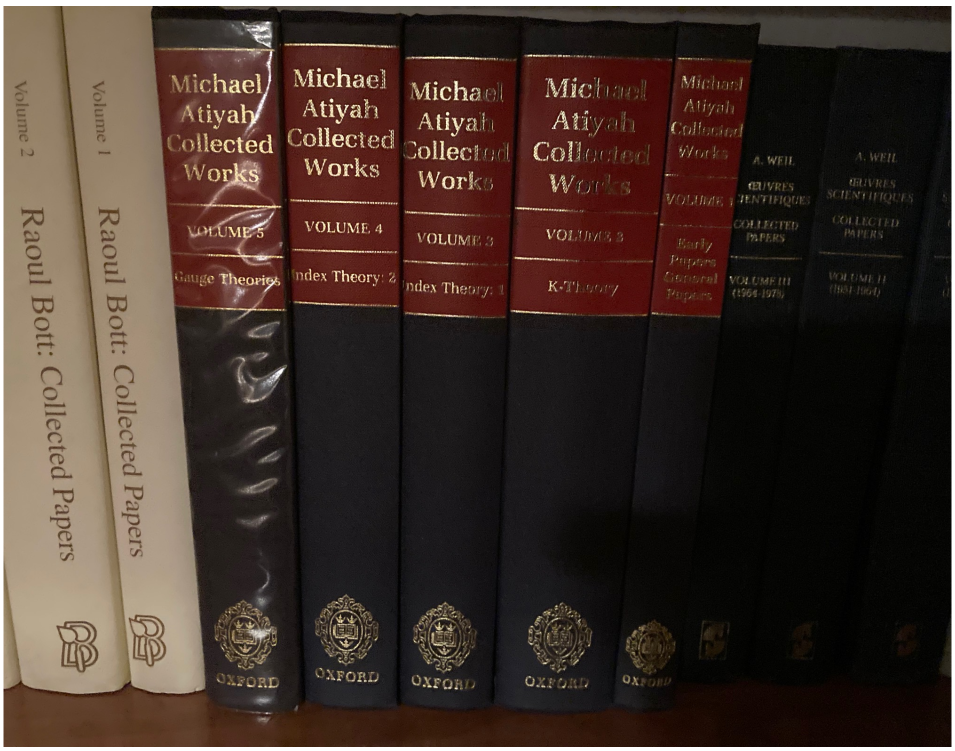
Torus actions on moduli spaces after Atiyah and Bott



Newton Institute

21 September 21

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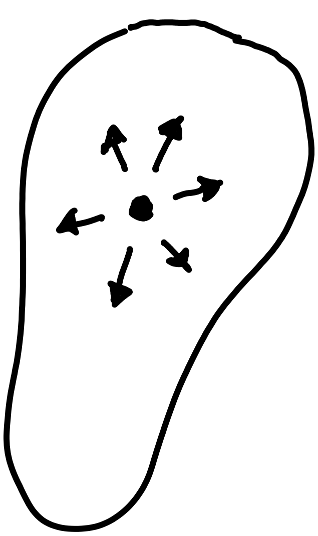
§1 We start with a formula of Bott 1967

- M a compact complex manifold
- $v \in H^0(TM)$ a holomorphic vector field with isolated non-degenerate zeros

for every zero $p \in M$ of v

$\Rightarrow$ flow along v yields an endomorphism



$$L_p : T_pM \to T_pM$$

with eigenvalues $m = \dim_{\mathbb{C}} M$

$$\lambda_1^p, \lambda_2^p, \dots, \lambda_m^p$$

all non-vanishing

The Bott Residue formula

elementary symmetric functions of $\{\lambda_i^p\}$

$$\int_M p(c_1, \ldots, c_m) = \sum_{p \in \mathrm{Zeros}(V)} \frac{p(e_1^p, \ldots, e_m^p)}{e_m^p}$$

$$e_m^p = \prod_{i=1}^{m} \lambda_i^p$$

polynomial in Chern classes of TM

Michigan MJ "Vector fields and characteristic Numbers"

Atiyah wrote the Review (see MathsciNet) and pointed out lines for two different proofs. The second is via equivariant geometry.

§2 We jump forward to 1984 where the idea takes a concrete form

Atiyah-Bott, Topology

"The moment map and equivariant cohomology"

Can be pursued in

- topology — Witten's complex
- symplectic geometry — Duistermaat-Heckman
- algebraic geometry (later 1996) — equivariant Chow, Totaro, Edidin-Graham

Atiyah and Bott write very modestly:

"our contribution is therefore mainly an expository one linking together various points of view."

see also Berline-Vergne 82

M nonsingular projective algebraic variety $/\mathbb{C}$

$T \times M \to M$ algebraic torus action

$T = \mathbb{C}^*$

- $H^*(M)$ singular cohomology,

  $H^*_T(M)$ equivariant cohomology

- $CH^*(M)$ Chow groups of algebraic cycles,

  $CH^*_T(M)$ equivariant Chow

What is $H_T^*(M)$?

Simplest path:

$T = \mathbb{C}^*$ acts on $\mathbb{C}^{k+1}$ by scaling all coordinates

Projective space $\quad \mathbb{CP}^k = \dfrac{\mathbb{C}^{k+1} \setminus \{0\}}{\mathbb{C}^*}$

Definition:

$$H_T^*(M) = \lim_{k \to \infty} H^*\left(M \times_T \left(\mathbb{C}^{k+1} \setminus \{0\}\right)\right)$$

quotient by T of the product $M \times \left(\mathbb{C}^{k+1} \setminus \{0\}\right)$

$H_T^*(M)$ is a module over $H^*(\mathbb{CP}^\infty) \cong \mathbb{Q}[u]$

nonsingular varieties

$$M^T = \coprod_\alpha M^T_\alpha \xhookrightarrow[\text{inclusion}]{\iota} M$$

T-fixed locus

Normal bundle $M^T_\alpha \subset M$

Atiyah-Bott 1984

$$[M] = \sum_\alpha \iota_* \frac{[M^T_\alpha]}{c^T_{top}(Nor_\alpha)}$$

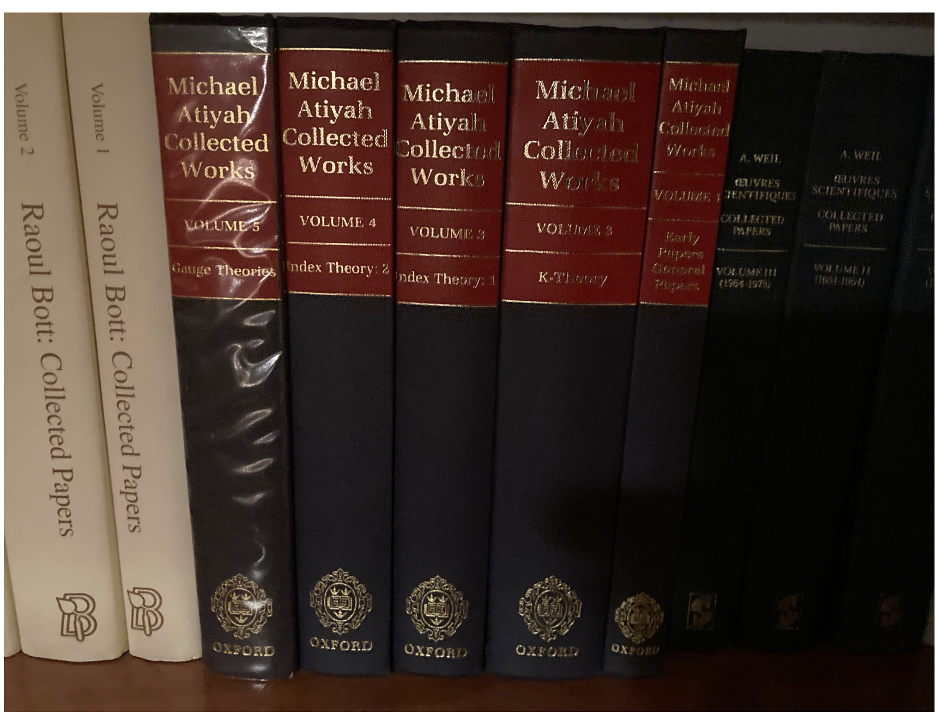
Holds in $H^*_T(M)_u$ or $CH^*_T(M)_u$

after localization (inverting $u \in H^*(\mathbb{CP}^\infty)$)

Recover Bott 1967

$$\begin{tikzcd}
\mathcal{H}_T^*(M^T) \arrow[rr, "\iota_*"] \arrow[dr, "\int"'] & \arrow[d, phantom, "\circlearrowright"] & \mathcal{H}_T^*(M) \arrow[dl, "\int"] \\
& \mathcal{H}_T^*(\bullet) \cong \mathbb{Q}(u) &
\end{tikzcd}$$

Look Here in Volume 5



§3 The Atiyah-Bott localization formula has been used effectively for toric and G/P geometries

- moduli space of stable maps

$$\overline{M}_{g,n}(X,\beta) \qquad X = \text{toric or } G/P$$

localization $\Rightarrow$

Calculate GW theory $\displaystyle\int_{[\overline{M}_{g,n}(X,\beta)]^{vir}}$ $\xrightarrow{\text{in terms of}}$ $\displaystyle\int_{\overline{M}_{g,n}}$

much better understood

[Ellingsrud-Strømme 92]
Kontsevich 94
Givental 96, Lian-Liu-Yau 97
Graber-P 99
Faber-P 00
Many others...

- moduli space of stable sheaves on

X = toric or G/P

Calculate DT theory

dim $X \leq 3$ or 4

$$\int_{[\mathcal{M}^X(v)]^{vir}} \quad \xrightarrow{\text{in terms of}} \quad \sum_{\substack{\text{Box} \\ \text{Configurations}}}$$

much better understood

MNOP 03

Maulik-Oblomkov 08

Bryan, Pixton, R. Thomas

Kool - Göttsche

Marian- Oprea, Szenes

Many others ...

Both moduli directions are conceptually clear:

localization reduces integration on complicated moduli spaces to simpler geometries

§4 We can consider instead the inverse problem:

Let $m > 0$ be an integer.

Suppose we have a list

$$(M_1, N_1), \ldots, (M_i, N_i), \ldots, (M_k, N_k)$$

nonsingular projective variety

vector bunde $N_i \to M_i$ with fiberwise $\mathbb{C}^*$-action

where $\dim M_i + \operatorname{rank} N_i = m$.

<u>Question</u>: Does there exist a nonsingular projective variety X of $\dim m$ with a $\mathbb{C}^*$-action satisfying

$$X^{\mathbb{C}^*} \cong \bigsqcup_i (M_i, N_i) \ ?$$

$\mathbb{C}^*$-fixed locus

The most elementary case of the question:

all $\mathcal{M}_i = \bullet$ are points

$\mathcal{N}_i$ = m-dim representation of $\mathbb{C}^*$

with weights $w^i_1, \ldots, w^i_m \in \mathbb{Z}$

By Atiyah-Bott $\Rightarrow$

Let $f(c_1, \ldots, c_m)$ be a polynomial

$\boxed{c_i \text{ has degree } i}$ homogeneous of degree $< m$

If X exists, then

elementary symmetric functions of $\{w^i_j\}$

$$(\star) \qquad \sum_{i=1}^{k} \frac{f(e^i_1, \ldots, e^i_m)}{e^i_m} = 0 .$$

Condition $(\star)$ provides an obstruction to affirmative answer

Observation: the existence of $\mathcal{X}$ imposes nontrivial conditions relating the different loci (M_i, N_i)

A dream plan: Suppose we are interested in a moduli space $\mathcal{M}$ with a conjectural property P

<u>no</u> $\mathbb{C}^*$-action on $\mathcal{M}$ is assumed

- Hope for the existence of another moduli space $\hat{\mathcal{M}}$ which has a known property $\hat{P}$

- Find a larger space $\mathcal{X}$ with a $\mathbb{C}^*$-action with

$\mathbb{C}^*$-fixed locus

$$\mathcal{X}^{\mathbb{C}^*} = \mathcal{M} \sqcup \hat{\mathcal{M}}$$

- Use the compatibilities imposed by Atiyah-Bott localization to prove $(\hat{P} \text{ for } \hat{M}) \Rightarrow (P \text{ for } M)$

In fact, the plan can be used:

- Holomorphic anomaly for the Calabi-Yau quintic 3-fold $X_5 \subset \mathbb{CP}^4$

  $z_0^5 + z_1^5 + \cdots + z_4^5 = 0$
  Fermat for example

  H-L Chang, S. Guo, J. Li 2018

  S. Guo, Janda, Ruan 2018 -
  Q. Chen

- Cohomological Abel-Jacobi theory

  Janda - P - Pixton - Zvonkine 2016, 2018

  Bae - Holmes - P - Schmitt - Schwarz 2020

§5 Holomorphic anomaly for X_5

Gromov-Witten invariants

virtual curve count of genus g degree d curves on X_5

$$N_{g,d} = \int_{[\overline{M}_g(X_5,d)]^{vir}} 1 \in \mathbb{Q}$$

$$\Downarrow$$

$Q(q)$ mirror map

Define $\mathcal{F}_g(Q) = \sum_{d=0}^{\infty} N_{g,d} Q^d$

$\mathcal{F}_g^B(q) = \mathcal{F}_g(Q)$

Warning: I've made several simplifications (see next page)

<u>CLAIMS</u>

- $\mathcal{F}_g^B(q)$ is a polynomial

BCOV 93

in the series $A_2(q), A_4(q), A_6(q)$

Yamaguchi-Yau 04

- $$\frac{1}{C_0^2 C_1^2} \frac{\partial \mathcal{F}_g^B}{\partial A_2} - \frac{1}{5 C_0^2 C_1^2} \frac{\partial \mathcal{F}_g^B}{\partial A_4} + \frac{1}{50\, C_0^2 C_1^2} \frac{\partial F_g^B}{\partial A_6}$$

genus reduction

$$= \frac{1}{2} \sum_{i=1}^{g-1} \frac{\partial F_{g-i}^B}{\partial T} \frac{\partial F_i^B}{\partial T} + \frac{1}{2} \frac{\partial^2 F_{g-1}^B}{\partial T^2} .$$

Precise formulas:

$$Q(q) = \exp\left(\frac{I_1(q)}{I_0(q)}\right) = q\cdot\exp\left(\frac{5\sum_{d=1}^{\infty} q^d \frac{(5d)!}{(d!)^5}\left(\sum_{r=d+1}^{5d}\frac{1}{r}\right)}{\sum_{d=0}^{\infty} q^d\frac{(5d)!}{(d!)^5}}\right).$$

In order to state the holomorphic anomaly equations, we require several series in q. First, let

$$L(q) = (1-5^5q)^{-\frac{1}{5}} = 1 + 625q + 117185q^2 + \dots.$$

Let $\mathsf{D} = q\frac{d}{dq}$, and let

$$C_0(q) = I_0\,, \quad C_1(q) = \mathsf{D}\left(\frac{I_1}{I_0}\right),$$

where I_0 and I_1 are the hypergeometric series appearing in the mirror map for the true quintic theory. We define

$$\begin{aligned}
K_2(q) &= -\frac{1}{L^5}\frac{\mathsf{D}C_0}{C_0}\,,\\
A_2(q) &= \frac{1}{L^5}\left(-\frac{1}{5}\frac{\mathsf{D}C_1}{C_1} - \frac{2}{5}\frac{\mathsf{D}C_0}{C_0} - \frac{3}{25}\right),\\
A_4(q) &= \frac{1}{L^{10}}\Bigg(-\frac{1}{25}\left(\frac{\mathsf{D}C_0}{C_0}\right)^2 - \frac{1}{25}\left(\frac{\mathsf{D}C_0}{C_0}\right)\left(\frac{\mathsf{D}C_1}{C_1}\right)\\
&\quad + \frac{1}{25}\mathsf{D}\left(\frac{\mathsf{D}C_0}{C_0}\right) + \frac{2}{25^2}\Bigg),\\
A_6(q) &= \frac{1}{31250L^{15}}\Bigg(4 + 125\mathsf{D}\left(\frac{\mathsf{D}C_0}{C_0}\right) + 50\left(\frac{\mathsf{D}C_0}{C_0}\right)\left(1 + 10\mathsf{D}\left(\frac{\mathsf{D}C_0}{C_0}\right)\right)\\
&\quad -5L^5\left(1 + 10\left(\frac{\mathsf{D}C_0}{C_0}\right) + 25\left(\frac{\mathsf{D}C_0}{C_0}\right)^2 + 25\mathsf{D}\left(\frac{q\frac{d}{dq}C_0}{C_0}\right)\right)\\
&\quad +125\mathsf{D}^2\left(\frac{\mathsf{D}C_0}{C_0}\right) - 125\left(\frac{\mathsf{D}C_0}{C_0}\right)^2\left(\left(\frac{\mathsf{D}C_1}{C_1}\right) - 1\right)\Bigg).
\end{aligned}$$

Let T be the standard coordinate mirror to $t = \log(q)$,

$$T = \frac{I_1(q)}{I_0(q)}.$$

Then $Q(q) = \exp(T)$ is the mirror map.

page from LHO-P 2018

How does the dream plan go here?

- Introduce a formal quintic theory which satisfies the Holomorphic Anomaly exactly.
- Find a master space which connects the actual quintic to the formal quintic by the localization idea.
- Prove (HA for formal quintic) $\Rightarrow$ (HA for quintic)

A few words about what such a formal quintic theory can look like:

$$\overline{\mathcal{M}}_g(X_5, d) \subset \overline{\mathcal{M}}_g(\mathbb{P}^4, d)$$

Since $X_5 \subset \mathbb{P}^4$

Kontsevich : for $g=0$,

$$N_{0,d} = \int_{[\overline{\mathcal{M}}_0(X_5,d)]^{vir}} 1 = \int_{\overline{\mathcal{M}}_0(\mathbb{P}^4,d)} e^T\left(H^0(C, f^*\mathcal{O}_{\mathbb{P}^4}(5))\right)$$

Apply Bott 67 directly

What about $g>0$?

Easy to consider:

$$\widetilde{N}_{g,d}(\lambda_K) = \int_{[\overline{\mathcal{M}}_g(\mathbb{P}^4,d)]^{vir}} \frac{e^T\left(H^0(C, f^*\mathcal{O}_{\mathbb{P}^4}(5))\right)}{e^T\left(H^1(C, f^*\mathcal{O}_{\mathbb{P}^4}(5))\right)}$$

almost formal quintic theory

Weights of $\mathbb{C}^*$

makes sense only after localization

$\widetilde{N}_{g,d}(\lambda_k)$ is not a number but lies

in $\mathbb{Q}(\lambda_0, \ldots, \lambda_4)$ ← rational functions in the weights of $\mathbb{C}^* \curvearrowright \mathbb{C}^5$ where $\mathbb{P}^4 = \mathbb{P}(\mathbb{C}^5)$

New idea: $\widetilde{N}_{g,d} = \widetilde{N}_{g,d}(\lambda_k = \exp(2\pi i \cdot k/5))$

Hyenho Lho - P 18

definition of formal quintic

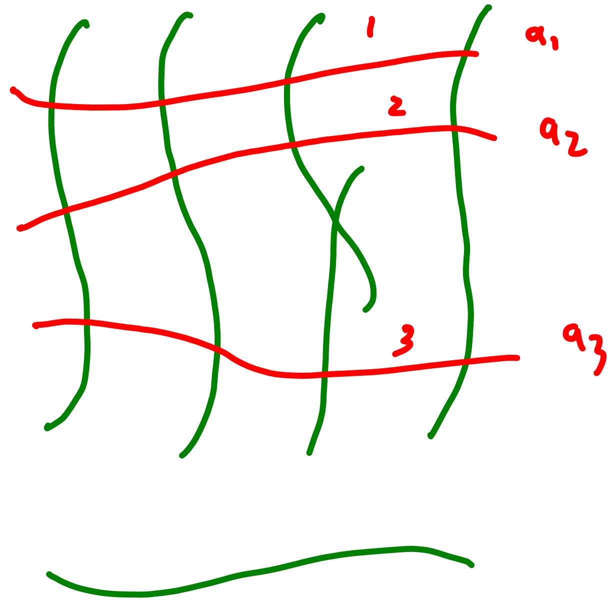
motivated in part by calculations of Zagier-Zinger

Theorm (Lho-P) 2018: Formal quintic theory satisfies holomorphic anomaly equation (in exactly the form expected for the true quintic theory).

§6 Cohomological Abel-Jacobi theory

Consider a family of pointed nodal curves

$$\begin{array}{c} \mathcal{C} \\ {\scriptstyle p_1,\ \dots,\ p_n} \uparrow \ \downarrow {\scriptstyle \pi} \\ S \end{array}$$



Curves connected, marking in smooth locus

with two additional items:

- Line bundle of degree d $\longrightarrow$ $\begin{array}{c} \mathcal{L} \\ | \\ \mathcal{C} \\ \downarrow {\scriptstyle \pi} \\ S \end{array}$

- A vector of integers $A = (a_1, \dots, a_n)$ with $\sum_{i=1}^{n} a_i = d$

Codim g

There should be an Abel-Jacobi locus of points $(C, p_1, \ldots, p_n) \in S$ where

$$\mathcal{O}_C\left(\sum_{i=1}^{n} a_i p_i\right) \cong \mathcal{L}_C$$

not a closed condition

These ideas lead to a natural operational Chow class

$$AJ_{g,A} : CH_*(S) \to CH_{*-g}(S)$$

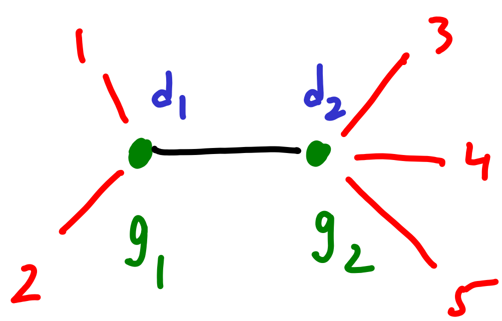
can also be viewed as a cohomology class

defined via classical intersection theory in the versal deformation space of $(C, p_1, \ldots, p_n, \mathcal{L}_C)$

Question: Find a universal formula for $AJ_{g,A}$.

But a formula in terms of what?

- Graphs:

$\Gamma =$  $\in G_{g,n,d}$

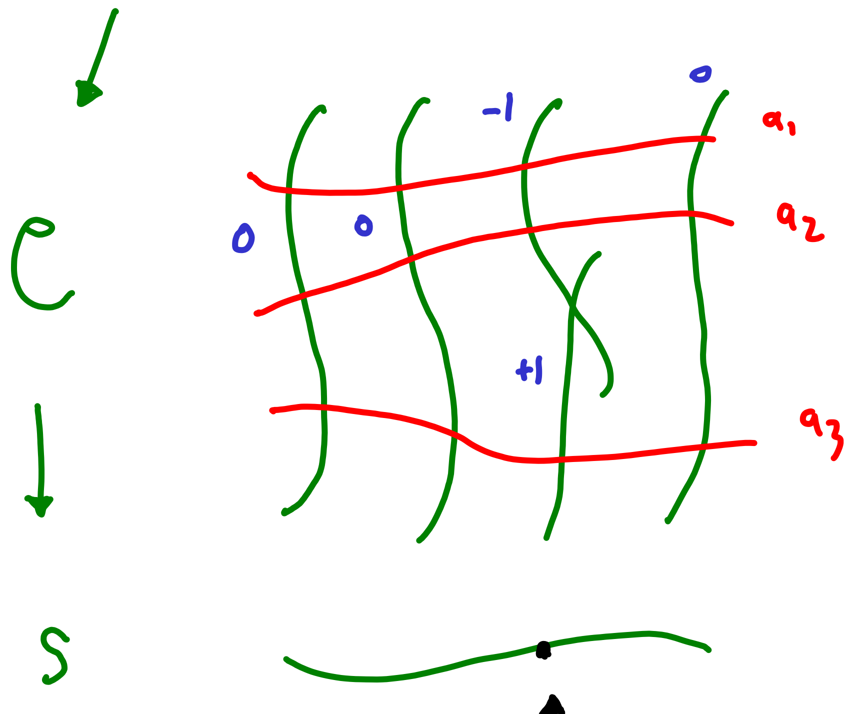
set of all graphs

No stability condition

Γ yields a class on S by closure of the locus with dual graph Γ

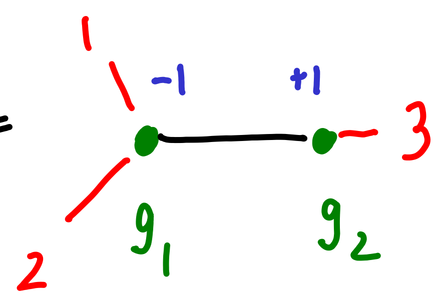
Example

$\mathcal{L}$ deg 0



$a_1 = -2$
$a_2 = +1$
$a_3 = +1$

locus with dual graph

$\Gamma =$ 

- Further tautological Classes:

cotangent line classes $\rightarrow$ ψ_i markings, ψ_j half edges

$$\xi_i = c_1(p_i^* \mathcal{L}) \leftarrow \text{marking } i$$

$$\eta(v) = \pi_*\left(c_1(\mathcal{L})^2\right) \leftarrow \text{vertex } v$$

for $r \in \mathbb{N}_+$

- Weightings mod r of $\Gamma \in G_{g,n,d}$

$$w : \text{HalfEdges}(\Gamma) \to \{0, 1, 2, \dots, r-1\}$$

(i) $\quad w(i) = a_i \mod r$

(ii) $\quad w(h) + w(h') = 0 \mod r$

when $\overset{h}{\bullet\!\!-\!\!-\!\!-}\overset{h'}{-\!\!-\!\!-\!\!\bullet}$ form an edge

(iii) $\quad \displaystyle\sum_{h \vdash v} w(h) = d(v) \mod r$

Let $W_{\Gamma, r}$ be the set of all weightings mod r of Γ.

$W_{\Gamma, r}$ is a finite set of cardinality $r^{h^1(\Gamma)}$

Let $P^r_{g,A}$ be the degree g part of

$$\sum_{\Gamma \in G_{g,n,d}} \sum_{w \in W_{\Gamma,r}} \frac{1}{|\mathrm{Aut}\,\Gamma|} \frac{1}{r^{h^1(\Gamma)}} \cdot$$

$$i_{\Gamma *}\left[\prod_{i=1}^{n} \exp\left(\frac{a_i^2}{2}\psi_i + a_i \xi_i\right) \cdot \prod_{v \in \mathrm{Vert}(\Gamma)} \exp\left(-\frac{1}{2}\eta(v)\right)\right.$$

Version of Pixton's formula

$$\left. \cdot \prod_{e=(h,h') \in \mathrm{Edge}(\Gamma)} \frac{1-\exp\left(-\frac{w(h)w(h')}{2}\cdot(\psi_h + \psi_{h'})\right)}{\psi_h + \psi_{h'}} \right]$$

Claim: for $r \gg 0 \Rightarrow$ dependence upon r is polynomial

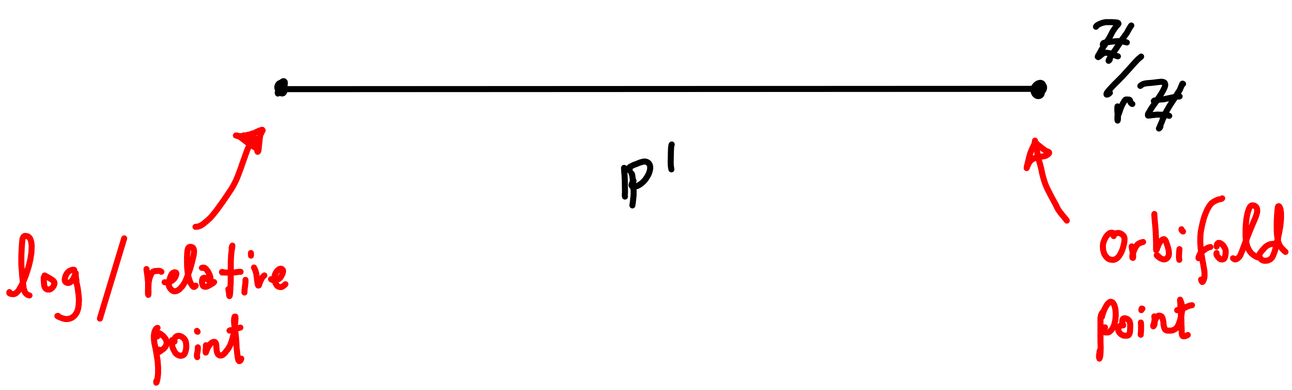
Theorem BHPSS 2020: $AJ_{g,A} = P^{r=0}_{g,A}$

How does the dream plan go here?

Chiodo

- Pixton's formula arises naturally in the orbifold Gromov-Witten theory of $B\,\mathbb{Z}/r\mathbb{Z}$

- The moduli space of relative maps is connected to $\overline{M}(B\,\mathbb{Z}/r\mathbb{Z})$ by localization for the target



- $AJ_{g,A}$ is expressible in terms of relative Gromov-Witten theory.

Janda, P, Pixton, Zvonkine 2016, 2018

Atiyah's last visit to ETH Zürich (January 2016)

Photo just before his Abel in Zürich lecture

Hosted by the Forschungsinstitut für Mathematik

* Thanks to Andrea Waldburger (FIM) for the photos!

... and at the Apéro after

The End