ARITHMETICAL COMPACTIFICATION OF MIXED SHIMURA VARIETIES

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For Mohamed Elham Mohamed Sheiry,

who was the first to teach me real mathematics.

Introduction

Let $\mathcal{H}\subset\mathbb{C}$ denote the so-called "complex upper half plane", i.e. the set of all $z\in\mathbb{C}$ with (strictly) positive imaginary part. The map

$$SL_2(\mathbb{Z})\times\mathcal{H}\to\mathcal{H},\ \left(\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right),z\right)\mapsto\frac{az+b}{cz+d}$$

defines a holomorphic left operation of $SL_2(\mathbb{Z})$ on $\mathcal{H}$. For any subgroup of finite index $\Gamma\subset SL_2(\mathbb{Z})$, the quotient $\Gamma\backslash\mathcal{H}$ inherits the structure of a Riemann surface. This Riemann surface is not compact, but by adjoining a finite number of additional points, the so-called <u>cusps</u>, it can be embedded in a compact Riemann surface (see [Sh] ch.1). In particular, there is a smooth projective algebraic curve $\bar{X}$ over $\mathbb{C}$, a non-empty Zariski-open subset $X\subset\bar{X}$, and an isomorphism $X(\mathbb{C})\cong\Gamma\backslash\mathcal{H}$, all of this unique up to isomorphism.

Let us now assume that Γ is a congruence subgroup. This signifies that Γ contains the subgroup

$$\Gamma(N) := \{\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in SL_2(\mathbb{Z})\,|\,a\equiv d\equiv 1,\ b\equiv c\equiv 0 \bmod N\}$$

for some integer N. Then $\Gamma\backslash\mathcal{H}$ is a moduli space for elliptic curves with a certain level structure (which depends on Γ). For instance, if $\Gamma=\Gamma(N)$, then $\Gamma\backslash\mathcal{H}$ is in bijection with the set of all isomorphy classes of pairs (E,x,y), where E is an elliptic curve over $\mathbb{C}$ and (x,y) is a basis of the subgroup $E[N]$ of all N-torsion points of E, such that $e_N(x,y)=e^{2\pi i/N}$. Here e_N denotes the canonical pairing $E[N]\times E[N]\to\mu_N$. The moduli problem associated to Γ can be described in algebraic terms, and it can be shown in a purely algebraic way that there exists a corresponding (coarse or fine) moduli scheme M over a certain explicit number field K. In the case $\Gamma=\Gamma(N)$ we have $K=\mathbb{Q}(e^{2\pi i/N})\subset\mathbb{C}$, while for arbitrary $\Gamma\supset\Gamma(N)$ any subfield of $\mathbb{Q}(e^{2\pi i/N})$ may occur.

This modular interpretation of $\Gamma\backslash\mathcal{H}$ determines an isomorphism $X \cong M \times_K \mathbb{C}$. In other words, it defines a "model" of X over K. Let $\overline{M}$ be the (up to isomorphism) unique non-singular projective curve over K that contains M as an open dense subset. Then the above isomorphism extends to an isomorphism $\overline{X} \cong \overline{M} \times_K \mathbb{C}$, that is, the model of X extends to one of $\overline{X}$. Thus the following problem arises naturally: Describe the structure of this model on, and near, the cusps $\overline{X} \setminus X$.

An answer could have the following form. Let D be the (unique) reduced closed subscheme of $\overline{M}$ with support in $\overline{M} \setminus M$. Via the isomorphism

$$D(\overline{\mathbb{Q}}) = D(\mathbb{C}) \cong \overline{X}(\mathbb{C}) \setminus X(\mathbb{C}) = \overline{X}(\overline{\mathbb{Q}}) \setminus X(\overline{\mathbb{Q}})$$

induced from the above, the model defines an action of $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$ on the cusps. Conversely this action determines the structure of D. More abstractly, it would be desirable to have an explicit scheme D_1 over K and an explicit isomorphism $D \cong D_1$. This latter formulation generalizes to the infinitesimal neighborhoods: one would like to have an explicit formal scheme $\mathfrak{D}_1$ over K, and an explicit isomorphism of $\mathfrak{D}_1$ with the formal completion of $\overline{M}$ along D.

In the case under consideration, such descriptions follow from a modular interpretation of $\overline{X}(\mathbb{C})$. This involves a degenerate version of elliptic curves with N-structure, the so-called generalized elliptic curves. For the formal completion, the Tate-curve can be used (see [DR], and 10.17-22 below).

In these notes we carry out the same program for arbitrary Shimura varieties. To explain the complications arising in the general case, fix a hermitian symmetric domain $\mathcal{H}$ of non-compact type. Also fix a semisimple algebraic group G over $\mathbb{Q}$, and an isomorphism from

$G(\mathbb{R})^o$ to the group of all holomorphic automorphisms of $\mathcal{H}$. Let Γ be an arithmetic subgroup of $G(\mathbb{Q}) \cap G(\mathbb{R})^o$, then $\Gamma\backslash\mathcal{H}$ again carries a natural structure of normal complex space. If Γ is sufficiently small, then $\Gamma\backslash\mathcal{H}$ is already a complex manifold. For simplicity we assume this to be case.

In general, $\Gamma\backslash\mathcal{H}$ is not compact. It possesses different possible compactifications as a normal complex space, with different advantages. Only in the case $G \cong PGL_{2,\mathbb{Q}}$ they happen to coincide. Logically and historically, the Baily-Borel compactification comes first. Its nice properties are that it depends on no extra data, and that it is uniquely characterized as the minimal normal compactification. In any case, it can be defined intrinsically and is a projective variety. It possesses a natural stratification in terms of quotients $\Gamma'\backslash\mathcal{H}'$ for certain other hermitian symmetric domains $\mathcal{H}'$ (the so-called boundary components), and arithmetic subgroups Γ' of $\mathrm{Aut}(\mathcal{H}')$ (see [BB], [AMRT], and 6.2-3 below).

Unfortunately, the Baily-Borel compactification may have serious singularities along the boundary. A nice resolution of these singularities is the toroidal compactification. It is defined locally as a torus embedding. The cone decompositions involved in this definition are not canonical, so there are different toroidal compactifications associated to different systems of cone decompositions. However, for suitable choices the resulting complex space is projective and smooth, and the boundary is the union of smooth divisors with at most normal crossings. The toroidal compactification dominates the Baily-Borel compactification, and possesses a natural stratification compatible with that of the Baily-Borel compactification. But the strata now have a more general form: they are torus-torsors over families of abelian varieties over the strata of the Baily-Borel compactification.

These compactifications carry a natural algebraic structure, and induce an algebraic structure on $\Gamma\backslash\mathcal{H}$. The next step is to ask for a model of these algebraic varieties over a number field. Here one gets stronger results if one considers not only $\Gamma\backslash\mathcal{H}$, but a certain finite disjoint union of such quotients. For this, G is replaced by a central extension with a torus, $\mathcal{H}$ by a finite disjoint union of hermitian symmetric domains with a transitive holomorphic action of $G(\mathbb{R})$, and instead of $\Gamma\backslash\mathcal{H}$ one considers a double quotient $G(\mathbb{Q})\backslash\mathcal{H}\times(G(\mathbb{A}_f)/K_f)$. Here $\mathbb{A}_f$ denotes the finite adeles of $\mathbb{Q}$, K_f is an open compact subgroup of $G(\mathbb{A}_f)$, and $G(\mathbb{Q})$ acts on both factors. This double quotient is indeed of the desired form, and constitutes the set of complex points of a <u>Shimura variety</u>. Let us denote the associated algebraic variety over $\mathbb{C}$ by $M_{\mathbb{C}}$, the Baily-Borel compactification by $\overline{M}_{\mathbb{C}}$, and the toroidal compactification associated to a system $\mathcal{S}$ of cone decompositions by $M_{\mathcal{S},\mathbb{C}}$.

The advantage of this adelic definition is that one gets a model over a number field that is independent of K_f, called the <u>reflex field</u>. It is determined by G, $\mathcal{H}$, and a little extra data concerning the center of G. Some Shimura varieties have a natural interpretation as moduli spaces of abelian varieties with some extra structure. For instance, generalizing the special case considered above, the group of symplectic similitudes $CSp_{2g,\mathbb{Q}}$ gives rise to the Siegel modular variety parametrizing principally polarized abelian varieties of dimension g with level structure. Such a modular interpretation induces a model M of $M_{\mathbb{C}}$ over the reflex field. However, not every Shimura variety has such a purely algebraic modular interpretation. Nevertheless, following Shimura the models defined by modular interpretation can be uniquely characterized in an intrinsic way (see [D1]). Taking this characterization as a definition, one arrives at the so-called <u>canonical model</u> of an arbitrary Shimura variety. Its uniqueness is not too hard to prove ([D1]), but its existence in the most general case has been established only recently (see [Mi1]).

The first of the questions we have to examine is: *Under what condition does the canonical model extend to a model of $\overline{M}_{\mathbb{C}}$ or of $M_{\mathcal{S},\mathbb{C}}$?* For $\overline{M}_{\mathbb{C}}$ it always extends uniquely to a model $\overline{M}$, since the Baily-Borel compactification can be defined intrinsically. For the toroidal compactification the answer depends on $\mathcal{S}$, and is linked with the other questions. In any case, there is up to isomorphism at most one extension $M_{\mathcal{S}}$, so we do not get into trouble if we directly turn to the next question: *Which model is induced on the boundary?*

In the case of the Baily-Borel compactification, the stratification of the boundary suggests a natural guess: the boundary of the extended canonical model $\overline{M}$ should possess a stratification in terms of canonical models of other Shimura varieties. This indeed turns out to be the case (see our *first main theorem 12.3*). The complete answer includes an explicit description of the Shimura varieties that occur at the boundary (see 6.3).

For the toroidal compactification one has to work much harder to get the analogous results. The chief reason for this is that, although the boundary strata are fibred over (other) Shimura varieties, they are more general objects. Thus, in order to be able to describe the boundary within the same framework, one has to generalize the concept of Shimura variety.

A close look at the group theoretical data involved in the definition of the toroidal compactification (see [AMRT] ch.III) suggests that the desired generalization involves an extension of a reductive group (associated to a Shimura variety) by a unipotent group of a certain type. Such groups arise naturally as normal subgroups of maximal parabolic subgroups of our original group G. Along another line of thought, note that every Shimura variety, even if it does not possess a (known) algebraic modular interpretation, at least parametrizes, as a complex manifold, suitable variations of polarized pure Hodge structures or combinations thereof. If the group is not reductive, these "combinations" can no longer be direct sums, but should be mixed Hodge structures! These considerations naturally lead to a concept of mixed Shimura varieties that parametrize variations of mixed Hodge structures, all of whose pure constituents are polarizable.

In the chapters 1-3 we develop a theory of mixed Shimura varieties. We consider mixed Shimura varieties in their own right, not only insofar as they occur in the boundary of other Shimura varieties. Nevertheless the desired properties (variation of mixed Hodge structures, i.e. Griffiths' transversality, and polarizability of all pure constituents) imply that they also are torus-torsors over families of abelian varieties over usual (henceforth called pure) Shimura varieties (see 3.12ff). They have a rich structure; in particular the group operations, torus actions, etc. on the fibres can all be described in terms of abstractly defined morphisms of Shimura varieties. Moreover, for pure Shimura varieties that possess an interpretation as moduli schemes of polarized abelian varieties, the universal families (of abelian varieties, and of polarizing line bundles) themselves can be described as mixed Shimura varieties (see 10.7). This phenomenon can be viewed as a modular interpretation for these mixed Shimura varieties. A mixed Shimura variety is pure when the associated algebraic group is reductive. The treatment of chapter 1 follows [D2] §1.

The next problem we have to deal with is how to make mixed Shimura varieties algebraic. Like for pure Shimura varieties, this problem can be solved by suitable projective compactifications. Such compactifications are also interesting in connection with the modular interpretation: one can describe degenerations of abelian varieties in terms of compactifications of mixed Shimura varieties. As, for instance,

[M4] suggests, these compactifications should be toroidal. In the special case of elliptic curves we indeed give such a description (see 10.17-22).

All in all, we see that toroidal compactifications should be constructed not only for pure, but for arbitrary mixed Shimura varieties. This is done in the chapters 4 and 6, which rely heavily on [AMRT]. Chapter 5 contains an assortment of results about torus embeddings, which will be needed throughout the remaining chapters. This construction is not more complicated than for pure Shimura varieties; on the contrary the general setting accounts for a certain coherence. The essential new ingredient is an explicit way of associating certain mixed Shimura varieties to a given (possibly pure) one. These are closely related with what in the usual theory is called rational boundary components, which is why we give them the same name. The rational boundary components are in general "more mixed" than the original mixed Shimura variety. The toroidal compactification is then constructed as in [AMRT]. First one considers for every rational boundary component a torus embedding, with respect to a certain cone decomposition and the torus that acts naturally on the unipotent fibre. If the cone decompositions have been chosen compatibly, these partial compactifications can be glued together to form the desired toroidal compactification.

We carry out this construction with an eye toward the question for the model of the boundary. Since we eventually want to describe a stratification of the boundary in terms of other mixed Shimura varieties, in particular in adelic language, it is desirable that the construction already reflects this. Unfortunately this adds (mainly notational) complexity to the constructions of the chapters 6-9 and 12. The benefit is that we directly get a stratification of the toroidal compactification in terms of the mixed Shimura varieties occurring as rational boundary components, and quotients thereof. This stratification is studied in chapter 7.

In order to algebraize mixed Shimura varieties and their compactifications, we have to find compactifications that are projective. In other words we have to construct ample line bundles. While for pure Shimura varieties (in [AMRT] ch.IV §2) such ample line bundles can be described in terms of "piecewise linear strictly convex rational functions" on the cone decompositions, the presence of abelian varieties in the fibres of mixed Shimura varieties necessitates a more invariant construction (see 8.15, 9.39). As it turns out, one can use the torus-torsor structure on mixed Shimura varieties to turn certain toroidal compactifications of mixed Shimura varieties into line bundles. (In the case of a modular interpretation this has been mentioned above.) In other words, certain morphisms $M'_{\mathfrak{S}',\mathbb{C}} \to M_{\mathfrak{S},\mathbb{C}}$ of toroidal compactifications of mixed Shimura varieties "are" line bundles in a canonical way (see 8.6).

In chapter 8 we derive a criterion for such a line bundle to be ample (along the unipotent fibres). What remains to achieve the algebraization, is the construction of cone decompositions satisfying this criterion. This is done in chapter 9. We prove that for every mixed Shimura variety $M_{\mathbb{C}}$, there exists a mixed Shimura variety $M'_{\mathbb{C}}$ and cone decompositions $\mathfrak{S}$, $\mathfrak{S}'$, so that $M_{\mathfrak{S},\mathbb{C}}$ is compact, $M'_{\mathfrak{S}',\mathbb{C}} \to M_{\mathfrak{S},\mathbb{C}}$ is a line bundle, and some combination with the canonical sheaf is ample. Moreover, $\mathfrak{S}$ can be chosen such that $M_{\mathfrak{S},\mathbb{C}}$ is smooth, the complement $M_{\mathfrak{S},\mathbb{C}} \smallsetminus M_{\mathbb{C}}$ is a union of smooth divisors with at most normal crossings, and such $\mathfrak{S}$ can be made arbitrarily fine. In particular, both $M_{\mathbb{C}}$ and such $M_{\mathfrak{S},\mathbb{C}}$ are algebraic varieties. We also prove that "most" other toroidal compactifications $M_{\mathfrak{S},\mathbb{C}}$ are algebraic varieties, and that all of them are algebraic spaces in the sense of [K].

Following the same program as above, we next have to define canonical models for mixed Shimura varieties. This is done in the chapters 10-11. The definition and the proof of uniqueness are exactly as for pure Shimura varieties. The modular interpretation yields a canonical model for the mixed Shimura varieties associated to the moduli problem of polarized abelian varieties with level structure. Here, as in [D1], the main theorem of Shimura and Taniyama about abelian varieties with complex multiplication plays the essential role. Using this, along the lines of [D1], the existence of canonical models in general is reduced to the case of pure Shimura varieties, where it is known by [Mi1].

Having developed the theory of mixed Shimura varieties thus far, we are now in the position to formulate and to prove the main result for the toroidal compactification of a mixed Shimura variety. Recall that chapter 7 gives a stratification in terms of other mixed Shimura varieties. By the results of chapter 9, this stratification is in fact algebraic. Just as for the Baily-Borel compactification, the main theorem (12.4) asserts that (if the cone decompositions satisfies a certain arithmeticity condition, see 6.4) the canonical model extends to the toroidal compactification, and that this stratification descends to the reflex field.

Since the neighborhoods of a stratum in the toroidal compactification are so nice, without much extra effort we get an even better assertion. By 7.17 some neighborhood of a stratum is essentially isomorphic to a neighborhood of a stratum in a simple torus embedding of a rational boundary component. By 9.37, this description can be algebraized, yielding an isomorphism between the respective formal completions (over $\mathbb{C}$). The last part of our main theorem (12.4 (c)) asserts that this isomorphism of formal schemes also descends to the reflex field. These results constitute a complete answer to the questions put forward above.

The main theorem is proved in chapter 12, in the following way. First, using the moduli scheme of generalized elliptic curves (see 10.20), we prove it in the special case of elliptic modular curves (associated to the group $GL_{2,\mathbb{Q}}$). (In particular, as mentioned above, the universal family of generalized elliptic curves "is" the canonical model of a certain toroidal compactification of a mixed Shimura variety.) We then embed twists of such modular curves into an arbitrary mixed Shimura variety (like Hirzebruch-Zagier cycles in Hilbert modular surfaces). Finally a density argument, much as in the intrinsic characterization of canonical models, but this time for the formal neighborhood of a boundary component, extends the result to the general case.

The results of these notes have several applications. Quite immediate are the consequences for q-expansions of (certain) automorphic forms. Consider, for a given mixed Shimura variety M, a space of sections of a vector bundle F constructed out of sheaves of differential forms, possibly with logarithmic poles at infinity, or with some other boundary condition. Our description of the formal neighborhood $\mathfrak{M}$ of a boundary stratum M_1 implies a similar description for the pullback $\mathfrak{F}$ of F. Moreover, since $\mathfrak{M}$ has an explicit description in terms of a torus embedding, there is a canonical decomposition of $\Gamma(\mathfrak{M},\mathfrak{F})$ as a product of $\Gamma(M_1,F_\nu)$ for certain explicit sheaves F_ν on M_1. The map

$$\Gamma(M_{\mathfrak{F}},F) \rightarrow \Gamma(\mathfrak{M},\mathfrak{F}) \rightarrow \prod_\nu \Gamma(M_1,F_\nu)$$

now associates to a section of F its "q-expansion coefficients." These "coefficients" are an analog of Jacobi modular forms. Our results immediately imply a q-expansion principle for rationality of such automorphic forms over the reflex field: A section over $\mathbb{C}$ descends to the reflex field if and only if its q-expansion coefficients do so at a suitable set of boundary components (see 12.18–20). In a similar form, for the Baily-Borel compactification, this result has been obtained by Harris [Ha1], [Ha2] for arbitrary vector-valued automorphic forms.

In another direction these results have consequences for the <u>ℓ-adic cohomology</u> of (mixed) Shimura varieties. The explicit description of formal neighborhoods of the boundary should make it possible to describe the pure constituents of, say, $H^*_{\text{ét}}(M_{\bar{\mathbb{Q}}},\mathbb{Q}_\ell)$ in terms of some pure cohomology (e.g. intersection cohomology, cuspidal cohomology, or the like) of its boundary components (including itself) with values in some other explicit ℓ-adic sheaves. The author plans to take this up in a sequel to these notes. This would of course be a prerequisite for studying the mixed motives that occur in $H^*_{\text{ét}}(M_{\bar{\mathbb{Q}}},\mathbb{Q}_\ell)$.

Two more technical remarks are in order. In the explanations above we have tacitly used the following slight generalization of the definition of a pure Shimura variety. In the usual definition, $\mathcal{H}$ is a $G(\mathbb{R})$-conjugacy class of certain homomorphisms $\mathbb{C}^\times \to G(\mathbb{R})$. The generalization consists in allowing a finite $G(\mathbb{R})$-equivariant covering $\mathfrak{X} \to \mathcal{H}$, with $G(\mathbb{R})$ acting transitively on $\mathfrak{X}$. This generalization makes a difference for the field of definition of a connected component of a Shimura variety, and is necessary to stratify a normal compactification in the same framework. From the point of view of global abelian class field theory, it is in fact the more natural choice.

The other remark concerns questions of sign. In order to avoid a source of errors, we have found it useful to work without any globally fixed choice of a square root of -1. In fact, complex analysis by no means depends on such a choice! We hope to have been successful in this.

As explained above, our results rely on the modular interpretation of certain (mixed) Shimura varieties. This modular approach, of course, achieves much more: it can be carried out not only over a number field, but over suitable rings of finite type over $\mathbb{Z}$. This has been done in several instances: Deligne and Rapoport [DR] for elliptic modular curves, Rapoport [Rap] for Hilbert modular varieties, and Chai [C] and Faltings [F2] for the Siegel modular variety. A modular interpretation of certain mixed Shimura varieties (in terms of "1-motifs") has been described in Brylinski's thesis [Br]. In these cases, a part of the results of this thesis have been known.

The toroidal compactification of pure Shimura varieties has been treated over $\mathbb{C}$ in [AMRT] and [N]. That it descends to $\overline{\mathbb{Q}}$ has been proved in [F1], using a rigidity argument. After the present author had obtained his results, the following two manuscripts came to his attention: (a) a survey article by Milne [Mi2] that independently suggests the concept of mixed Shimura variety along a similar line of ideas; and (b) a manuscript by Harris [Ha3] containing similar results for pure Shimura varieties.

Finally, the author has the pleasure to thank, most of all, Professor G. Harder for suggesting the present research, for providing a creative atmosphere, and for his constant support. He is obliged to J. S. Milne for terminological and other suggestions, and to P. Deligne, Th. Höfer, N. M. Katz, and M. Rapoport for useful conversations. He is indebted to the DAAD and the Universität Bonn for financial support. Last, but not least, he is grateful to S. Fette for friendly providing him with countless cups of tea.

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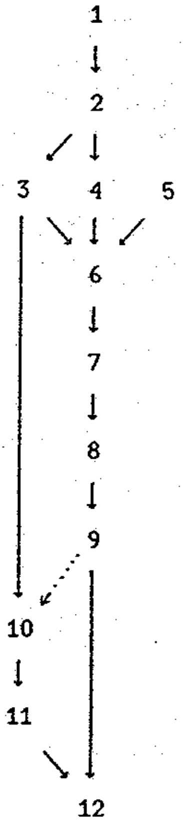
Leitfaden



<u>solid arrow</u>: logical dependence

<u>dotted arrow</u>: logical dependence, but not strictly necessary for understanding

"Just follow that line forever," said the Mathemagician, "and when you reach the end, turn left. There you'll find the land of Infinity, where the tallest, the shortest, the smallest, and the most and the least of everything are kept." ... Up he went - very quickly at first - then more slowly - then in a little while even more slowly than that - and finally, after many minutes of climbing up the endless stairway, one weary foot was barely able to follow the other. Milo suddenly realized that with all his efforts he was no closer to the top than when he began, and not a great deal further from the bottom.

Norton Juster, The phantom tollbooth

L'arithmétique ça fatigue, ça énerve.

Eugène Ionesco, La leçon

0. Notations and conventions

0.1. General: The set of connected components of a topological space X is denoted by $\pi_0(X)$. For any two sets X, Y, the set-theoretic difference is denoted by $X \setminus Y$. This is not to be confused with elementwise difference or with left quotient: If $(A,+)$ is an abelian group, $a \in A$ and $X, Y \subset A$, then the elementwise sum or difference is denoted by $X+Y$, $X-Y$, $a-X$, etc. Likewise for an arbitrary group $(G,\cdot)$.

0.2. Fields and Schemes: If $T \to S$ is a morphism of schemes, and X a scheme over S, then we write $X_T := X \times_S T$. If $S = \mathrm{Spec}(K)$ and $T = \mathrm{Spec}(L)$ for a field extension L/K, then we write this also in the form $X_L = X \times_K L$. If Y_L is a subscheme of X_L which is derived from a subscheme of X by base extension, then by abuse of terminology we say that it is defined over K. An "L-subobject" (e.g. "L-subgroup", "L-parabolic subgroup", "L-simple subgroup") of an object X is a subscheme of X_L that possesses the indicated property.

Let L/K be a field extension and X is a scheme over L. By a model of X over L we mean a scheme X_o over K together with an isomorphism $X_{o,L} \cong X$ over L. The Weil restriction of X with respect to L/K is denoted by $\mathcal{R}_{L/K} X$. If X is a group scheme (general, reductive, semisimple, or a torus), then so is $\mathcal{R}_{L/K} X$.

$\mathbb{R}$ is the field of real numbers, $\mathbb{C}$ a fixed algebraic closure of $\mathbb{R}$. The symbol $\mathcal{A}$ denotes an arbitrary solution in $\mathbb{C}$ of the equation $z^2+1=0$. We do not fix a particular choice of $\mathcal{A}$. Wherever this symbol occurs, it will be clear that the whole assertion is independent of such a choice. In the expression $2\pi\mathcal{A}$, the symbol π denotes the unique positive real number for which $2\pi\mathcal{A}$ is a generator of kernel of the exponential function $\exp: \mathbb{C} \to \mathbb{C}^\times$ (Clearly this does not depend on $\mathcal{A}$).

If X is a variety over a field K that is contained in $\mathbb{R}$, then we call X of compact type if and only if $X(\mathbb{R})$ is compact.

All number fields K are considered as subfields of $\mathbb{C}$, in other words they are given together with a distinguished embedding into $\mathbb{C}$. We denote by $\overline{K}$ the algebraic closure of K in $\mathbb{C}$, so that the absolute Galois group $\mathrm{Gal}(\overline{K}/K)$ is a quotient of $\mathrm{Aut}(\mathbb{C})$. The ring of adeles of K is denoted by $\mathbb{A}_K$, the subring of finite adeles by $\mathbb{A}_{K,f}$, and we abbreviate $\mathbb{A}:=\mathbb{A}_{\mathbb{Q}}$ and $\mathbb{A}_f:=\mathbb{A}_{\mathbb{Q},f}$.

0.3. Groups: Let G be a group and $g,g'\in G$. The left action of G by conjugation on itself is denoted by $\mathrm{int}_G(g)\colon g'\mapsto g\cdot g'\cdot g^{-1}$. The commutator is $[g,g'] := g\cdot g'\cdot g^{-1}\cdot g'^{-1}$. An almost direct (almost semidirect) product in a category of groups is any finite quotient of the usual direct (semidirect) product.

If A is a topological or algebraic group, then A^o is the connected component of the identity of A. The center of A is denoted by $Z(A)$, this is a closed, but not necessarily connected subgroup. If A is a topological group, then $\pi_o(A)$ is isomorphic to A/A^o.

Let G be a linear algebraic group over a field K. By a representation of G we always mean a finite dimensional representation over K. If $L\subset \mathrm{Lie}\, G$ is a K-sub Liealgebra, then $\exp L$ denotes the unique connected K-subgroup of G with Lie algebra L.

By a reductive group over a field K we always mean a connected reductive linear algebraic group over K. In particular, every semisimple group and every torus will be connected. If G is a reductive group, then $G^{der}=[G,G]$ and $G^{ad}=G/Z(G)$ denote its derived, resp. adjoint group. The adjoint representation of G on Lie G will be denoted by Ad_G, its derivative (the induced left action of Lie G on itself) by ad_G.

Let P be a connected linear algebraic group over K, and U its unipotent radical. A Levi-decomposition is a decomposition of P into a semidirect product $P=U\rtimes G$. Any two Levi-decompositions are conjugate under $U(K)$. Also, let H be a reductive group over K, and $\varphi_1, \varphi_2: H\to P$ two homomorphisms so that the composites $H\to P\rightrightarrows P/U$ are equal. Then φ_1 and φ_2 are conjugate under $U(K)$.

0.4. Group actions: If a group G acts from the left hand side on a set X, we denote by $G\backslash X$ the set of all G-orbits in X. Likewise for X/H if a group H acts from the right hand side, and consequently $G\backslash X/H$ if two such actions are given. Fortunately we shall not need actions from above or below. A left homogeneous space under G is a set X with a transitive left action of G. If G is a real Lie-group, then such X is a C^∞-manifold in a canonical way.

Let Γ be a discrete group acting on a locally compact (hence Hausdorff) topological space X. By definition Γ acts properly discontinuously on X, if and only if for any two compact subsets K and K' of X the set $\{\gamma\in\Gamma \mid \gamma K\cap K'\neq\emptyset\}$ is finite. An equivalent condition is that any two points in X possess neighborhoods U and V such that the set $\{\gamma\in\Gamma \mid \gamma U\cap V\neq\emptyset\}$ is finite. If this condition holds, then $\Gamma\backslash X$, endowed with the quotient topology, is again Hausdorff (even locally compact) and locally isomorphic to a quotient of X by a finite group. In particular the stabilizer of any point is finite.

Contrary to the usual terminology we also say that a group Γ acts properly discontinuously if the action factors through a quotient $\overline{\Gamma}$ which acts properly discontinuously in the usual sense. The point is that the kernel of the action is allowed to be infinite.

0.5. Arithmetic subgroups: Let G be a linear algebraic group over $\mathbb{Q}$. A subgroup of $G(\mathbb{Q})$ is called a congruence subgroup if and only if it is of the form $G(\mathbb{Q})\cap K_f$ for some open compact subgroup $K_f \subset G(\mathbb{A}_f)$. A subgroup of $G(\mathbb{Q})$ is called an arithmetic subgroup if and only if it is commensurable with some (hence with every) congruence subgroup.

According to [B] 17.1 an element $g \in GL_n(\mathbb{Q})$ is called neat, if the subgroup of $\bar{\mathbb{Q}}^\times$ that is generated by the eigenvalues of g is torsion free. If G is a linear algebraic group, then an element $g \in G(\mathbb{Q})$ is called neat if its image in some faithful representation of G is neat. It is easy to check that this condition then holds for every, not necessarily faithful, representation of G (see [B] 17.3). A subgroup of $G(\mathbb{Q})$ is called neat if all its elements are neat. In particular such a group is torsion free. Every subgroup of a neat subgroup is neat, and the image of a neat subgroup under a homomorphism $G \to H$ is again neat. Finally every sufficiently small congruence subgroup of $G(\mathbb{Q})$ is neat (by [B] 17.4, or 0.6 below).

0.6. Neatness and adelic groups: We want to extend the notion of neatness to subgroups $K_f \subset G(\mathbb{A}_f)$, with the same functorial properties, and so that $G(\mathbb{Q})\cap K_f$ is neat whenever K_f is neat. First consider an element $g_f=(g_p)_p \in GL_n(\mathbb{A}_f)$. For every p let Γ_p be the subgroup of $\bar{\mathbb{Q}}_p^\times$ generated by all eigenvalues of g_p. Let $\bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$ be some embedding, and consider the torsion part $(\bar{\mathbb{Q}}^\times \cap \Gamma_p)_{tors}$. Since every subgroup of $\bar{\mathbb{Q}}^\times$ consisting of roots of unity is normalized by $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, this group does not depend on the choice of the embedding $\bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$. We now define g_f to be neat if and only if

$$\bigcap_p (\bar{\mathbb{Q}}^\times \cap \Gamma_p)_{tors} = \{1\}.$$

As before, if G is a linear algebraic group, we call an element $g_f \in G(\mathbb{A}_f)$ neat if its image in some faithful representation of G is neat. If ρ is a