I was of three minds,
Like a tree
In which there are three blackbirds.

Wallace Stevens

13/2 WAYS OF COUNTING CURVES

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Abstract. In the past 20 years, compactifications of the families of curves in algebraic varieties X have been studied via stable maps, Hilbert schemes, stable pairs, unramified maps, and stable quotients. Each path leads to a different enumeration of curves. A common thread is the use of a 2-term deformation/obstruction theory to define a virtual fundamental class. The richest geometry occurs when X is a nonsingular projective variety of dimension 3.

We survey here the 13/2 principal ways to count curves with special attention to the 3-fold case. The different theories are linked by a web of conjectural relationships which we highlight. Our goal is to provide a guide for graduate students looking for an elementary route into the subject.

Contents

0. Introduction

§Counting. Let X be a nonsingular projective variety (over $\mathbb{C}$), and let $\beta \in H_2(X, \mathbb{Z})$ be a homology class. We are interested here in counting the algebraic curves of X in class β. For example, how many twisted cubics in $\mathbb{P}^3$ meet 12 given lines? Mathematicians such as Hurwitz, Schubert, and Zeuthen have considered such questions since the 19^{th} century. Towards the end of the 20^{th} century and continuing to the present, the subject has been greatly enriched by new insights from symplectic geometry and topological string theory.

Under appropriate genericity conditions, counting solutions in algebraic geometry often yields deformation invariant answers. A simple example is provided by Bezout's Theorem concerning the intersections of plane curves. Two generic algebraic curves in $\mathbb{C}^2$ of degrees d_1 and d_2 intersect transversally in finitely many points. Counting these points yields the topological intersection number $d_1 d_2$. But in nongeneric situations, we can find fewer solutions or an infinite number. The curves may intersect with tangencies in a smaller number of points (remedied by counting intersection points with multiplicities). If the curves intersect "at infinity", we will again find fewer intersection points in $\mathbb{C}^2$ whose total we do not consider to be a "sensible" answer. Instead, we compactify $\mathbb{C}^2$ by $\mathbb{P}^2$ and count there. Finally, the curves may intersect in an entire component. The technique of excess intersection theory is required then to obtain the correct answer. Compactification and transversality already play a important role in the geometry of Bezout's Theorem.

Having deformation invariant answers for the enumerative geometry of curves in X is desirable for several reasons. The most basic is the possibility of deforming X to a more convenient space. To achieve deformation invariance, two main issues must be considered:

(i) compactification of the moduli space $\mathcal{M}(X, \beta)$ of curves $C \subset X$ of class β,
(ii) transversality of the solutions.

What we mean by the moduli space $\mathcal{M}(X, \beta)$ is to be explained and will differ in each of the sections below. Transversality concerns both the possible excess dimension of $\mathcal{M}(X, \beta)$ and the transversality of the constraints.

§Compactness. For Bezout's Theorem, we compactify the geometry so intersection points running to infinity do not escape our counting. The result is a deformation invariant answer.

A compact space $\mathcal{M}(X, \beta)$ which parameterises all nonsingular embedded curves in class β will usually have to contain singular curves of some sort. Strictly speaking, the compact moduli spaces $\mathcal{M}(X, \beta)$ will often not be compactifications of the spaces of nonsingular embedded curves — the latter need not be dense in $\mathcal{M}(X, \beta)$. For instance $\mathcal{M}(X, \beta)$ might be nonempty when there are *no* nonsingular embedded curves. The singular strata are important for deformation invariance. As we deform X, curves can "wander off to infinity" in $\mathcal{M}(X, \beta)$ by becoming singular.

§Transversality. A simple question to consider is the number of elliptic cubics in $\mathbb{P}^2$ passing through 9 points $p_1, \ldots, p_9 \in \mathbb{P}^2$. The linear system

$$\mathbb{P}(H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(3))) \cong \mathbb{P}^9$$

provides a natural compactification of the moduli space. Each p_i imposes a single linear condition which determines a hyperplane

$$\mathbb{P}^8_i \subset \mathbb{P}^9,$$

of curves passing through p_i. For general p_i, these 9 hyperplanes are transverse and intersect in a single point. Hence, we expect our count to be 1. But if the p_i are the 9 intersection points of two cubics, then we obtain an entire pencil of solutions given by the linear combinations of the two cubics.

An alternative way of looking at the same enumerative question is the following. Let

$$\epsilon : S \to \mathbb{P}^2$$

be the blow-up of $\mathbb{P}^2$ at 9 points p_i and consider curves in the class

$$\beta = 3H - E_1 - E_2 - \ldots - E_9$$

where H is the ϵ pull-back of the hyperplane class and the E_i are the exceptional divisors. In general there will be a unique elliptic curve embedded in class β. But if the 9 points are the intersection of two cubics, then S is a rational elliptic surface via the pencil

$$\pi : S \to \mathbb{P}^1.$$

How to sensibly "count" the pencil of elliptic fibres on S is not obvious.

A temptation based on the above discussion is to define the enumeration of curves by counting after taking a generic perturbation of the geometry. Unfortunately, we often do not have enough perturbations to make the situation fully transverse. A basic rigid example is given by counting the intersection points of a (-1)-curve with itself on a surface. Though we cannot algebraically move the curve to be transverse to itself, we know another way to get the "sensible" answer of topology: take the Euler number -1 of the normal bundle. In curve counting, there is a similar excess intersection theory approach to getting a sensible, deformation invariant answer using *virtual fundamental classes*.

For the rational elliptic surface S, the base $\mathbb{P}^1$ is a natural compact moduli space parameterising the elliptic curves in the pencil. The count of elliptic fibres is the Euler class of the *obstruction bundle* over the pencil $\mathbb{P}^1$. Calculating the obstruction bundle to be $\mathcal{O}_{\mathbb{P}^1}(1)$, we recover the answer 1 expected from deformation invariance.

Why is the obstruction bundle $\mathcal{O}_{\mathbb{P}^1}(1)$? In Section $1\frac{1}{2}$, a short introduction to the deformation theory of maps is presented. Let $E \subset S$ be the fibre of π over $[E] \in \mathbb{P}^1$. Let ν_E be the normal bundle of E in S. The obstruction space at $[E] \in \mathbb{P}^1$ is

$$H^1(E, \nu_E) = H^1(E, \mathcal{O}_E) \otimes \mathcal{O}_{\mathbb{P}^1}(2)|_{[E]} \ .$$

The term $H^1(E, \mathcal{O}_E)$ yields the dual of the Hodge bundle as E varies and is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(-1)$. Hence, we find the obstruction bundle to be $\mathcal{O}_{\mathbb{P}^1}(1)$.

We will discuss virtual classes in the Appendix. We should think loosely of $\mathcal{M}(X, \beta)$ as being cut out of a nonsingular ambient space by a set of equations. The expected, or *virtual*, dimension of $\mathcal{M}(X, \beta)$ is the dimension of the ambient space minus the number of equations. If the derivatives of the equations are not linearly independent along the common zero locus $\mathcal{M}(X, \beta)$, then $\mathcal{M}(X, \beta)$ will be singular or have dimension higher than expected. In practice, $\mathcal{M}(X, \beta)$ is very rarely nonsingular of the expected dimension. We should think of the virtual class as representing the fundamental cycle of the "correct" moduli space (of dimension *equal* to the virtual dimension) inside the actual moduli space. The virtual class may be considered to give the result of perturbing the setup to a transverse geometry, even when such perturbations do not actually exist.

§Overview. A nonsingular embedded curve $C \subset X$ can be described in two fundamentally different ways:

(i) as an algebraic map $C \to X$
(ii) as the zero locus of an ideal of algebraic functions on X.

In other words, C can be seen as a *parameterised* curve with a map or an *unparameterised* curve with an embedding. Both realisations arise naturally in physics — the first as the worldsheet of a string moving in X, the second as a D-brane or boundary condition embedded in X.

Associated to the two basic ways of thinking of curves, there are two natural paths for compactifications. The first allows the map f to degenerate badly while keeping the domain curve as nice as possible. The second keeps the map as an embedding but allows the curve to degenerate arbitrarily.

We describe here $6\frac{1}{2}$ methods for defining curve counts in algebraic geometry. We start in Section $\frac{1}{2}$ with a discussion of the successes and limitations of the naive counts pursued by the 19^{th} century geometers (and followed for more than 100 years). Since such counting is not always well-defined and has many drawbacks, we view the naive approach as only $\frac{1}{2}$ a method.

In Sections $1\frac{1}{2}$ – $6\frac{1}{2}$, six approaches to deformation invariant curve counting are presented. Two (stable maps and unramified maps) fall in class (i), three (BPS invariants, ideal sheaves, stable pairs) in class (ii), and one (stable quotients) straddles both classes (i-ii). The compactifications and virtual class constructions are dealt with differently in the six cases. Of course, each of the six has advantages and drawbacks.

There are several excellent references covering different aspects of the material surveyed here in much greater depth, see for instance [23, 43, 64, 78, 100]. Also, there are many beautiful directions which we do not cover at all. For example, mirror symmetry, integrable hierarchies, descendent invariants, 3-dimensional partitions, and holomorphic symplectic geometry all play significant roles in the subject. Though orbifold and relative geometries have been very important for the development of the ideas presented here, we have chosen to omit a discussion. Our goal is to describe the $6\frac{1}{2}$ counting theories as simply as possible and to present the web of relationships amongst them.

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$\frac{1}{2}$. Naive counting of curves

Let X be a nonsingular projective variety, and let $\beta \in H_2(X, \mathbb{Z})$ be a homology class. Let $C \subset X$ be a nonsingular embedded (or immersed) curve of genus g and class β. The expected dimension of the family of genus g and class β curves containing C is

$$3g-3+\chi(T_X|_C)=\int_C c_1(X)+(\dim_{\mathbb{C}}X-3)(1-g)\,. \tag{1.1}$$

The first term on the left comes from the complex moduli of the genus g curve,

$$\dim_{\mathbb{C}}\overline{\mathcal{M}}_g=3g-3\,.$$

The second term arises from infinitesimal deformations of C which do not change the complex structure of C. More precisely,

$$\chi(T_X|_C)=h^0(C,T_X|_C)-h^1(C,T_X|_C)$$

where $H^0(C,T_X|_C)$ is the space of such deformations (at least when C has no continuous families of automorphisms). The "expectation" amounts to the vanishing of $H^1(C,T_X|_C)$. Indeed if $H^1(C,T_X|_C)$ vanishes, the family of curves is nonsingular of expected dimension at C, see [54]. We will return to this deformation theory in Section $1\frac{1}{2}$.

If the open family of embedded (or immersed) curves of genus g and class β is of pure expected dimension (1.1), then naive classical curve counting is sensible to undertake. We can attempt to count the actual numbers of embedded (or immersed) curves of genus g and class β in X subject to incidence conditions.

The main classical[1] examples where naive curve counting with simple incidence is reasonable to consider constitute a rather short list:

(i) Counting Hurwitz coverings of $\mathbb{P}^1$ and curves of higher genus,
(ii) Severi degrees in $\mathbb{P}^2$ and $\mathbb{P}^1\times\mathbb{P}^1$ in all genera,
(iii) Counting genus 0 curves in general blow-ups of $\mathbb{P}^2$,
(iv) Counting genus 0 curves in homogeneous spaces such as $\mathbb{P}^n$, Grassmannians, and flag varieties,
(v) Counting lines on complete intersections in $\mathbb{P}^n$,
(vi) Counting curves of genus 1 and 2 in $\mathbb{P}^3$.

The Hurwitz covers of $\mathbb{P}^1$ (or higher genus curves),

$$C\to\mathbb{P}^1,$$

are neither embeddings nor immersions, but rather are counts of ramified maps, see [74] for an introduction. Nevertheless (i) fits naturally in the list of classical examples. The Severi degrees (ii) are the numbers of immersed curves of genus g and class β passing through the expected number of points on a surface. Particularly for the case of

[1]We do not attempt here to give a complete classical bibliography. Rather, the references we list, for the most part, are modern treatments.

$\mathbb{P}^2$, the study of Severi degrees has a long history [16, 37, 87]. Counting genus 0 curves on blow-ups (iii) is equivalent to imposing multiple point singularities for plane curves, see [38] for a treatment. Genus 0 curves behave very well in homogeneous spaces, so the questions (iv) have been considered since Schubert and Zeuthen [90, 107]. Examples of (v) include the famous 27 lines on a cubic surface and the 2875 lines on a quintic 3-fold, see [23]. The genus 1 and 2 enumerative geometry of space curves was much less studied by the classical geometers, but still can be viewed in terms of naive counting.

For particular genera and classes on other varieties X, the families of curves might be pure of expected dimension. The above list addresses the cases of more uniform behavior. Until new ideas from symplectic geometry and topological string theory were introduced in the 1980s and 90s, the classical cases (i-vi) were the main topics of study in enumerative geometry. The subject was an important area, especially for the development of intersection theory in algebraic geometry. See [30] for a historical survey. However, because of the restrictions, we treat naive counting as only $\frac{1}{2}$ of an enumerative theory here.

New approaches to enumerative geometry by tropical methods have been developed extensively in recent years [46, 73]. However, the lack of a virtual fundamental class in tropical geometry restricts the direct[2] applications at the moment to the classical cases.

The counting of rational curves on algebraic $K3$ surfaces is almost a classical question. A $K3$ surface with Picard number 1 has finitely many rational curves in the primitive class (even though the expected dimension of the family of rational curves is -1 by (1.1)). As proved in [18], for a general $K3$ of Picard number 1, all the primitive rational curves are nodal. A proposal for the count was made by Yau and Zaslow [106] in terms of modular forms. The proofs by Beauville [2] and Bryan-Leung [11] certainly use modern methods. The counting of rational curves in all (including imprimitive) classes on $K3$ surfaces shows the fully non-classical nature of the question [52].

$1\frac{1}{2}$. Gromov-Witten theory

§Moduli. Gromov-Witten theory provided the first modern approach to curve counting which dealt successfully with the issues of compactification and transversality. The subject has origins in Gromov's work on pseudo-holomorphic curves in symplectic geometry [39] and papers

[2]Tropical methods do interact in an intricate way with virtual curve counts on Calabi-Yau 3-folds in the program of Gross and Siebert [40, 41, 42] to study mirror symmetry.

of Witten on topological strings [104]. Contributions by Kontsevich, Manin, Ruan, and Tian [56, 57, 88, 89] played an important role in the early development.

In Gromov-Witten theory, curves are viewed as parameterised with an algebraic map

$$C \to X\,.$$

The compactification strategy is to admit only nodal singularities in the domain while allowing the map to become rather degenerate. More precisely, define $\overline{\mathcal{M}}_g(X,\beta)$ to be the moduli space of *stable maps*:

$$\left\{ f\colon C \to X \;\middle|\; \begin{array}{c} C \text{ a nodal curve of arithmetic genus } g, \\ f_*[C]=\beta, \text{ and } \mathrm{Aut}(f) \text{ finite} \end{array} \right\}.$$

The map f is invariant under an automorphism ϕ of the domain C if

$$f = f \circ \phi\,.$$

By definition, $\mathrm{Aut}(f) \subset \mathrm{Aut}(C)$ is the subgroup of elements for which f is invariant. The finite automorphism condition for a stable map implies the moduli space $\overline{\mathcal{M}}_g(X,\beta)$ is naturally a Deligne-Mumford stack.

The compactness of $\overline{\mathcal{M}}_g(X,\beta)$ is not immediate. A proof can be found in [32] using standard properties of semistable reduction for curves. In Section $3\frac{1}{2}$ below, we will discuss nontrivial limits in the space of stable maps, see for instance (3.1) and (3.3).

§Deformation theory. We return now to the deformation theory for embedded curves briefly discussed in Section $\frac{1}{2}$. The deformation theory for arbitrary stable maps is very similar.

Let $C \subset X$ be a nonsingular embedded curve with normal bundle ν_C. The Zariski tangent space to the moduli space $\overline{\mathcal{M}}_g(X,\beta)$ at the point $[C \to X]$ is given by $H^0(C,\nu_C)$. Locally, we can lift a section of ν_C to a section of $T_X|_C$ and deform C along the lift to first order. Since globally ν_C is not usually a summand of $T_X|_C$ but only a quotient, the lifts will differ over overlaps by vector fields along C. The deformed curve will have a complex structure whose transition functions differ by these vector fields. In other words, from

$$0 \to T_C \to T_X|_C \to \nu_C \to 0,$$

we obtain the sequence

$$0 \to H^0(C,T_C) \to H^0(C,T_X|_C) \to H^0(C,\nu_C) \to H^1(C,T_C) \tag{1.1}$$

which expresses how deformations in $H^0(C, \nu_C)$ change the complex structure on C through the boundary map to $H^1(C, T_C)$. The kernel

$$H^0(C, T_X|_C)/H^0(C, T_C)$$

consists of the deformations given by moving C along vector fields in X, thus preserving the complex structure of C, modulo infinitesimal automorphisms of C. Similarly, obstructions to deformations lie in $H^1(C, \nu_C)$.

The expected dimension $\chi(\nu_C) = h^0(\nu_C) - h^1(\nu_C)$ of the moduli space is given by the calculation

$$\chi(\nu_C) = \int_C c_1(X) + (\dim_{\mathbb{C}} X - 3)(1 - g) \tag{1.2}$$

obtained from sequence (1.1). If $H^1(C, T_X|_C)$ vanishes, so does the obstruction space $H^1(C, \nu_C)$. Formula (1.2) then computes the actual dimension of the Zariski tangent space.

For arbitrary stable maps $f : C \to X$, we replace the dual of ν_C by the complex

$$\{f^*\Omega_X \to \Omega_C\} \tag{1.3}$$

on C. If C is nonsingular and f is an embedding, the complex (1.3) is quasi-isomorphic to its kernel ν_C^*. The deformations/obstructions of f are governed by

$$\mathrm{Ext}^i\left(\{f^*\Omega_X \to \Omega_C\}, \mathcal{O}_C\right) \tag{1.4}$$

for $i = 0, 1$. Similarly the deformations/obstructions of f with the curve C fixed are governed by $\mathrm{Ext}^i\left(f^*\Omega_X, \mathcal{O}_C\right) = H^i(f^*T_X)$.

Since the Ext groups (1.4) vanish for $i \neq 0, 1$, the deformation/obstruction theory is 2-term. The moduli space admits a virtual fundamental class[3]

$$[\overline{\mathcal{M}}_g(X, \beta)]^{vir} \in H_*(\overline{\mathcal{M}}_g(X, \beta), \mathbb{Q})$$

of complex dimension equal to the virtual dimension

$$\mathrm{ext}^0 - \mathrm{ext}^1 = \int_\beta c_1(X) + (\dim_{\mathbb{C}} X - 3)(1 - g)\,. \tag{1.5}$$

An introduction to the virtual fundamental class is provided in the Appendix.

[3]The virtual fundamental class is algebraic, so should be more naturally considered in the Chow group $A_*(\overline{\mathcal{M}}_g(X, \beta), \mathbb{Q})$.

§Invariants. To obtain numerical invariants, we must cut the virtual class from dimension (1.5) to zero. The simplest way is by imposing incidence conditions: we count only those curves which pass though fixed cycles in X. Let

$$\mathcal{C} \to X \times \overline{\mathcal{M}}_g(X, \beta)$$

be the universal curve. We would like to intersect $\mathcal{C}$ with a cycle α pulled back from X. Transversality issues again arise here, so we use Poincaré dual cocyles.[4] Let

$$f\colon \mathcal{C} \to X \quad \text{and} \quad \pi\colon \mathcal{C} \to \overline{\mathcal{M}}_g(X, \beta)$$

be the universal map and the projection to $\overline{\mathcal{M}}_g(X, \beta)$ respectively. Let

$$\widetilde{\alpha} = \pi_*\big(f^*\mathrm{PD}(\alpha)\big) \in H^*(\overline{\mathcal{M}}_g(X, \beta)).$$

If α is a cycle of real codimension a, then $\widetilde{\alpha}$ is a cohomology class[5] in degree $a - 2$. When transversality is satisfied, $\widetilde{\alpha}$ is Poincaré dual to the locus of curves in $\overline{\mathcal{M}}_g(X, \beta)$ which intersect α. After imposing sufficiently many incidence conditions to cut the virtual dimension to zero, we define the *Gromov-Witten invariant*

$$N^{\mathsf{GW}}_{g,\beta}(\alpha_1, \ldots, \alpha_k) = \int_{[\overline{\mathcal{M}}_g(X,\beta)]^{vir}} \widetilde{\alpha}_1 \wedge \ldots \wedge \widetilde{\alpha}_k \ \in \mathbb{Q}\,.$$

We view the Gromov-Witten invariant[6] $N_{g,\beta}$ as counting the curves in X which pass through the cycles α_i. The deformation invariance of $N_{g,\beta}$ follows from construction of the virtual class. We are free to deform X and the cycles α_i in order to compute $N_{g,\beta}$.

The projective variety X may be viewed as a symplectic manifold with symplectic form obtained from the projective embedding. In fact, $N_{g,\beta}$ can be defined on any symplectic manifold X by picking a compatible almost complex structure and using pseudo-holomorphic maps of curves. The resulting invariants do not depend on the choice of compatible almost complex structure, so define invariants of the symplectic structure.[7]

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[4]Even if two submanifolds do not intersect transversally, the integral of the Poincaré dual cohomology class of one over the other still gives the correct topological intersection.

[5]The cohomological push-forward here uses the fact that π is an lci morphism. Alternatively flatness can be used [28].

[6]We drop the superscript GW when clear from context.

[7]The role of the symplectic structure in the definition of the invariants is well hidden. Via Gromov's results, the symplectic structure is crucial for the compactness of the moduli space of stable maps.

We can try to perturb the almost complex structures to make the moduli space transverse of the correct dimension. But even when embedded pseudo-holomorphic curves in X are well-behaved, their multiple covers invariably are not. Even within symplectic geometry, the correct treatment of Gromov-Witten theory currently involves virtual classes.

§Advantages. Gromov-Witten theory is defined for spaces X of all dimensions and has been proved to be a symplectic invariant (unlike most of the theories we will describe below). As the first deformation invariant theory constructed, Gromov-Witten theory has been intensively studied for more than 20 years — by now there are many exact calculations and significant structural results related to integrable hierarchies and mirror symmetry.

Since the moduli space of stable maps $\overline{\mathcal{M}}_g(X, \beta)$ lies over the moduli space $\overline{\mathcal{M}}_g$ of stable curves, Gromov-Witten theory is intertwined with the geometry of $\overline{\mathcal{M}}_g$. Relations in the cohomology of $\overline{\mathcal{M}}_{g,n}$ yield universal differential equations for the generating functions of Gromov-Witten invariants. The most famous case is the WDVV equation [26, 103] obtained by the linear equivalence of the boundary strata of $\overline{\mathcal{M}}_{0,4}$. The WDVV equation implies the associativity of the quantum cohomology ring of X defined via the genus 0 Gromov-Witten invariants. For example, associativity for $\mathbb{P}^3$ implies 80160 twisted cubics meet 12 general lines [25, 32]. Higher genus relations such as Getzler's [33] in genus 1 and the BP equation [8] in genus 2 also exist.

Gromov-Witten theory has links in many directions. When X is a curve, Gromov-Witten theory is related to counts of Hurwitz covers [75]. For the Severi degrees of curves in $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$, Gromov-Witten theory agrees with naive counts (when the latter are sensible). For surfaces of general type, Gromov-Witten theory links beautifully with Seiberg-Witten theory [95]. For 3-folds, there is a subtle and surprising relationship between Gromov-Witten theory and the sheaf counting theories discussed here in later sections. The relation with mirror symmetry [15, 34, 66] is a high point of the subject.

§Drawbacks. The theory is extremely hard to compute: even the Gromov-Witten theories of varieties of dimensions 0 and 1 are very complicated. The theory of a point is related to the KdV hierarchy [104], and the theory of $\mathbb{P}^1$ is related to the Toda hierarchy [75]. While such connections are beautiful, using Gromov-Witten theory to actually count curves is difficult, essentially due to the nonlinearity of maps from curves to varieties. The sheaf theories considered in the next sections concern more linear objects.

Because of the finite automorphisms of stable maps, Gromov-Witten invariants are typically rational numbers. An old idea in Gromov-Witten theory is that underlying the rational Gromov-Witten invariants should be integer-valued curve counts. For instance, consider a stable map $f \in \overline{\mathcal{M}}_g(X, \beta)$ double covering an image curve $C \subset X$ in class $\beta/2$. Suppose, for simplicity, f and C are rigid and unobstructed. Then, f counts $1/2$ towards the Gromov-Witten invariant $N_{g,\beta}(X)$ because of its $\mathbb{Z}/2$-stabiliser. Underlying this rational number is an integer 1 counting the embedded curve C in class $\beta/2$.

§Serious difficulties. For the case of 3-folds, Gromov-Witten theory is *not* enumerative in the naive sense in genus $g > 0$ due to degenerate contributions. The departure from naive counting happens already in positive genus for $\mathbb{P}^3$.

Let X be a 3-fold. The formula for the expected dimension of the moduli space of stable maps (1.5) is not genus dependent. Consider a nonsingular embedded rigid rational curve

$$\mathbb{P}^1 \subset X \tag{1.6}$$

in homology class β. The curve not only contributes 1 to $N_{0,\beta}$, but also contributes in a complicated way to $N_{g\geq 1,\beta}$. By attaching to the $\mathbb{P}^1$ any stable curve C at a nonsingular point, we obtain a stable map in the same class β which collapses C to a point. The contribution of (1.6) to the Gromov-Witten invariants $N_{g\geq 1,\beta}$ of X must be computed via integrals over the moduli spaces of stable curves. The latter integrals are hard to motivate from the point of view of curve counting.

A rather detailed study of the *Hodge integrals* over the moduli spaces of curves which arise in such degenerate contributions in Gromov-Witten theory has been pursued [29, 76]. A main outcome has been an understanding of the relationship of Gromov-Witten theory to naive curve counting on 3-folds in the Calabi-Yau and Fano cases. The conclusion is a precise conjecture expressing integer counts in terms of Gromov-Witten invariants (see the BPS conjecture in the next section). The sheaf counting theories developed later are now viewed as a more direct path to the integers underlying Gromov-Witten theory in dimension 3.

What happens in higher dimensions? Results of [53, 85] for spaces X of dimensions 4 and 5 show a similar underlying integer structure for Gromov-Witten theory. However, a direct interpretation of the integer counts (in terms of sheaves or other structures) in dimensions higher than 3 awaits discovery.

$2\frac{1}{2}$. Gopakumar-Vafa / BPS invariants

§Invariants. BPS invariants were introduced for Calabi-Yau 3-folds by Gopakumar-Vafa in [35, 36] using an M-theoretic construction. The multiple cover calculations [29, 76] in Gromov-Witten theory provided basic motivation. The definitions and conjectures related to BPS states were generalised to arbitrary 3-folds in [76, 77]. While the original approach to the subject is not yet on a rigorous footing, the hope is to define curve counting invariants which avoid the multiple cover and degenerate contributions of Gromov-Witten theory. The BPS counts should be the integers underlying the rational Gromov-Witten invariants of 3-folds.

To simplify the discussion here, let X be a Calabi-Yau 3-fold. Gopakumar and Vafa consider a moduli space $\mathcal{M}$ of D-branes supported on curves in class β. While the precise mathematical definition is not clear, for a nonsingular embedded curve $C \subset X$ of genus g and class

$$[C] = \beta \in H_2(X, \mathbb{Z}),$$

the D-branes are believed to be (the pushforward to X of) line bundles on C of a fixed degree, with moduli space a Jacobian torus diffeomorphic to T^{2g}. For singular curves, the D-brane moduli space should be a type of relative compactified Jacobian over the "space" of curves of class β.

Mathematicians have tended to interpret $\mathcal{M}$ as a moduli space of stable sheaves with 1-dimensional support in class β and holomorphic Euler characteristic $\chi = 1$. The latter condition is a technical device to rule out strictly semistable sheaves. Over nonsingular curves C, the moduli space is simply $\mathrm{Pic}_g(C)$. For singular curves, more exotic sheaves in the compactified Picard scheme arise. For nonreduced curves, we can find higher rank sheaves supported on the underlying reduced curve. The support map $\mathcal{M} \to B$, taking such a sheaf to the underlying support curve, is also required for the geometric path to the BPS invariants. Here, B is an appropriate (unspecified) parameter space of curves in X. For instance, there is certainly such a support map to the Chow variety of 1-cycles in X.

Let us now imagine that we are in the ideal situation where the parameter space $B = \coprod_i B_i$ is a disjoint union of connected components over which the map $\mathcal{M} \to B$ is a product,

$$\mathcal{M} = \coprod \mathcal{M}_i \quad \text{and} \quad \mathcal{M}_i = B_i \times F_i$$

with fibres F_i. The supposition is not ridiculous: the virtual dimension of curves in a Calabi-Yau 3-fold is 0, so we might hope that B is a finite set of points. Then,

$$H^*(\mathcal{M}) = \bigoplus_i H^*(B_i) \otimes H^*(F_i)\,. \tag{2.1}$$

When each B_i parameterises nonsingular curves of genus g_i only,

$$H^*(F_i) = H^*(T^{2g_i}) = (H^*(S^1))^{\otimes 2g_i}$$

has normalised Poincaré polynomial

$$P_y(F_i) = y^{-g_i}(1+y)^{2g_i}\,.$$

Here, we normalise by shifting cohomological degrees by $-\dim_{\mathbb{C}}(F_i)$ to make $P_y(F_i)$ symmetric about degree 0. Then, $P_y(F_i)$ is a palindromic Laurent polynomial invariant under $y \leftrightarrow y^{-1}$ by Poincaré duality.

For more general F_i, the normalised Poincaré polynomial $P_y(F_i)$ is again invariant under $y \leftrightarrow y^{-1}$ if $H^*(F_i)$ satisfies even dimensional Poincaré duality. Therefore, $P_y(F_i)$ may be written as a finite integral combination of terms $y^{-r}(1+y)^{2r}$, since the latter form a basis for the palindromic Laurent polynomials. Thus we can express $H^*(F_i)$ as a virtual combination of cohomologies of even dimensional tori. For instance, a cuspidal elliptic curve is topologically S^2 with

$$P_y = y^{-1}(1+y^2) = (y^{-1}+2+y) - 2 = P_y(T^2) - 2P_y(T^0)\,.$$

Cohomologically, we interpret the cuspidal elliptic curve as 1 Jacobian of a genus 1 curve minus 2 Jacobians of genus 0 curves.

To tease the "number of genus r curves" in class β from (2.1), Gopakumar-Vafa write

$$\sum_i (-1)^{\dim B_i} e(B_i) P_y(F_i) \quad \text{as} \quad \sum_r n_r(\beta) y^{-r}(1+y)^{2r} \tag{2.2}$$

and define the integers $n_r(\beta)$ to be the BPS invariants counting genus r curves in class β. In Section $3\frac{1}{2}$, we will see that when B is nonsingular and can be broken up into a finite number of points by a generic deformation, that number of points is $(-1)^{\dim B}e(B)$, see for instance (3.8). In other words, the virtual class of B consists of $(-1)^{\dim B}e(B)$ points, explaining the first term in (2.2).

The Künneth decomposition (2.1) does *not* hold for general $\mathcal{M} \to B$, but can be replaced by the associated Leray spectral sequence. According to [44], the *perverse* Leray spectral sequence on intersection cohomology is preferable since it collapses and its terms satisfy the Hard Lefschetz theorem (which replaces the Poincaré duality used above).

At least when B is nonsingular, $\mathcal{M}$ is reduced with sufficiently mild singularities, and

$$\pi : \mathcal{M} \to B$$

is equidimensional of fibre dimension $f = \dim \mathcal{M} - \dim B$, we can take

$$y^{-f} \sum_j (-1)^{\dim B} e({}^p R^j \pi_* \mathcal{IC}(\underline{\mathbb{C}})) y^j = \sum_r n_r(\beta) y^{-r} (1+y)^{2r} \tag{2.3}$$

as the Hosono-Saito-Takahashi definition[8] of the BPS invariants $n_r(\beta)$.

The entire preceding discussion of BPS invariants is only motivational. We have not been precise about the definition of the moduli space B. Moreover, the hypotheses imposed in the above constructions are rarely met (and when the hypotheses fail, the constructions are usually unreasonable or just wrong). Nevertheless, there should exists BPS invariants $n_{g,\beta} \in \mathbb{Z}$ "counting" curves of genus g and class β in X.

In addition to the M-theoretic construction, Gopakumar and Vafa have made a beautiful prediction of the relationship of the BPS counts to Gromov-Witten theory. For Calabi-Yau 3-folds, the conjectural formula is

$$\begin{aligned} \sum_{g \geq 0,\ \beta \neq 0} N^{\mathsf{GW}}_{g,\beta} u^{2g-2} v^\beta \ &= \\ &\sum_{g \geq 0,\ \beta \neq 0} n_{g,\beta} u^{2g-2} \sum_{d>0} \frac{1}{d} \left(\frac{\sin(du/2)}{u/2} \right)^{2g-2} v^{d\beta} \,. \end{aligned} \tag{2.4}$$

The trigonometric terms on the right are motivated by multiple cover formulas in Gromov-Witten theory [29, 76]. The entire geometric discussion can be bypassed by *defining* the BPS invariants via Gromov-Witten theory by equation (2.4). A precise conjecture [12] then arises.

BPS conjecture I. *For the $n_{g,\beta}$ defined via Gromov-Witten theory and formula* (2.4)*, the following properties hold:*

(i) $n_{g,\beta} \in \mathbb{Z}$,
(ii) for fixed β, the $n_{g,\beta}$ vanish except for finitely many $g \geq 0$.

---

[8]The original sources [35, 36, 44] make a great deal of use of $\mathfrak{sl}_2 \times \mathfrak{sl}_2$-actions on the cohomology of $\mathcal{M}$, but the end result is equivalent to the above intuitive description: decompose the fibrewise cohomology of $\mathcal{M}$ into the cohomologies of Jacobian tori, then take signed Euler characteristics in the base direction.

For other 3-folds X, when the virtual dimension is positive

$$\int_\beta c_1(X) > 0\,,$$

incidence conditions to cut down the virtual dimension to 0 must be included. This case will be discussed in Section $5\frac{1}{2}$ below. The conjectural formula for the BPS counts is similar, see (5.2).

§Advantages. For 3-folds, BPS invariants should be the ideal curve counts. The BPS invariants are integer valued and coincide with naive counts in many cases where the latter make sense. For example, the BPS counts (defined via Gromov-Witten theory) agree with naive curve counting in $\mathbb{P}^3$ in genus 0, 1, and 2. The definition via Gromov-Witten theory shows $n_{g,\beta}$ is a symplectic invariant.

For Calabi-Yau 3-folds X, the BPS counts do not always agree with naive counting. A trivial example is the slightly different treatment of an embedded super-rigid elliptic curve $E \subset X$, see [76]. Such an E contributes a single BPS count to each multiple degree $n[E]$. A much more subtle BPS contribution is given by a super-rigid genus 2 curve C in class $2[C]$ [13, 14]. We view BPS counting now as more fundamental than naive curve counting (and equivalent to, but *not* always equal to, naive counting).

§Drawbacks. The main drawback is the murky foundation of the geometric construction of the BPS invariants. For nonreduced curves, the contributions of the higher rank moduli spaces of sheaves on the underlying support curves remain mysterious. The real strength of the theory will only be realised after the foundations are clarified. For example, properties (i) and (ii) of the BPS conjecture should be immediate from a geometric construction. The definition via Gromov-Witten theory is far from adequate.

A significant limitation of the BPS counts is the restriction to 3-folds. However, calculations [53, 85] show some hope of parallel structures in higher dimensions, see also [47].

§Serious difficulties. The geometric foundations appear very hard to establish. There is no likely path in sight (except in genus 0 where Katz has made a rigorous proposal [49], see Section $4\frac{1}{2}$). The Hosono-Saito-Takahashi approach does not incorporate the virtual class (the term $(-1)^{\dim B}e(B)$ is a crude approximation for the virtual class of the base B) and fails in general.

Developments concerning motivic invariants [48, 58] and the categorification of invariants with cohomology theories instead of Euler characteristics appear somewhat closer to the methods required in the Calabi-Yau 3-fold case. For instance, Behrend has been working to categorify his constructible function [3] to give a perverse sheaf that could replace $\mathcal{IC}(\underline{\mathbb{C}})$ in the HST definition, perhaps yielding a deformation invariant theory. Even then, why formula (2.4) should hold is a mystery.

An approach to BPS invariants via stable pairs (instead of Gromov-Witten theory) will be discussed in Section $4\frac{1}{2}$ below. The BPS invariants $n_{g,\beta}$ are there again defined by a formula similar to (2.4). The stable pairs perspective is better than the Gromov-Witten approach and has led to substantial recent progress [19, 71, 72, 92, 97]. Nevertheless, the hole in the subject left by the lack of a direct geometric construction is not yet filled.

$3\frac{1}{2}$. Donaldson-Thomas theory

§Moduli. Instead of considering maps of curves into X, we can instead study embedded curves. Let a *subcurve* $Z \subset X$ be a subscheme of dimension 1. The *Hilbert scheme* compactifies embedded curves by allowing them to degenerate to arbitrary subschemes. Let $I_n(X,\beta)$ be the Hilbert scheme parameterising subcurves $Z \subset X$ with

$$\chi(\mathcal{O}_Z) = n \in \mathbb{Z} \quad \text{and} \quad [Z] = \beta \in H_2(X)\,.$$

Here, χ denotes the holomorphic Euler characteristic and $[Z]$ denotes the class of the subcurve (involving only the 1-dimensional components). By the above conditions, $I_n(X,\beta)$ parameterises subschemes which are unions of possibly nonreduced curves and points in X.

We give a few examples to show how the Hilbert scheme differs from the space of stable maps. First, consider a family of nonsingular conics

$$C_{t\neq 0} = \{x^2 + ty = 0\} \subset \mathbb{C}^2 \tag{3.1}$$

as a local model which can, of course, be further embedded in any higher dimension. The natural limit as $t \to 0$,

$$C_0 = \{x^2 = 0\} \subset \mathbb{C}^2, \tag{3.2}$$

is indeed the limit in the Hilbert scheme. The limit (3.2) is the y-axis with multiplicity two thickened in the x-direction.

In the stable map case, the limit of the family (3.1) is very different. There we take the limit of the associated map from $\mathbb{C}$ to C_t given by[9]

$$\xi \mapsto (-t^{1/2}\xi, \xi^2)\,.$$

[9]The formula gives a well-defined map only modulo automorphisms of the curve — specifically the automorphism $\xi \mapsto -\xi$.

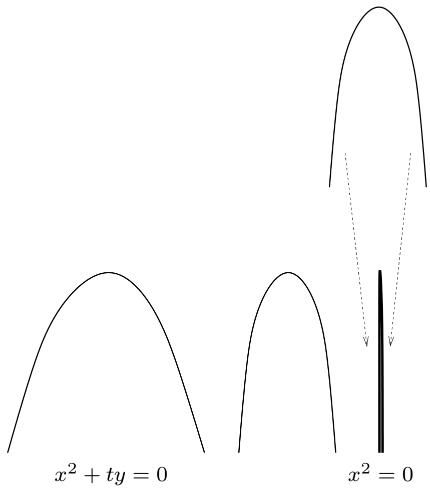

FIGURE 1. The degeneration (3.1) with the limiting stable map double covering $x = 0$.

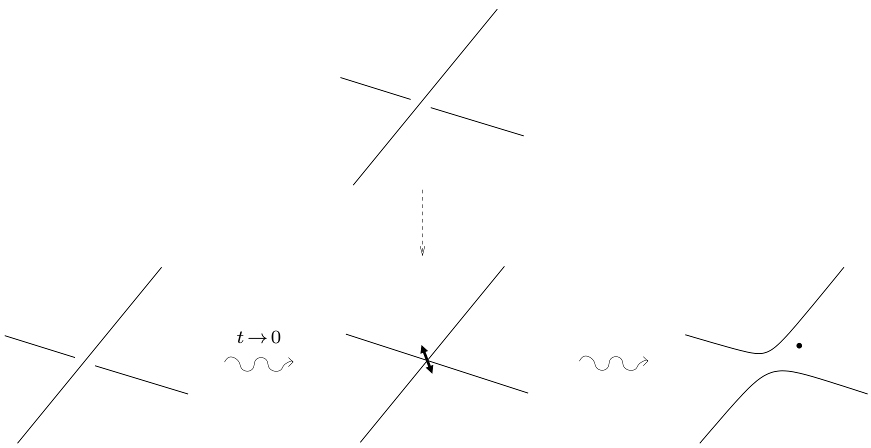

FIGURE 2. The family (3.3) with the subscheme limit below and the stable map limit above. On the right is a deformation of the limit subscheme with a free point breaking off.

The result is the *double cover* $\xi \mapsto (0, \xi^2)$ of the y-axis. So the thickened scheme in the Hilbert scheme is replaced by the double cover. The latter is an orbifold point in the space of stable maps with $\mathbb{Z}/2$-stabiliser given by $\xi \mapsto -\xi$.

In the next example, we illustrate the phenomenon of genus change which occurs only in dimension at least 3. A global model is given by a twisted rational cubic in $\mathbb{P}^3$ degenerating to a plane cubic of genus 1

[86]. An easier local model $C_t \subset \mathbb{C}^3$ has 2 components: the x-axis in the plane $z = 0$, and the y-axis moved up into the plane $z = t$,

$$C_t = \{x = 0 = z\} \sqcup \{y = 0 = z - t\} \subset \mathbb{C}^3\,, \tag{3.3}$$

see Figure 2. As a stable map, we take the associated inclusion of two copies of the affine line $\mathbb{C}$. The stable map limit at $t = 0$ takes the same domain $\mathbb{C} \sqcup \mathbb{C}$ onto the x- and y- axes, an embedding away from the origin where the map is 2:1. In other words, the limit stable map[10] is the normalisation of the image

$$\{xy = 0 = z\} \subset \mathbb{C}^3\,. \tag{3.4}$$

In the Hilbert scheme, the limit of the family (3.3) is rather worse. The ideal of C_t is

$$(x, z) \cdot (y, z - t) \;=\; \big(xy, x(z - t), yz, z(z - t)\big)\,.$$

We take the limit as $t \to 0$. The flat limit here happens to be the ideal generated by the limit of the above generators. The limit ideal does not contain z:

$$(xy, xz, yz, z^2) \;\subsetneq\; (xy, z)\,. \tag{3.5}$$

However, after multiplying z by any element of the maximal ideal (x, y, z) of the origin, we land inside the limit ideal. Therefore, the limit curve is given by $\{xy = 0 = z\}$ with a scheme-theoretic embedded point added at the origin pointing along the z-axis — in the direction along which the two components came together. The embedded point "makes up for" the point lost in the intersection and ensures that the family of curves is flat over $t = 0$.

In a further flat family, the embedded point can break off, and the curve can be smoothed $\{xy = \epsilon,\ z = 0\}$ to a curve of higher genus. In the Hilbert scheme, we have all 1-dimensional subschemes made up of curves *and* points, with curves of different genus balanced by extra free points. The constant n in $I_n(X, \beta)$ is $1 - g + k$ for a reduced curve of arithmetic genus g with k free and embedded points added, so we can increase g at the expense of increasing k by the same amount.

§Deformation theory. Hilbert schemes of curves can have arbitrary dimensional components and terrible singularities. Worse still, the natural deformation/obstruction theory of the Hilbert scheme does not lead to a virtual class. However, if we restrict attention to 3-folds X

[10] There is another stable map given by the embedding of the image (3.4). In a compact global model, the latter would be a map from a nodal stable curve of genus one larger so would *not* feature in the compactification of the family we are considering.

and view $I_n(X,\beta)$ as a moduli space of *sheaves*, then we obtain a different obstruction theory which does admit a virtual class. The latter observation is the starting point of Donaldson-Thomas theory.

Given a 1-dimensional subscheme $Z \subset X$, the associated ideal sheaf $\mathscr{I}_Z$ is a stable sheaf with Chern character

$$(1, 0, -\beta, -n) \in H^0 \oplus H^2 \oplus H^4 \oplus H^6$$

and trivial determinant. Conversely, all such stable sheaves with trivial determinant can be shown to embed in their double duals $\mathcal{O}_X$ and thus are all ideal sheaves. Hence, $I_n(X,\beta)$ is a moduli space of sheaves, at least set theoretically. With more work, an isomorphism of schemes can be established, see [82, Theorem 2.7].

The moduli space of sheaves $I_n(X,\beta)$ also admits a virtual class [96, 68]. The main point is that deformations and obstructions are governed by

$$\mathrm{Ext}^1(\mathscr{I}_Z,\mathscr{I}_Z)_0 \quad \text{and} \quad \mathrm{Ext}^2(\mathscr{I}_Z,\mathscr{I}_Z)_0 \tag{3.6}$$

respectively, where the subscript 0 denotes the trace-free part governing deformations with *fixed determinant*. Since $\mathrm{Hom}(\mathscr{I}_Z,\mathscr{I}_Z) = \mathbb{C}$ consists of only the scalars, the trace-free part vanishes. By Serre duality,

$$\mathrm{Ext}^3(\mathscr{I}_Z,\mathscr{I}_Z) \cong \mathrm{Hom}(\mathscr{I}_Z,\mathscr{I}_Z \otimes K_X)^* \cong H^0(K_X)^* \cong H^3(\mathcal{O}_X)\,.$$

The last group $H^3(\mathcal{O}_X)$ is removed when taking trace-free parts. Hence, the terms (3.6) are the only nonvanishing trace-free Exts, and there are *no* higher obstruction spaces. The Exts (3.6) govern a perfect obstruction theory of virtual dimension equal to

$$\mathrm{ext}^1(\mathscr{I}_Z,\mathscr{I}_Z)_0 - \mathrm{ext}^2(\mathscr{I}_Z,\mathscr{I}_Z)_0 = \int_\beta c_1(X)\,,$$

compare (1.5). If the virtual dimension is positive, insertions are needed to produce invariants [68].

On Calabi-Yau 3-folds, moduli of sheaves admit a particularly nice deformation-obstruction theory [27, 96]. The deformation and obstruction spaces (3.6) are *dual* to each other,

$$\mathrm{Ext}^2(\mathscr{I}_Z,\mathscr{I}_Z)_0 \cong \mathrm{Ext}^1(\mathscr{I}_Z,\mathscr{I}_Z)_0^*\,, \tag{3.7}$$

by Serre duality. Any moduli space of sheaves on a Calabi-Yau 3-fold can be realized as the critical locus of a holomorphic function on an ambient nonsingular space: the holomorphic Chern-Simons functional in infinite dimensions [105, 27] or locally on an appropriate finite dimensional slice [48]. Since the moduli space is the zero locus of a closed 1-form, the obstruction space is the cotangent space at any point of moduli space. More generally, Behrend [3] calls obstruction theories satisfying the global version of (3.7) *symmetric*. The condition is equivalent to asking for the moduli space to be locally the zeros of an *almost closed* 1-form on a smooth ambient space — a 1-form with exterior derivative vanishing scheme theoretically on the moduli space.[11]

If the moduli space of sheaves is nonsingular (but of too high dimension), then the symmetric obstruction theory forces the obstruction bundle to be globally isomorphic to the cotangent bundle. The virtual class, here the top Chern class of the obstruction bundle, is then just the signed topological Euler characteristic of the moduli space

$$(-1)^{\dim I_n(X,\beta)} e(I_n(X,\beta)) \,. \tag{3.8}$$

Remarkably, Behrend shows that for any moduli space $\mathcal{M}$ with a symmetric obstruction theory there is a constructible function

$$\chi^B \colon \mathcal{M} \to \mathbb{Z}$$

with respect to which the weighted Euler characteristic gives the integral of the virtual class [3]. Therefore, each point of the moduli space contributes in a local way to the global invariant, by $(-1)^{\dim \mathcal{M}}$ for a nonsingular point and by a complicated number taking multiplicities into account for singular points. When $\mathcal{M}$ is locally the critical locus of a function, the number is $(-1)^{\dim \mathcal{M}}(1-e(F))$ where F is the Milnor fibre of our point. Unfortunately, how to find a parallel approach to the virtual class when X is not Calabi-Yau is not currently known.

Integration against the virtual class of $I_n(X,\beta)$ yields the Donaldson-Thomas invariants. In the Calabi-Yau case, no insertions are required:

$$I_{n,\beta} \;=\; \int_{[I_n(X,\beta)]^{vir}} 1 \;=\; e\big(I_n(X,\beta),\chi^B\big) \,.$$

Since $I_n(X,\beta)$ is a scheme (ideal sheaves have no automorphisms) and $[I_n(X,\beta)]^{vir}$ is a cycle class with $\mathbb{Z}$-coefficients, the invariants $I_{n,\beta}$ are *integers*. Deformation invariance of $I_{n,\beta}$ follows from properties of the virtual class.

§MNOP conjectures. A series of conjectures linking the Donaldson-Thomas theory of 3-folds to Gromov-Witten theory were advanced in [68, 69]. For simplicity, we restrict ourselves here to the Calabi-Yau case.

For fixed curve class $\beta \in H_2(X,\mathbb{Z})$, the Donaldson-Thomas *partition function* is

$$\mathsf{Z}^{\mathrm{DT}}_\beta(q) = \sum_n I_{n,\beta} q^n \,.$$

[11] By [84], the condition is strictly weaker than asking for the moduli space to be locally the zeros of a closed 1-form.

Since $I_n(X,\beta)$ is easily seen to be empty for n sufficiently negative, the partition function is a Laurent series in q. To count just curves, and not points and curves, MNOP form the *reduced* generating function [68] by dividing by the contribution of just points:

$$\mathsf{Z}^{\mathrm{red}}_\beta(q) = \frac{\mathsf{Z}^{\mathrm{DT}}_\beta(q)}{\mathsf{Z}^{\mathrm{DT}}_0(q)}\,. \tag{3.9}$$

MNOP first conjectured the degree $\beta=0$ contribution can be calculated as

$$\mathsf{Z}^{\mathrm{DT}}_0(q) = M(-q)^{e(X)}\,,$$

where M is the MacMahon function,

$$M(q) = \prod_{n\geq 1}(1-q^n)^{-n}\,,$$

the generating function for 3d partitions. Proofs can now be found in [7, 63, 60]. Second, MNOP conjectured $\mathsf{Z}^{\mathrm{red}}_\beta(q)$ is the Laurent expansion of a *rational function* in q, invariant[12] under $q \leftrightarrow q^{-1}$. Therefore, we can substitute $q=-e^{iu}$ and obtain a real-valued function of u. The main conjecture of MNOP in the Calabi-Yau case is the following.

GW/DT Conjecture: $\mathsf{Z}^{\mathrm{GW}}_\beta(u) = \mathsf{Z}^{\mathrm{red}}_\beta(-e^{iu})$.

The conjecture asserts a precise equivalence relating Gromov-Witten to Donaldson-Thomas theory. Here,

$$\mathsf{Z}^{\mathrm{GW}}_\beta(u) = \sum_{g\geq 0} N^\bullet_{g,\beta}\; u^{2g-2}$$

is the generating function of *disconnected* Gromov-Witten invariants $N^\bullet_{g,\beta}$ defined just as in Section $1\frac{1}{2}$ by relaxing the condition that the curves be connected, but excluding maps which contract connected components to points. Equivalently, $\mathsf{Z}^{\mathrm{GW}}_\beta(u)$ is the exponential of the generating function of connected Gromov-Witten invariants $N_{g,\beta}$,

$$\sum_{\beta\neq 0}\mathsf{Z}^{\mathrm{GW}}_\beta(u)v^\beta = \sum_{\beta\neq 0,\, g\geq 0} N^\bullet_{g,\beta}\; u^{2g-2}v^\beta = \exp\Bigg(\sum_{\beta\neq 0,\, g\geq 0} N_{g,\beta}\, u^{2g-2}v^\beta\Bigg).$$

A version of the GW/DT correspondence with insertions for non Calabi-Yau 3-folds can be found in [69]. Various refinements, involving theories relative to a divisor, or equivariant with respect to a group action, are

[12] The Laurent series itself need not be $q\leftrightarrow q^{-1}$ invariant. For instance the rational function $q(1+q)^{-2}$ is invariant, but the associated Laurent series $q-2q^2+3q^3-\ldots$ is not.

also conjectured. All of these conjectures have been proved for toric 3-folds in [70].

The GW/DT conjecture should be viewed as involving an analytic continuation and series expansion about two different points ($q = 0$ and $q = -1$, corresponding to $u = 0$). Therefore, the conjecture cannot be understood term by term[13] — to determine a single invariant on one side of the conjecture, knowledge of all of the invariants on the other side is necessary.

The overall shape of the conjecture is clear: the two different ways of counting curves in a fixed class β are entirely equivalent, with *integers* determining the Gromov-Witten invariants of 3-folds. By [82, Theorem 3.20], the integrality prediction of the GW/DT correspondence is entirely equivalent to the integrality prediction of the Gopakumar-Vafa formula (2.4).

§Advantages. The integrality of the invariants is a significant advantage of using the Hilbert schemes $I_n(X, \beta)$ to define a counting theory. Also, the virtual counting of subschemes, at least in the Calabi-Yau 3-fold case, fits into the larger context of counting higher rank bundles, sheaves, and objects of the derived category of X. The many recent developments in wall-crossing [48, 58] apply to this more general setting. We will see an example in the next section.

Behrend's constructible function sometimes makes computations (in the Calabi-Yau case at least) more feasible — we can use cut and paste techniques to reduce to more local calculations. See for instance [4].

§Drawbacks. The theory only works for nonsingular projective varieties of dimension at most 3. While the Hilbert scheme of curves is always well-defined, the deformation/obstruction theory fails to be 2-term in higher dimensions. By contrast, Gromov-Witten theory is well-defined in all dimensions and is proved to be a symplectic invariant. While we expect Donaldson-Thomas theory to have a fully symplectic approach, how to proceed is not known.

In Gromov-Witten theory, the genus expansion makes a connection to the moduli of curves (independent of X). The Euler characteristic n plays a parallel role in Donaldson-Thomas theory, but is much less useful. While there are very good low genus results in Gromov-Witten theory, there are few analogues for the Hilbert scheme.

[13]When combined with the Gopakumar-Vafa formula (2.4) and the relationship to the stable pairs discussed below, the GW/DT conjecture will become rather more comprehensible, see (4.9).

Behrend's constructible function approach for the Calabi-Yau case is difficult to use. For example, the constructible functions even for toric Calabi-Yau 3-folds have not been determined.[14] So far, Behrend's theory has been useful mainly for formal properties related to motivic invariants and wall-crossing. For more concrete calculations involving Behrend's functions see [4, 5].

§Serious difficulties. For the GW/DT correspondence, the division by $\mathsf{Z}_0(q)$ confuses the geometric interpretation of the invariants. In fact, the subschemes of X with free points make the theory rather unpleasant to work with. This "compactification" of the space of embedded curves is much larger than the original space, adding enormous components with free points. In practice, the free points lead to constant technical headaches (which play little role in the main development of the invariants).

It is tempting to think of working with the closure of the "good components" of the Hilbert scheme instead, but such an approach would not have a reasonable deformation theory nor a virtual class. However, a certain birational modification of the idea does work and will be discussed in the next section.

$4\frac{1}{2}$. Stable pairs

§Limits revisited. Consider again the family of Figure 2. For $t \neq 0$, denote the disjoint union (3.3) by

$$C_t = C_t^1 \cup C_t^2 \,.$$

The ideal sheaf $\mathscr{I}_{C_t}$, central to the Hilbert scheme analysis, is just the kernel of the surjection

$$\mathcal{O}_X \to \mathcal{O}_{C_t^1} \oplus \mathcal{O}_{C_t^2} \,. \tag{4.1}$$

For the moduli of stable pairs, the map itself (not just the kernel) will be fundamental. We will take a natural limit of the map given by

$$\mathcal{O}_X \to \mathcal{O}_{C_0^1} \oplus \mathcal{O}_{C_0^2} \tag{4.2}$$

where the limits of the component curves are

$$C_0^1 = \{x = 0 = z\} \quad \text{and} \quad C_0^2 = \{y = 0 = z\} \,.$$

The result is a map which is *not* a surjection at the origin (where C_1 and C_2 intersect and the sheaf on the right has rank 2). In the limit, there is a nonzero cokernel, the structure sheaf of the origin $\mathcal{O}_0$, which

[14] Amazingly, we do not even know whether the constructible functions are non-constant in the toric Calabi-Yau case!

accounts for the extra point lost in the intersection. Losing surjectivity replaces the embedded point arising in the limit of ideal sheaves (3.5).

The cokernels of the above maps (4.1) are *not* flat over $t = 0$ even though the sheaves $\mathcal{O}_{C_t^1} \oplus \mathcal{O}_{C_t^2}$ *are* flat. Similarly the kernels of the maps (4.1) are *not* flat over $t = 0$. In fact, at $t = 0$, we get the ideal (xy, z) of $C_0^1 \cup C_0^2$ which we already saw in (3.5) is not the flat limit of the ideal sheaves of C_t.

§Moduli. The limit (4.2) is an example of a stable pair. The moduli of stable pairs provides a different sheaf-theoretic compactification of the space of embedded curves. The moduli space is intimately related to the Hilbert scheme, but is much more efficient.

Let X be a nonsingular projective 3-fold. A *stable pair* (F, s) is a coherent sheaf F with dimension 1 support in X and a section $s \in H^0(X, F)$ satisfying the following stability condition:

- F is *pure*, and
- the section s has zero dimensional cokernel.

Let C be the scheme-theoretic support of F. Condition (i) means all the irreducible components of C are of dimension 1 (no 0-dimensional components). By [82, Lemma 1.6], C has no embedded points. A stable pair

$$\mathcal{O}_X \to F$$

therefore defines a Cohen-Macaulay curve C via the kernel $\mathscr{I}_C \subset \mathcal{O}_X$ and a 0-dimensional subscheme of C via the support of the cokernel[15].

To a stable pair, we associate the Euler characteristic and the class of the support C of F,

$$\chi(F) = n \in \mathbb{Z} \quad \text{and} \quad [C] = \beta \in H_2(X, \mathbb{Z})\,.$$

For fixed n and β, there is a projective moduli space of stable pairs $P_n(X, \beta)$ [82, Lemma 1.3] by work of Le Potier [59]. While the Hilbert scheme $I_n(X, \beta)$ is a moduli space of curves plus free and embedded points, $P_n(X, \beta)$ should be thought of as a moduli space of curves plus points *on the curve* only. Even though points still play a role (as the example (3.3) shows), the moduli of stable pairs is much smaller than $I_n(X, \beta)$.

[15] When C is Gorenstein (for instance if C lies in a nonsingular surface), stable pairs supported on C are in bijection with 0-dimensional subschemes of C. More precise scheme theoretic isomorphisms of moduli spaces are proved in [83, Appendix B].

§Deformation theory. To define a flexible counting theory, a compactification of the family of curves in X should admit a 2-term deformation/obstruction theory and a virtual class. As in the case of $I_n(X,\beta)$, the most immediate obstruction theory of $P_n(X,\beta)$ does *not* admit such a structure. For $I_n(X,\beta)$, a solution was found by considering a subscheme C to be equivalent to a sheaf $\mathscr{I}_C$ with trivial determinant. For $P_n(X,\beta)$, we consider a stable pair to define an object of $D^b(X)$, the quasi-isomorphism equivalence class of the complex

$$I^\bullet = \{\mathcal{O}_X \xrightarrow{s} F\}\,. \tag{4.3}$$

For X of dimension 3, the object $I^\bullet$ determines the stable pair [82, Proposition 1.21], and the fixed-determinant deformations of $I^\bullet$ in $D^b(X)$ match those of the pair (F,s) to all orders [82, Theorem 2.7]. The latter property shows the scheme $P_n(X,\beta)$ may be viewed as a moduli space of objects in the derived category.[16] We can then use the obstruction theory of the complex $I^\bullet$ in place of the obstruction theory of the pair.

The deformation/obstruction theory for complexes is governed at $[I^\bullet]\in P_n(X,\beta)$ by

$$\mathrm{Ext}^1(I^\bullet,I^\bullet)_0 \quad \text{and} \quad \mathrm{Ext}^2(I^\bullet,I^\bullet)_0\,. \tag{4.4}$$

Formally, the outcome is parallel to (3.6). The obstruction theory (4.4) has all the attractive properties of the Hilbert scheme case: 2 terms, a virtual class of dimension $\int_\beta c_1(X)$, and a description via the χ^B-weighted Euler characteristics in the Calabi-Yau case.

§Invariants. After imposing incidence conditions (when $\int_\beta c_1(X)$ is positive) and integrating against the virtual class, we obtain stable pairs invariants for 3-folds X. In the Calabi-Yau case, the invariant is the length of the virtual class:

$$P_{n,\beta} = \int_{[P_n(X,\beta)]^{vir}} 1 \;=\; e\big(P_n(X,\beta),\chi^B\big)\,.$$

For fixed curve class $\beta\in H_2(X,\mathbb{Z})$, the stable pairs partition function is

$$\mathsf{Z}^{\mathrm{P}}_\beta(q) = \sum_n P_{n,\beta}q^n.$$

Again, elementary arguments show the moduli spaces $P_n(X,\beta)$ are empty for sufficiently negative n, so $\mathsf{Z}^{\mathrm{P}}_\beta$ is a Laurent series in q. Since

[16] Studying the moduli of objects in the derived category is a young subject. Usually, such constructions lead to Artin stacks. The space $P_n(X,\beta)$ is a rare example where a component of the moduli of objects in the derived category is a scheme (uniformly for all 3-folds X).

the free points are now confined to the curve instead of roaming over X, we do not have to form a reduced series as in (3.9). In fact, we conjecture [82, Conjecture 3.3] the partition function $\mathsf{Z}^{\mathrm{P}}_\beta$ to be precisely the reduced theory of Section $3\frac{1}{2}$.

DT/Pairs Conjecture: $\mathsf{Z}^{\mathrm{P}}_\beta(q) \;=\; \mathsf{Z}^{\mathrm{red}}_\beta(q)$.

The DT/Pairs correspondence is expected for all 3-folds X with the incidence conditions playing no significant role [69]. Using the definition $\mathsf{Z}^{\mathrm{red}}_\beta = \mathsf{Z}^{\mathrm{DT}}_\beta / \mathsf{Z}^{\mathrm{DT}}_0$, we find

$$\sum_m P_{n-m,\beta} \cdot I_{m,0} = I_{n,\beta}\,. \tag{4.5}$$

Relation (4.5) should be interpreted as a wall-crossing formula for counting invariants in the derived category of coherent sheaves $D^b(X)$ under a change of stability condition.

For invariants of Calabi-Yau 3-folds, wall-crossing has been studied intensively in recent years, and we give only the briefest of descriptions. Ideal sheaves parameterised by $I_n(X,\beta)$ are Gieseker stable. We can imagine changing the stability condition[17] to destabilise the ideal shaves with free and embedded points. If Z is a 1-dimensional subscheme, then Z has a maximal pure dimension 1 subscheme C defining a sequence

$$0 \to \mathscr{I}_Z \to \mathscr{I}_C \to Q \to 0\,,$$

where Q is the maximal 0-dimensional subsheaf of $\mathcal{O}_Z$. In $D^b(X)$, we equivalently have the exact triangle

$$Q[-1] \to \mathscr{I}_Z \to \mathscr{I}_C\,. \tag{4.6}$$

We can imagine the stability condition crossing a wall on which the phase (or slope) of $Q[-1]$ equals that of $\mathscr{I}_C$. On the other side of the wall, $\mathscr{I}_Z$ will be destabilised by (4.6). Meanwhile, extensions E in the opposite direction

$$\mathscr{I}_C \to E \to Q[-1] \tag{4.7}$$

will become stable. But stable pairs are just such extensions! The exact sequence

$$0 \to \mathscr{I}_C \to \mathcal{O}_X \xrightarrow{s} F \to Q \to 0$$

yields the exact triangle

$$\mathscr{I}_C \to I^\bullet \to Q[-1]\,.$$

[17]Ideally, we would work with Bridgeland stability conditions [9], but that is not currently possible. The above discussion can be made precise using the limiting stability conditions of [1, 98], or even Geometric Invariant Theory [94].

The moduli space of pairs $P_n(X,\beta)$ should give precisely the space of stable objects for the new stability condition.

The formula (4.5) for $I_{n,\beta} - P_{n,\beta}$ should follow from the more general wall-crossing formulae of [48, 58]. The m^{th} term in (4.5) is the correction from subschemes Z whose maximal 0-dimensional subscheme (or total number of free and embedded points) is of length m. It involves both the space $\mathbb{P}(\mathrm{Ext}^1(\mathscr{I}_C, Q[-1]))$ of extensions (4.6) and the space $\mathbb{P}(\mathrm{Ext}^1(Q[-1], \mathscr{I}_C))$ of extensions (4.7). Though both are hard to control, they contribute to the wall-crossing formula through the difference in their Euler characteristics[18], which is the topological number

$$\chi(\mathscr{I}_C, Q) = \ \text{length}\,(Q)\ = m\,.$$

The above sketch has now been carried out at the level of (unweighted) Euler characteristics [99, 94] and for χ^B-weighted Euler characteristics in [10] in the Calabi-Yau case. The upshot is the DT/Pairs conjecture is now proved for Calabi-Yau 3-folds. The rationality of $\mathsf{Z}^{\mathrm{red}}_\beta(q)$ and the symmetry under $q \leftrightarrow q^{-1}$ is also proved [10].

§Example. Via the Behrend weighted Euler characteristic approach to the invariants of a Calabi-Yau 3-fold, we can talk about the contribution of a single curve $C \subset X$ to the stable pairs generating function $\mathsf{Z}^P_\beta(q)$. No such discussion is possible in Gromov-Witten theory.

If C is nonsingular of genus g, then the stable pairs supported on C with $\chi = 1-g+n$ are in bijection with $\mathrm{Sym}^n C$ via the map taking a stable pair to the support of the cokernel Q. Therefore, C contributes[19]

(4.8)
$$\mathsf{Z}^{\mathrm{P}}_C(q) \ = \ c\sum_n (-1)^{n-g} e(\mathrm{Sym}^n C) q^{1-g+n} \ = \ (-1)^g c\, q^{1-g}(1+q)^{2g-2}.$$

The rational function on the right is invariant under $q \leftrightarrow q^{-1}$. We view the symmetry as a manifestation of Serre duality (discussed below). Control of the free points in stable pair theory makes the geometry more transparent. The same calculation for $\mathsf{Z}^{\mathrm{red}}_C(q)$ based on the Hilbert scheme is much less enlightening. The above calculation is closely related to the BPS conjecture for stable pairs.

---

[18]Really, we need to weight by the restriction of the Behrend function χ^B. To make the above analysis work then requires χ^B to satisfy the identities of [48, 58]. In fact, the automorphisms of Q make the matter much more complicated than we have suggested.

[19]The Behrend function restricted to $\mathrm{Sym}^n C$ can be shown [83, Lemma 3.4] to be the constant $(-1)^{n-g}c$, where $c = \chi^B(\mathcal{O}_C)$ is the Behrend function of the moduli space of torsion sheaves evaluated at $\mathcal{O}_C$.

§Stable pairs and BPS invariants. By a formal argument [82, Section 3.4], the stable pairs partition function can be written uniquely in the following special way:

$$\begin{aligned}\mathsf{Z}^{\mathrm{P}}(q,v) &:= 1 + \sum_{\beta\neq 0} \mathsf{Z}^{\mathrm{P}}_{\beta}(q) v^{\beta} \\ &= \exp\left(\sum_{r} \sum_{\gamma\neq 0} \sum_{d\geq 1} \tilde{n}_{r,\gamma} \frac{(-1)^{(1-r)}}{d} (-q)^{d(1-r)} (1-(-q)^d)^{2r-2} v^{d\gamma} \right),\end{aligned}$$

where the $\tilde{n}_{r,\gamma}$ are integers and vanish for fixed γ and r sufficiently large.

We can compose the various conjectures to link the BPS counts of Gopakumar and Vafa to the stable pairs invariants. The form we get from the conjectures is almost exactly as above:

$$\begin{aligned}&\mathsf{Z}^{\mathrm{P}}(q,v) = \\ &\qquad \exp\left(\sum_{r\geq 0} \sum_{\gamma\neq 0} \sum_{d\geq 1} n_{r,\gamma} \frac{(-1)^{(1-r)}}{d} (-q)^{d(1-r)} (1-(-q)^d)^{2r-2} v^{d\gamma} \right).\end{aligned}$$

The only difference is the restriction on the r summation. Hence, we can *define* the BPS state counts by stable pairs invariants via the $\tilde{n}_{r,\gamma}$!

BPS conjecture II. *For the $\tilde{n}_{r,\beta}$ defined via stable pairs theory, the vanishing*

$$\tilde{n}_{r<0,\beta} = 0$$

holds for $r < 0$.

By its construction, the approach to defining the BPS states counts via stable pairs satisfies the full integrality condition and half of the finiteness of BPS conjecture I. We therefore regard the stable pairs perspective as better than the path via Gromov-Witten theory. Still, a direct construction of the BPS invariants along the lines discussed in Section $2\frac{1}{2}$ would be best of all.[20]

[20]For curves with only reduced plane curve singularities, both constructions of BPS numbers have been shown to coincide after making the $\chi^B = (-1)^{\dim}$ approximation to the virtual class [71, 72].

For irreducible classes[21], the BPS formula for the stable pairs invariants can be written as

$$\mathsf{Z}^{\mathrm{P}}_{\beta}(q) = \sum_{r\geq 0}^{g} n_{r,\beta}\, q^{1-r}(1+q)^{2r-2}, \tag{4.9}$$

with $n_{r,\beta} = 0$ for all sufficiently large r. There is a beautiful interpretation of (4.9) in the light of (4.8): to the stable pairs invariants, the curves in class β look like a disjoint union of a finite number $n_{r,\beta}$ of nonsingular curves of genus r.

We can prove directly that the partition function $\mathsf{Z}^{\mathrm{P}}_{\beta}$ can be written in the form (4.9). For $r \geq 1$, the functions $q^{1-r}(1+q)^{2r-2}$,

$$1, \qquad q^{-1}+2+q, \qquad q^{-2}+4q^{-1}+6+4q+q^2, \qquad q^{-3}+\ldots$$

form a natural $\mathbb{Z}$-basis for the Laurent polynomials invariant under $q \leftrightarrow q^{-1}$. For $r = 0$, the coefficients of the Laurent series do not satisfy the same symmetry,

$$q(1+q)^{-2} \;=\; q - 2q^2 + 3q^3 - 4q^4 + \ldots \;=\; \sum_{n\geq 1} (-1)^{n-1} n q^n .$$

To prove (4.9), it is therefore equivalent to show the coefficients $P_{n,\beta}$ of the partition function satisfy not the $q \leftrightarrow q^{-1}$ symmetry but

$$P_{n,\beta} = P_{-n,\beta} + c(-1)^{n-1} n \tag{4.10}$$

for some constant c.

Relation (4.10) is a simple consequence of Serre duality for the fibres of the Abel-Jacobi map. By forgetting the section, we obtain a map from stable pairs to stable sheaves[22],

$$\begin{array}{ccc} P_n(X,\beta) & \longrightarrow & \mathcal{M}_n(X,\beta)\,, \\ (F,s) & \mapsto & F\,. \end{array}$$

The fibre of the map is $\mathbb{P}(H^0(F))$ with weighted Euler characteristic[23] $(-1)^{n-1} c \cdot h^0(F)$. There is an isomorphism

$$\begin{array}{ccc} \mathcal{M}_n(X,\beta) & \longrightarrow & \mathcal{M}_{-n}(X,\beta)\,, \\ F & \mapsto & F^{\vee}\,. \end{array}$$

[21] A class $\beta \in H_2(X,\mathbb{Z})$ is irreducible if it cannot be written as a sum $\alpha+\gamma$ of nonzero classes containing algebraic curves.

[22] The irreducibility of β implies the sheaves with arise are stable since F has rank 1 on its irreducible support.

[23] As proved in [83, Theorem 4], the Behrend function is constant on $\mathbb{P}(H^0(F))$ with value $(-1)^{n-1}c$ where $c = \chi^B(\mathcal{O}_C)$. On a first reading, the Behrend function can be safely ignored here.

where $F^\vee = \mathscr{E}xt^2(F, K_X)$. If F is the push-forward of a line bundle L from a nonsingular curve C, then $F^\vee$ is the push-forward of $L^* \otimes \omega_C$, see [83] for details. The fibre $\mathbb{P}(H^0(F^\vee))$ over $F^\vee$ is $\mathbb{P}(H^1(F)^*)$ by Serre duality, with weighted Euler characteristic $(-1)^{-n-1}c \cdot h^1(F)$.

To prove relation (4.10), we calculate the difference between the two above contributions:

$$(-1)^{n-1}c(h^0(F) - h^1(F)) = (-1)^{n-1}c\chi(F) = (-1)^{n-1}cn\,.$$

Summation over the space of stable sheaves (in the sense of Euler characteristics) yields the relation

$$P_{n,\beta} - P_{-n,\beta} = (-1)^{n-1}n\, e(\mathcal{M}_n(X,\beta), c)\,. \tag{4.11}$$

The weighted Euler characteristics

$$e(\mathcal{M}_n(X,\beta), c) = e(\mathcal{M}_{n+1}(X,\beta), c) \tag{4.12}$$

are independent of n: tensoring with a degree 1 line bundle relates sheaves supported on C with $\chi = n$ to those with $\chi = n+1$. We have proved the relation (4.10).

The above argument shows the coefficient $n_{0,\beta}$ of $q(1+q)^{-2}$ is the χ^B-weighted Euler characteristic of $\mathcal{M}_n(X,\beta)$. In fact, Katz [49] had previously proposed the DT invariant of $\mathcal{M}_1(X,\beta)$ as a good definition of $n_{0,\beta}$ for *any* class β, not necessarily irreducible. Naively, Katz's definition sees only the rational curves because for a curve of higher genus the action of the Jacobian on the moduli space of sheaves forces the (weighted) Euler characteristic of the latter to be zero. Katz's proposal can be viewed as a weak analogue of the genus by genus methods in Gromov-Witten theory.

Identity (4.11) is easily seen to be another wall-crossing formula [1, 83, 98]. In [100], Toda has extended the above analysis to all curve classes by extending the methods of Joyce [48] and the ideas of Kontsevich-Soibelman [58] on BPS formulations of general sheaf counting. His main result reduces BPS conjecture II to an analogue of identity (4.12) for DT invariants for dimension 1 sheaves for *all* classes β.[24]

§Advantages. The stable pair theory has the advantages of the ideal sheaf theory – integer invariants conjecturally equivalent to the rational Gromov-Witten invariants – but with the bonus of eliminating the free points on X. The geometry of the BPS conjectures is more clearly explained by stable pairs than any of the other approaches.

[24] When β is not irreducible, sheaf stability issues change the definition of the DT invariant, see [100, Conjecture 6.3] for details.

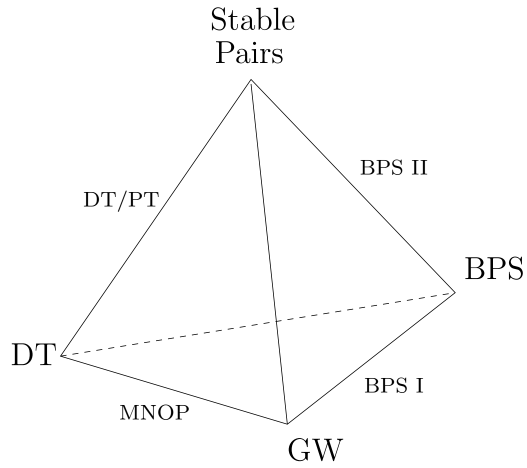

FIGURE 3. Conjectures connecting curve counting theories

If descendent insertions (coming from higher Chern classes of the tautological bundles) are considered, the theory of stable pairs behaves much better than the parallel constructions for the moduli of stable maps or the Hilbert scheme. For example, the descendent partition functions for stable pairs are rational in q. See [80] for proofs in toric cases and further discussion.

At least for 3-folds, stable pairs appears to be the best counting theory to consider at the moment. The main hope for a better approach lies in the direct geometric construction of the BPS counts.

§Drawbacks. Just as for the Donaldson-Thomas theory of ideal sheaves, the stable pairs invariants have only been constructed on nonsingular projective varieties of dimension at most 3. While we expect a parallel theory for symplectic invariants, how to proceed is not clear.

§Serious difficulties. In the theory of stable pairs, free points are allowed to move along the support curve C. The free points are necessary to probe the geometry of the curve (and the associated BPS contributions in all genera) but in a rather roundabout way. An alternative opened up by the Behrend function might be to work with open moduli spaces (on which the arithmetic genus does not jump), but deformation invariance then becomes problematic.

As we have said repeatedly, a rigorous and sheaf theoretic approach to BPS invariants (at least for 3-folds and possibly in higher dimensions as well) would be highly desirable.

$5\frac{1}{2}$. Stable unramified maps

§Singularities of maps. A difficulty which arises in Gromov-Witten theory is the abundance of collapsed components. In the moduli space of higher genus stable maps to $\mathbb{P}^1$ of degree 1, the entire complexity comes from such collapsed components attached to a degree 1 map of a genus 0 curve to $\mathbb{P}^1$. Collapsed contributions have to be removed to arrive at the integer counts underlying Gromov-Witten theory.

A map f from a nodal curve C to a nonsingular variety X is *unramified* at a nonsingular point $p \in C$ if the differential

$$df : T_{C,p} \to T_{X,f(p)}$$

is injective. If f is unramified at p, the component of C on which p lies cannot be collapsed.

The idea of stable unramified maps, introduced by Kim, Kresch, and Oh [51], is to control both the domain (allowing only nodal curves) *and* the singularities of the maps (essentially unramified and with no collapsed components). The price for these properties is paid in the complexity of the target space X. The target cannot remain inert, but must be allowed to degenerate.

§Degenerations. Let X be a nonsingular projective variety of dimension n. The Fulton-MacPherson [31] configuration space $X[k]$ compactifies the moduli of k distinct labelled points on X. The Fulton-MacPherson compactification may be viewed as a higher dimensional analogue of the geometry of marked points on stable curves — when the points attempt to collide, the space X degenerates and the colliding points are separated in a bubble.

The possible degenerations of X which occur are easy to describe. We start with the trivial family

$$\pi : X \times \triangle_0 \to \triangle_0 \,,$$

with fibre X over the disk $\triangle_0$ with base point 0. Next, we allow an iterated sequence of finitely many blow-ups of the total space $X \times \triangle_0$ at points which, at each stage,

(i) lie over $0 \in \triangle_0$ and
(ii) lie in the smooth locus of the morphism to $\triangle_0$.

After the sequence of blow-ups is complete, we take the fibre $\widetilde{X}$ of the resulting total space $\widetilde{X \times \triangle_0}$ over $0 \in \triangle_0$. The space $\widetilde{X}$, a *Fulton-MacPherson degeneration* of X, is a normal crossings divisor in the total space.

The Fulton-MacPherson degeneration $\widetilde{X}$ contains a distinguished component X_+ which is a blow-up of the original X at distinct points. The other components of $\widetilde{X}$ are simply blow-ups of $\mathbb{P}^n$. Of the latter, there are two special types

(i) *ruled* components ($\mathbb{P}^n$ blown-up at 1 point),
(ii) *end* components ($\mathbb{P}^n$ blown-up at 0 points).

The singularities of $\widetilde{X}$ occur only in the intersections of the components.

By construction, there is a canonical morphism

$$\rho : \widetilde{X} \to X$$

which blows-down X_+ and contracts the other components of $\widetilde{X}$. The automorphisms of $\widetilde{X}$ which commute with ρ can only be non-trivial on the components of type (i) and (ii).

For the moduli of stable unramified maps, the target X is allowed to degenerate to any Fulton-MacPherson degeneration $\widetilde{X}$.

§Moduli. Let X be a nonsingular projective variety of dimension n. The moduli space $\mathcal{M}_g(X,\beta)$ of *stable unramified maps* to X parameterises the data

$$C \xrightarrow{f} \widetilde{X} \xrightarrow{\rho} X$$

satisfying the following conditions:

(i) C is a connected nodal curve of arithmetic genus g,
(ii) $\widetilde{X}$ is a Fulton-MacPherson degeneration of X with canonical contraction ρ,
(iii) $\rho_* f_*[C] = \beta \in H_2(X,\mathbb{Z})$,
(iv) the nonsingular locus of $\widetilde{X}$ pulls-back to exactly the nonsingular locus of C,
$$f^{-1}(\widetilde{X}^{ns}) = C^{ns}\,,$$
(v) f is *unramified* on C^{ns},
(vi) at each node $q \in C$, the two incident branches $B_1, B_2 \subset C$ map to two *different* components $Y_1, Y_2 \subset \widetilde{X}$ and meet the intersection divisor at q with equal multiplicities,
$$\Big[B_1 \cdot Y_1 \cap Y_2\Big]_{Y_1,q} = \Big[B_2 \cdot Y_1 \cap Y_2\Big]_{Y_2,q}\,,$$
(vii) for each ruled component $R \subset \widetilde{X}$, there is a component of C which is mapped by f to R with image *not* equal to a fibre of the ruling,
(viii) for each end component $E \subset \widetilde{X}$, there is a component of C which is mapped by f to E with image *not* equal to a straight line.

By (v), the map f is unramified everywhere except possibly at the nodes of C (which must map to the singular locus of $\widetilde{X}$). Constraint (vi) is the standard *admissibility* condition for infinitesimal smoothing which arises in relative Gromov-Witten theory [45, 61, 62]. Conditions (vii) and (viii) serve to stabilize the components of $\widetilde{X}$ with automorphisms over ρ.

The moduli space $\mathcal{M}_g(X, \beta)$ of unramified maps is a proper Deligne-Mumford stack. The unramified map limits of our two simple examples of degenerations (3.1) and (3.3) are easily described. For (3.1), the stable map limit is a double cover which is ramified over two branch points. In the unramified limit, we take the Fulton-MacPherson degeneration which blows up these points in X and adds projective space components. The proper transform of the double cover is then attached to nonsingular plane conics in the two added projective spaces. The conics are tangent to the intersection divisors at the points hit by the double cover. For (3.3), the limit is the same as in Gromov-Witten theory: the normalisation of the image (3.4) in the trivial Fulton-MacPherson degeneration of X.

A central result of [51] is the identification of the deformation/obstruction theory of an unramified map mixing the (unobstructed) deformation theory of Fulton-MacPherson degenerations with the usual deformation theory of maps to X. The deformation/obstruction theory is 2-term, and a virtual class is constructed on $\mathcal{M}_g(X, \beta)$ of dimension

$$\int_\beta c_1(X) + (\dim_{\mathbb{C}} X - 3)(1-g)$$

as in Gromov-Witten theory.

There is no difficulty to include marked points in the definition of unramified maps [51]. Via incidence conditions imposed at the markings, a full set of unramified invariants can be constructed for any X.

§Connections to BPS counts: CY case. How do the unramified invariants relate to all the other counting theories we have discussed? Since unramified invariants have been introduced very recently, not many calculations have been done. In the case of Calabi-Yau 3-folds X, an attempt [91] at finding the analogue of the Aspinwall-Morrison formula for multiple covers of an embedded $\mathbb{P}^1 \subset X$ with normal bundle $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ showed the invariant was different for double covers.

A full transformation relating the unramified theory to the other Calabi-Yau counts has not yet been proposed. Surely such a transformation exists and has an interesting form.

Question: What is the relationship between unramified invariants and Gromov-Witten theory for the Calabi-Yau 3-folds?

§Connections to BPS counts: positive case. Let X be a nonsingular projective 3-fold and let $\beta \in H_2(X, \mathbb{Z})$ be a curve class satisfying

$$\int_\beta c_1(X) > 0\,. \tag{5.1}$$

Let $\gamma_1, \ldots, \gamma_n \in H^*(X, \mathbb{Z})$ be integral cohomology classes Poincaré dual to cycles in X defining incidence conditions for curves. We require the dimension constraint

$$n + \int_\beta c_1(X) = \sum_{i=1}^n \mathrm{codim}_{\mathbb{C}}(\gamma_i)$$

to be satisfied. Let

$$N^{\mathrm{UR}}_{g,\beta}(\gamma_1, \ldots, \gamma_n) \in \mathbb{Q}$$

be the corresponding genus g unramified invariant.

The BPS state counts of Gopakumar and Vafa were generalized from the Calabi-Yau to the positive case in [77, 76]. The BPS invariants $n_{g,\beta}(\gamma_1, \ldots, \gamma_n)$ are *defined* via Gromov-Witten theory by:

$$\sum_{g \geq 0} N^{\mathrm{GW}}_{g,\beta}(\gamma_1, \ldots, \gamma_n)\ u^{2g-2} =$$

$$\sum_{g \geq 0} n_{g,\beta}(\gamma_1, \cdots, \gamma_n)\ u^{2g-2} \left(\frac{\sin(u/2)}{u/2} \right)^{2g-2+\int_\beta c_1(X)}. \tag{5.2}$$

Zinger [108] proved the above definition yields *integers* $n_{g,\beta}(\gamma_1, \ldots, \gamma_n)$ which vanish for sufficiently high g (depending upon β) when the positivity (5.1) is satisfied. The following conjecture[25] connects the unramified theory to BPS counts.

BPS conjecture III: $N^{\mathrm{UR}}_{g,\beta}(\gamma_1, \ldots, \gamma_n) = n_{g,\beta}(\gamma_1, \ldots, \gamma_n)$.

The above simple BPS relation should be true because the moduli space of unramified maps avoids all collapsed contributions. If proved, unramified maps may be viewed as providing a direct construction of the BPS counts in the positive case.

---

[25] BPS conjecture III for unramified invariants was made by R.P. and appears in Section 5.2 of [51].

§Advantages. The main advantage of the unramified theory is the simple form of the singularities of the maps. In particular, avoiding collapsed components leads to (the expectations of) much better behavior than Gromov-Witten theory.

The theory also enjoys many of the advantages of Gromov-Witten theory: definition in all dimensions, relationship to the moduli of curves, and connection with naive enumerative geometry for $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$.

§Drawbacks. The Fulton-MacPherson degenerations add a great deal of complexity to calculations in the unramified theory. Even in modest geometries, a large number of components in the degenerations are necessary. In localization formulas, Hodge integrals on various Hurwitz/admissible cover moduli spaces occur (analogous to the standard Hodge integrals on the moduli space of curves appearing in Gromov-Witten theory). While the latter have been studied for a long time, the structure of the former has not been so carefully understood.

Unramified maps remove the degenerate contributions of Gromov-Witten theory, but keep the multiple covers. For Calabi-Yau 3-folds, the invariants are rational numbers. The BPS invariants are expected to underlie the theory, but how is not yet understood.

The unramified theory is expected to be symplectic, but the details have not been worked out yet.

§Serious difficulties. The theory has been studied for only a short time. Whether the complexity of the degenerating target is too difficult to handle remains to be seen.

$6\frac{1}{2}$. Stable quotients

§Sheaves on curves. We have seen compactifications of the family of curves on X via maps of nodal curves to X and via sheaves on X. The counting theory obtained from the moduli space of stable quotients [67], involving sheaves on nodal curves, takes a hybrid approach. The stable quotients invariants are directly connected to Gromov-Witten theory in many basic cases. However, the main application of stable quotients to date has been to the geometry of the moduli space of curves.

§Moduli. Let $(C, p_1, \ldots, p_n)$ be a connected nodal curve with nonsingular marked points. Let q be a quotient of the rank N trivial bundle C,

$$\mathbb{C}^N \otimes \mathcal{O}_C \xrightarrow{q} Q \to 0\,.$$

If the quotient sheaf Q is locally free at the nodes of C, then q is a *quasi-stable quotient*. Quasi-stability of q implies the associated kernel,

$$0 \to S \to \mathbb{C}^N \otimes \mathcal{O}_C \stackrel{q}{\to} Q \to 0\,,$$

is a locally free sheaf on C. Let r denote the rank of S.

Let C be a curve equipped with a quasi-stable quotient q. The data (C, q) determine a *stable quotient* if the $\mathbb{Q}$-line bundle

$$\omega_C(p_1 + \ldots + p_n) \otimes (\wedge^r S^*)^{\otimes \epsilon} \tag{6.1}$$

is ample on C for every strictly positive $\epsilon \in \mathbb{Q}$. Quotient stability implies $2g - 2 + n \geq 0$.

Viewed in concrete terms, no amount of positivity of S^* can stabilize a genus 0 component

$$\mathbb{P}^1 \stackrel{\sim}{=} P \subset C$$

unless P contains at least 2 nodes or markings. If P contains exactly 2 nodes or markings, then S^* *must* have positive degree.

§Isomorphism. Two quasi-stable quotients on a fixed curve C

$$\mathbb{C}^N \otimes \mathcal{O}_C \stackrel{q}{\to} Q \to 0, \qquad \mathbb{C}^N \otimes \mathcal{O}_C \stackrel{q'}{\to} Q' \to 0 \tag{6.2}$$

are *strongly isomorphic* if the associated kernels

$$S, S' \subset \mathbb{C}^N \otimes \mathcal{O}_C$$

are equal.

An *isomorphism* of quasi-stable quotients

$$\phi : (C, q) \to (C', q)$$

is an isomorphism of curves

$$\phi : C \stackrel{\sim}{\to} C'$$

with respect to which the quotients q and $\phi^*(q')$ are strongly isomorphic. Quasi-stable quotients (6.2) on the same curve C may be isomorphic without being strongly isomorphic.

The moduli space of stable quotients $\overline{\mathcal{Q}}_g(\mathbb{G}(r, N), d)$ parameterising the data

$$(C,\ 0 \to S \to \mathbb{C}^N \otimes \mathcal{O}_C \stackrel{q}{\to} Q \to 0)\,,$$

with *rank*$(S) = r$ and *deg*$(S) = -d$, is a proper Deligne-Mumford stack of finite type over $\mathbb{C}$. A proof, by Quot scheme methods, is given in [67].

Every stable quotient (C, q) yields a rational map from the underlying curve C to the Grassmannian $\mathbb{G}(r, N)$. If the quotient sheaf Q is locally free on all of C, then the stable quotient yields a regular map from C to the Grassmannian. Hence, we may view stable quotients as compactifying the space of maps of genus g curves to Grassmannians of class d times a line.

§Deformation theory. The moduli of stable quotients maps to the Artin stack of pointed domain curves

$$\nu^A : \overline{Q}_g(\mathbb{G}(r,N),d) \to \mathfrak{M}_{g,n} \, .$$

The moduli of stable quotients with fixed underlying curve

$$[C] \in \mathfrak{M}_{g,n}$$

is simply an open set of the Quot scheme of C. The deformation theory of the Quot scheme determines a 2-term obstruction theory on $\overline{Q}_g(\mathbb{G}(r,N),d)$ relative to ν^A given by $(R\,\mathrm{Hom}(S,Q))^\vee$.

More concretely, for the stable quotient,

$$0 \to S \to \mathbb{C}^N \otimes \mathcal{O}_C \xrightarrow{q} Q \to 0 \, ,$$

the deformation and obstruction spaces relative to ν^A are $\mathrm{Hom}(S,Q)$ and $\mathrm{Ext}^1(S,Q)$ respectively. Since S is locally free and C is a curve, the higher obstructions

$$\mathrm{Ext}^k(S,Q) = H^k(C, S^* \otimes Q) = 0, \quad k > 1$$

vanish.

A quick calculation shows the virtual dimension of the moduli of stable quotients equals the virtual dimension of the moduli of stable maps to $\mathbb{G}(r,N)$.

§Invariants. There is no difficultly in adding marked points to the moduli of stable quotients, see [67]. Therefore, we can define a theory of stable quotients invariants for Grassmannians. Similar targets such as flag varieties for $\mathbf{SL}_n$ admit a parallel development. An enumerative theory of stable quotients was sketched in [67] for complete intersections in such spaces. Hence, there is a stable quotients theory for the Calabi-Yau quintic in $\mathbb{P}^4$.

Since [67], the construction of stable quotient invariants has been extended to toric varieties [20] and appropriate GIT quotients [21]. The associated counting theories (well-defined with 2-term deformation/obstruction theories) should be regarded as depending not only on the target space, but *also* on the quotient presentation. The direction is related to the young subject of gauged Gromov-Witten theory (and in particular to the rapidly developing study of theories of Landau-Ginzburg type [17, 24]).

Question: What is the relationship between stable quotient invariants and Gromov-Witten theory for varieties?

For all flag varieties for $\mathbf{SL}_n$, the above question has a simple answer: the counting by stable quotients and Gromov-Witten theory agree exactly [67]. Perhaps exact agreement also holds for Fano toric varieties, see the conjectures in [20]. But in the non-Fano cases, and certainly for the Calabi-Yau quintic, the stable quotient theory is very different. There should be a wall-crossing understanding [101] of the transformations, but much work remains to be done.

§Advantages. Stable quotients provide a more efficient compactification than Gromov-Witten theory. In the case of projective space, there is a blow-down morphism

$$\overline{\mathcal{M}}_g(\mathbb{P}^{N-1}, d) \to \overline{\mathcal{Q}}_g(\mathbb{G}(1, N), d)$$

which pushes-forward the virtual class of the moduli of stable curves to the virtual class of the moduli of stable quotients [67]. A principal use of the moduli of stable quotients has been to explore the tautological rings of the moduli of curves [81] — and in particular to prove the Faber-Zagier conjecture for relations among the κ classes on $\mathcal{M}_g$ [79]. The efficiency of the boundary plays a crucial role in the analysis.

The difference between stable maps and stable quotients can be seen already for elliptic curves in projective space. For stable maps, the associated moduli space is singular with multiple components. A desingularization, by *blowing-up*, is described in [102] and applied to calculate the genus 1 Gromov-Witten invariants of the quintic Calabi-Yau in [109]. On the other hand, the moduli of stable quotients related to such elliptic curves is a nonsingular *blow-down* of the stable maps space [67]. The stable quotients moduli here is a much smaller compactification.[26] A parallel application to the genus 1 stable quotients invariants of the quintic Calabi-Yau is a very natural direction to pursue.

§Drawbacks. The stable quotients approach to the enumeration of curves, while valid for different dimensions, appears to require more structure on X (embedding, toric, or quotient presentations). The method is therefore not as flexible as Gromov-Witten theory.

Also, unlike Gromov-Witten theory, there is not yet a symplectic development. However, the connections with gauged Gromov-Witten theory may soon provide a fully symplectic path to stable quotients.

§Serious difficulties. The theory has been studied for only a short time. The real obstacles, beyond those discussed above, remain to be encountered.

[26] A geometric investigation by Cooper of the stable quotients spaces in genus 1 for projective spaces can be found in [22].

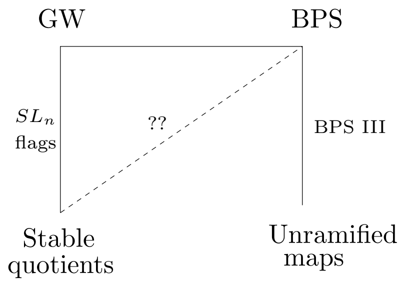

Figure 4. Conjectures relating curve counting theories

Appendix: Virtual classes

§Physical motivation. There are countless ways to compactify the spaces of curves in a projective variety X. What distinguishes the 6 main approaches we have described is the presence in each case of a virtual fundamental class.

Moduli spaces arising in physics should naturally carry virtual classes when cut out by a section (the derivative of an action functional) of a vector bundle over a nonsingular ambient space (the space of fields). While both the space and bundle are usually infinite dimensional, the derivative of the section is often Fredholm, so we can make sense of the difference in the dimensions. The difference is the *virtual dimension* of the moduli space – the number of equations minus the number of unknowns. The question, though, of what geometric objects to place in the boundary is often not so clearly specified in the physical theory.

As an example, the space of C^∞-maps from a Riemann surface C to X, modulo diffeomorphisms of C, is naturally an infinite dimensional orbifold away from the maps with infinite automorphisms. Taking $\overline{\partial}$ of such a map gives a Fredholm section of the infinite rank bundle with fibre $\Gamma(\Omega_C^{0,1}(f^*T_X))$ over the map f. The zeros of the section are the holomorphic maps

$$f\colon C \to X.$$

However, to arrive at the definition of a stable map requires further insights about nodal curves.

The Fredholm property allows us to take slices to reduce locally to the following finite dimensional model of the moduli problem.

§Basic model. Consider a nonsingular ambient variety A of dimension n. Let E be a rank r bundle on A with section $s \in \Gamma(E)$ with zero locus $\mathcal{M}$:

(7.3)
$$\begin{tikzcd} & E \arrow[d, shift right] \\ \mathcal{M} = Z(s) \subset & A \arrow[u, shift right, bend right, "s"'] \end{tikzcd}$$

Certainly, $\mathcal{M}$ has dimension $\geq n - r$. We define

$$\text{vdim}(\mathcal{M}) = n - r$$

to be the *virtual dimension* of $\mathcal{M}$.

The easiest case to understand is when s takes values in a rank r' subbundle

$$E' \subset E$$

and is transverse to the zero section in E'. Then, $\mathcal{M}$ is nonsingular of dimension

$$n - r' = \text{vdim}(M) + (r - r')\,.$$

If E splits as $E = E' \oplus E/E'$, we can write $s = (s', 0)$. We can then perturb s to the section

$$s_\epsilon = (s', \epsilon)$$

with new zero locus given by

$$Z(\epsilon) \subset \mathcal{M}\,.$$

In particular, if ϵ can be chosen to be transverse to the zero section of E/E', we obtain a smooth moduli space $Z(\epsilon)$ of the "correct" dimension $\text{vdim}(\mathcal{M})$ cut out by a transverse section s_ϵ of E. The fundamental class is

$$[Z(\epsilon)] = c_r(E)$$

in the (co)homology of A. If we work in the C^∞ category, we can always split E and pick such a transverse C^∞-section.

Even when E/E' has no algebraic sections (for instance if E/E' is negative), the fundamental class of $Z(\epsilon)$ is clearly $c_{r-r'}(E/E')$ in the (co)homology of $\mathcal{M}$. The "correct" moduli space, obtained when sufficiently generic perturbations of s exist or when we use C^∞ sections, has fundamental class given by the push-forward to A of the top Chern class of E/E'. The result is called the *virtual fundamental class*:

$$[\mathcal{M}]^{vir} = c_{r-r'}(E/E') \in A_{\text{vdim}}(\mathcal{M}) \to H_{2\text{vdim}}(\mathcal{M})\,.$$

Here, E/E', the cokernel of the derivative of the defining equations s, is called the *obstruction bundle* of the moduli space $\mathcal{M}$, for reasons we explain below.

More generally s need not be transverse to the zero section of any subbundle of E, and we must use the excess intersection theory of

Fulton-MacPherson [30]. The limit as $t \to \infty$ of the graph of ts defines a cone

$$C_s \subset E|_{\mathcal{M}} .$$

We define the virtual class to be the refined intersection of C_s with the zero section $0_E \colon \mathcal{M} \hookrightarrow E$ inside the total space of E:

$$[\mathcal{M}]^{vir} = 0_E^![C_s] \in A_{\mathrm{vdim}}(\mathcal{M}) \to H_{2\mathrm{vdim}}(\mathcal{M}) . \tag{7.4}$$

The result can also be expressed in terms of $c(E)s(C_s)$, where c is the total Chern class, and s is the Segre class.

In the easy split case with $s = (s', 0)$ discussed, C_s is precisely E'. We recover the top Chern class $c_{r-r'}(E/E')$ of the obstruction bundle for the virtual class.

§Deformation theory. While the basic model (7.3) for $\mathcal{M}$ rarely exists in practice (except in infinite dimensions), an infinitesimal version can be found when the moduli space admits a 2-term deformation/obstruction theory. The excess intersection formula (7.4) uses data only on $\mathcal{M}$ (rather than a neighbourhood of $\mathcal{M} \subset A$) and can be used in the infinitesimal context.

At a point $p \in \mathcal{M}$, the basic model (7.3) yields the following exact sequence of Zariski tangent spaces

$$0 \to T_p\mathcal{M} \to T_pA \xrightarrow{ds} E_p \to \mathrm{Ob}_p \to 0 . \tag{7.5}$$

So to first order, at the level of the Zariski tangent space, the moduli space looks like $\ker ds$ near $p \in \mathcal{M}$. Higher order neighbourhoods of $p \in \mathcal{M}$ are described by the implicit function theorem by the zeros of the nonlinear map $\pi(s)$, where π is the projection from E_p to Ob_p. The obstruction to prolonging a first order deformation of p inside $\mathcal{M}$ to higher order lies in Ob_p.[27]

The deformation and obstruction spaces, $T_p\mathcal{M}$ and Ob_p, have dimensions differing by the virtual dimension

$$\mathrm{vdim} = \dim A - \mathrm{rank}\, E$$

and are the cohomology of a complex of *vector bundles*[28]

$$B_0 \to B_1$$

[27]The obstruction space is not unique. Analogously, a choice of generators for the ideal of a subscheme $\mathcal{M} \subset A$ is not unique. For instance in our basic model we could have taken the obstruction bundle to be E/E' *or* zero. In each of the six approaches to curve counting, a natural *choice* of an obstruction theory is made.

[28]$B_0 = TA$ and $B_1 = E$ are vector bundles since A is smooth and E is a bundle.

over $\mathcal{M}$ restricted to p. The resolution of $T_p\mathcal{M}$ and Ob_p is the local infinitesimal method to express that $\mathcal{M}$ is cut out of a nonsingular ambient space by a section of a vector bundle.

Li and Tian [65] have developed an approach to handling deformation/obstruction theories over $\mathcal{M}$. If a global resolution $B_0 \to B_1$ exists, Li and Tian construct a cone $C_s \subset E_1$ and intersect with the zero cycle as in (7.4) to define a virtual class on $\mathcal{M}$. Due to base change issues, the technique is difficult to state briefly, but the upshot is that if the deformation and obstruction spaces of a moduli problem have a difference in dimension which is *constant* over $\mathcal{M}$ we can (almost always) expect a virtual cycle of the expected dimension.

§Behrend-Fantechi. We briefly describe a construction of the virtual class proposed by Behrend-Fantechi [6] which is equivalent and also more concise.

Dualising and globalising (7.5), we obtain the exact sequence of sheaves

$$E^*|_{\mathcal{M}} \xrightarrow{ds} \Omega_A|_{\mathcal{M}} \to \Omega_{\mathcal{M}} \to 0\,,$$

where the kernel of the leftmost map contains information about the obstructions. The sequence factors as

$$\begin{tikzcd}
E^*|_{\mathcal{M}} \arrow[r, "ds"] \arrow[d, "s"] & \Omega_A|_{\mathcal{M}} \arrow[d, equal] & \\
I/I^2 \arrow[r, "d"] & \Omega_A|_{\mathcal{M}} \arrow[r] & \Omega_{\mathcal{M}} \to 0\,,
\end{tikzcd}$$

where I is the ideal of $\mathcal{M} \subset A$ and the bottom row is the associated exact sequence of Kähler differentials. We write $E^*|_{\mathcal{M}} \xrightarrow{ds} \Omega_A|_{\mathcal{M}}$ as

$$B^{-1} \to B^0,$$

a 2-term complex of *vector bundles* because A is nonsingular and E is a bundle. The complex

$$\{I/I^2 \to \Omega_A|_{\mathcal{M}}\}$$

is (quasi-isomorphic to) the *truncated cotangent complex* $\mathbb{L}_{\mathcal{M}}$ of $\mathcal{M}$. Our data is what Behrend-Fantechi call a *perfect obstruction theory*: a morphism of complexes

$$B^\bullet \to \mathbb{L}_{\mathcal{M}}$$

which is an isomorphism on h^0 (the identity map $\Omega_{\mathcal{M}} \to \Omega_{\mathcal{M}}$) and a surjection on h^{-1} (because $E^* \to I/I^2$ is onto). The definition can also be interpreted in terms of classical deformation theory [6, Theorem 4.5].

Behrend-Fantechi show how a perfect obstruction theory leads to a cone in $B_1 = (B^{-1})^*$ which can be intersected with the zero section to give a virtual class of dimension

$$\mathrm{vdim} = \mathrm{rank}\, B^0 - \mathrm{rank}\, B^{-1}\,.$$

The virtual class is the usual fundamental class when the moduli space has the correct dimension and is the top Chern class of the obstruction bundle when $\mathcal{M}$ is nonsingular. The virtual class is also deformation invariant in an appropriate sense that would take too long to describe here.

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