Product loci

with Canning
Oprea
December 2022

- Whether $[\mathcal{A}_{g_1} \times \mathcal{A}_{g_2}] \in CH^*(\mathcal{A}_g)$ is always a tautological class in $R^*(\mathcal{A}_g)$ has been an open question.

A test of tautologicality:

Consider $[\mathcal{A}_1 \times \mathcal{A}_3] \in CH^3(\mathcal{A}_4)$

and $\quad$ Tor : $\mathcal{M}_4^{ct} \to \mathcal{A}_4$

If $[\mathcal{A}_1 \times \mathcal{A}_3] \in R^3(\mathcal{A}_4)$, then

$\mathrm{Tor}^* [\mathcal{A}_1 \times \mathcal{A}_3] \in \lambda$ ring of $R^*(\mathcal{M}_4^{ct})$.

We can calculate $\mathrm{Tor}^*[A_1 \times A_3]$ explicitly as a tautological class on the moduli of curves of compact type.

We know everything about $R^*(\mathcal{M}_4^{ct})$, so the implication can be tested.

The difficulty is to correctly analyse the scheme-theoretic inverse image

$$\mathrm{Tor}^{-1}(A_1 \times A_3) \subset \mathcal{M}_4^{ct}$$

$\mathrm{Tor}^{-1}(A_1 \times A_3)$ consists of 3 irreducible components of codimension 1, 2, 3 which intersect.

<u>Theorem</u> (Canning, Oprea, P):

$$\mathrm{Tor}^*[A_1 \times A_3] = 20\lambda_3 \in R^*(\mathcal{M}_4^{ct}),$$

The calculation is consistent with $[A_1 \times A_3]$ being tautological on A_4

Control of the λ ring is provided by admcycles

- Sketch of the calculation in 4 Steps:

I. Consider the fiber product of DM stacks:

$$\begin{tikzcd} \mathfrak{F} \arrow[r] \arrow[d] & A_1 \times A_3 \arrow[d] & \text{dim } 7 \\ M_4^{ct} \arrow[r, "\mathrm{Tor}"] & A_4 & \text{dim } 10 \\ \text{dim } 9 & & \end{tikzcd}$$

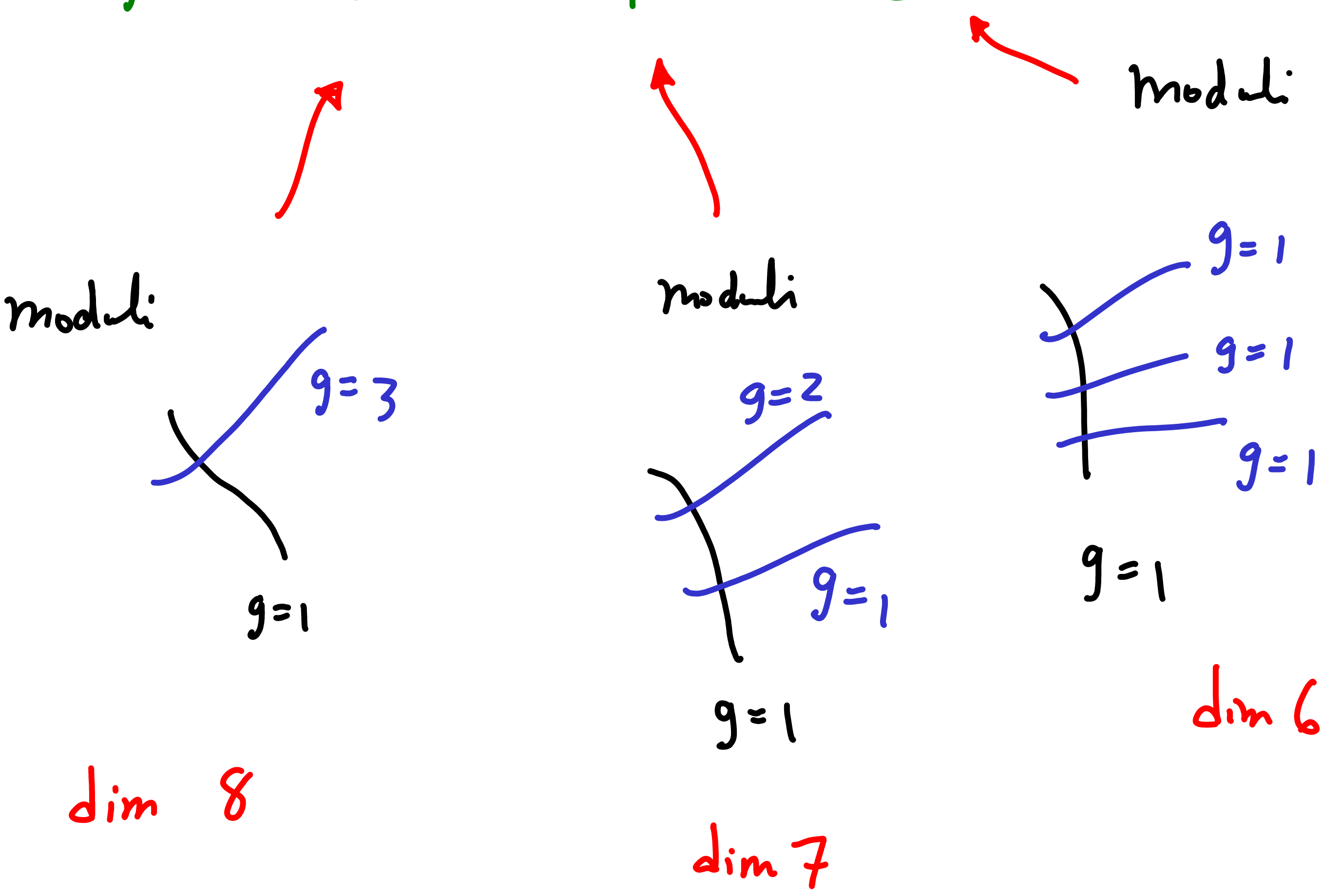
We want to compute

$$\mathrm{Tor}^! [A_1 \times A_3] \in CH_6(\mathfrak{F})$$

and then push-forward to M_4^{ct}.

II. The stack $\mathcal{P}$ has 3 irreducible components corresponding to compact type curves with a marked elliptic component.

$$\mathcal{P} = A \cup B \cup C$$



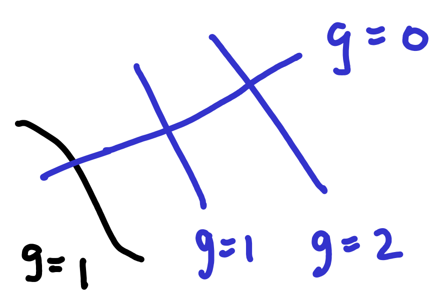
III. Claim: Away from the pairwise intersections $A \cap B$, $A \cap C$, $B \cap C$, P is nonsingular.

Proof by careful analysis of the differential of Tor.

Claim: The intersection loci are all of dim < 6 except for

$A \cap B = D$ moduli

dim 6



IV Conclusion

- $R^3(\mathcal{M}_4^{ct})$ has $\mathbb{Q}$-dim 6
- $R^3(A_4)$ has $\mathbb{Q}$-dim 2

generated by λ_1^3, λ_3

so the image of

$$\mathrm{Tor}^*: R^3(A_4) \to R^3(\mathcal{M}_4^{ct})$$

is the 2-dim span $\langle \lambda_1^3, \lambda_3 \rangle$.

- We represent

$$\mathrm{Tor}^![A_1 \times A_3] \in CH_6(\beta)$$

as an excess calculation

on $\quad \mathcal{P} \setminus (A \cap B) \cup (A \cap C) \cup (B \cap C)$

by the usual tangent and normal bundle data. Then:

$$\mathrm{Tor}^! [A_1 \times A_3] = \underbrace{X}_{\substack{c_2\left(\mathrm{Tor}^* \mathrm{Nor}_{A_1 \times A_3} / \mathrm{Nor}_A\right) \cap A \; + \; [C] \\ + \; c_1\left(\mathrm{Tor}^* \mathrm{Nor}_{A_1 \times A_3} / \mathrm{Nor}_B\right) \cap B}} - 3 \underbrace{[D]}_{\text{Correction for intersection locus}}$$

$$\begin{array}{c} \mathcal{P} \\ \varepsilon \downarrow \\ \mathcal{M}_4^{ct} \end{array}$$

- We show using adm cycles that

$$\varepsilon_*(X) - 3\,\varepsilon_*[D] = 20\lambda_3$$

in $R^3(\mathcal{M}_4^{ct})$. ■

- Projection calculation of Grushevsky-Hulek

Consider $A_4 \subset A_4^{\mathrm{Perf}}$ ← perfect cone compactification

The λ-ring of A_4^{Perf} is Gorenstein with socle in codimension 10.

The closure $\overline{A_1 \times A_3} \subset A_4^{\mathrm{Perf}}$ is a proper cycle of codim 3.

Define a linear form

$$L : R^3_\lambda\left(A_4^{\mathrm{Perf}}\right) \to \mathbb{Q}$$

λ-ring

$$L\left(\mathrm{Poly}(\lambda)\right) = \int_{[\overline{A_1 \times A_3}]} \mathrm{Poly}(\lambda)$$

By the Gorenstein property of $R^7_\lambda(A_4^{Perf})$, the linear form L can be realized by a unique class

$$\left[\overline{A_1 \times A_3}\right]^{taut} \in R^3_\lambda(A_4^{Perf})$$

called the tautological projection of $\left[\overline{A_1 \times A_3}\right]$:

$$L(\mathrm{Poly}(\lambda)) = \int_{[A_4^{Perf}]} \mathrm{Poly}(\lambda) \cdot \left[\overline{A_1 \times A_3}\right]^{taut}.$$

Grushevsky-Hulek Calculate:

$$\left[\overline{A_1 \times A_3}\right]^{taut} = 20\,\lambda_3 .$$

The connection between the Grushevsky-Hulek Calculation and ours:

if $\left[\overline{A_1 \times A_3}\right] \in R^3_\lambda\left(A_4^{Perf}\right)$,

then $\left[\overline{A_1 \times A_3}\right] = \left[\overline{A_1 \times A_3}\right]^{taut} \in R^3_\lambda\left(A_4^{Perf}\right)$

and $\quad \mathrm{Tor}^*\left[A_1 \times A_3\right] = 20\,\lambda_3 \in R^3\left(M_4^{ct}\right)$

would follow.

But whether $\left[\overline{A_1 \times A_3}\right]$ is tautological on A_4^{Perf} is unknown.

- Comments by Faber:

The cohomology Betti numbers of A_4 are mostly unknown. A summary of previous results is provided by

Hulek-Tommasi 2012:

$h^0 = 1$ $\quad h^{10} \geq 1$

$h^1 = 0$ $\quad h^{11} = 0$

$h^2 = 1$ $\quad h^{12} = 2$

$h^4 \geq 1$ $\quad h^{\geq 13} = 0$

$h^6 \geq 2$

$h^8 \geq 1$

Hulek-Tommasi 2017:

$$e(A_4) = 9,$$

So it is likely that all of the cohomology has been found.

- Deeper products

In A_4 there are product loci

$$A_1 \times A_3, \quad A_2 \times A_2, \quad A_1 \times A_1 \times A_2, \quad A_1 \times A_1 \times A_1 \times A_1.$$

Since $\mathrm{Tor}_*[M_4^{ct}]$ is a divisor class

in A_4, $\mathrm{Tor}_*[M_4^{ct}] = 8\lambda_1 \in R^1(A_4)$.

precise multiple not required for argument

(i) $\mathrm{Tor}_*[M_4^{ct}] \cdot [A_1 \times A_3] =$

$$\mathrm{Tor}_*\left(\mathrm{Tor}^*[A_1 \times A_3]\right) =$$

$$\mathrm{Tor}_*\left(\mathrm{Tor}^* 20\lambda_3\right) =$$

$$\mathrm{Tor}_*[M_4^{ct}] \cdot 20\lambda_3 \quad \in R^4(A_4)$$

Conclusion: $\lambda_1 \cdot [A_1 \times A_3] = 20 \lambda_1 \lambda_3$

in $R^*(A_4)$

(ii) $\lambda_1^2 \cdot [A_1 \times A_3] = 20 \lambda_1^2 \lambda_3$

$\lambda_1^3 \cdot [A_1 \times A_3] = 20 \lambda_1^3 \lambda_3$

proportional to $[A_1 \times A_1 \times A_2]$

and $[A_1 \times A_1 \times A_1 \times A_1]$

Conclusion:

predicted by Speculation II

$[A_1 \times A_1 \times A_2]$ and $[A_1 \times A_1 \times A_1 \times A_1]$

are in $R^*(A_4)$

Together, 3 new tautological classes.

- The extended tautological ring of A_g

$$R^*_{Ex}(A_g) \subset CH^*(A_g)$$

is defined as the subring generated by push-forwards under all maps

including $id: A_g \to A_g$

$$\varepsilon: A_{g_1} \times A_{g_2} \times \cdots \times A_{g_n} \longrightarrow A_g$$

of all products of λ classes on the factors A_{g_i}.

Is the extended ring larger?

$$R^*(A_g) \neq R^*_{Ex}(A_g) \ ?$$

It is easy to see from van der Geer's work

that $R_{Ex}^{>\binom{g}{2}}(A_g) = 0$.

Speculation II implies that the socle

is unchanged:

Speculation II $\Rightarrow$ $R_{Ex}^{\binom{g}{2}}(A_g) = \mathbb{Q}$.

Lots of open questions. For example:

Is the class of the Schottky locus in

$$R_{Ex}^{*}(A_g) \; ?$$

Extended classes don't increase $R^3(A_5)$

Some investigation is possible in $g=5$

The first calculation to do is

$$\mathrm{Tor}^*[\mathcal{M}_5^{ct}] \in R^3(\mathcal{M}_5^{ct}).$$

Self intersection of the Schottky locus

Speculation related to $[A_1 \times A_{g-1}]$

with Canning Oprea

January 2023

Question (Strong):

$$\left[\overline{A_1 \times A_{g-1}}\right] \overset{?}{=} k_g \cdot \lambda_{g-1} \in R^{g-1}(A_g)$$

constant

Evidence:

(A) True for $g = 1, 2, 3$

$g=1$: $k_1 = 1$ Trivial

$g=2$: $k_2 = 10$

$g=3$: $k_3 = 21$

follows from explicit understanding of A_2, A_3

(B) Since $\lambda_{g-1}^2 = 0 \in R^{g-1}(\mathcal{A}_g)$,

Conjecture (Strong) $\Rightarrow$

(i) $\lambda_{g-1} \cdot \left[\overline{\mathcal{A}_1 \times \mathcal{A}_{g-1}}\right] = 0 \in R^{g-1}(\mathcal{A}_g)$

(ii) $\left[\overline{\mathcal{A}_1 \times \mathcal{A}_{g-1}}\right]^2 = 0 \in R^{g-1}(\mathcal{A}_g)$

Both (i) + (ii) are true by direct calculation, (ii) requires some excess intersection analysis using the tangent bundle identifications.

(c) Abel-Jacobi Compatibility

Theorem (Canning, Oprea, P):

$$\mathrm{Tor}^*[A_1 \times A_3] = 20\lambda_3 \in R^3(\mathcal{M}_4^{ct}),$$

and a more complicated calculation

Theorem (Canning, Oprea, P):

$$\mathrm{Tor}^*[A_1 \times A_4] = 11\lambda_4 \in R^4(\mathcal{M}_5^{ct}),$$

As a result of these two calculations, we can formulate:

Question (Weak):

$$\mathrm{Tor}^* \left[\overline{A_1 \times A_{g-1}} \right] \overset{?}{=} k_g \cdot \lambda_{g-1} \in R^{g-1}(M_g^{ct})$$

What is the constant k_g ?

$$k_g = \frac{g}{6\,|\underbrace{B_{2g}}_{\text{Bernoulli}}|}$$

There are two pieces of conceptual evidence for the constant formula.

(A) Formula is true for

$g = 1, 2, 3, 4, 5.$

(B) Theorem: The tautological projection of $[\overline{A_1 \times A_{g-1}}]$ to

$R^{g-1}_{\lambda}(A_g^{Perf})$ is

$$\frac{g}{6\,|B_{2g}|}\,\lambda_{g-1}\,.$$

Update (Feb 2023):

$$\mathrm{Taut}^{*}\left[\overline{A_1 \times A_5}\right] \neq \kappa_6 \cdot \lambda_5 \in R^5(\mathcal{M}_6^{ct})$$

see my Berlin lecture at the

Northern German AG Seminar.