a direct product since W_1 acts trivially on U_0. Since all splittings of P' are conjugate under U_0, this splitting is canonical! Thus we have a canonical decomposition $U_1' = U_0 \times U_1$, and $B(\mathfrak{X}'^{o}, P_1')$ is contained in the second factor. In analogy to 5.20 we can now describe $\mathcal{S}'(\mathfrak{X}'^{o}, P_1', p_f')$ in terms of a function $\varphi: C^*(\mathfrak{X}^{o}, P_1) \to \mathbb{R}$, where the splitting associated to any $\sigma \in \mathcal{S}(\mathfrak{X}^{o}, P_1, p_f)$ is, with respect to the canonical decomposition of U_1', equal to $(-\lambda^{-1} \circ \varphi, \mathrm{id})$ on σ.

It is now easy to see that the condition (a) of 8.13 is equivalent to the strict convexity of φ in the sense of condition (2) of [AMRT] ch.IV §2.1 p.310. It can be shown that the condition (1) of [loc. cit.] is equivalent to the assertion that for all σ of top dimension the linear form $\lambda \circ (\mathrm{id} - e_\sigma \circ \pi_o)$ is strictly positive on $\overline{C(\mathfrak{X}^{o}, P_1)} \setminus \{0\}$. By 8.12 it thus implies our condition (b). In general we do not have equivalence, but this is only a very minor difference.

(b) More generally one can describe the line bundle of 8.13 in terms of a function φ as in [AMRT] ch.IV §2 if and only if the extension $1 \to U_0 \to P' \to P \to 1$ splits. In fact, we have described the line bundle locally as an equivariant torus embedding for a short exact sequence of tori $1 \to \mathbb{G}_m \to T' \to T \to 1$, and a global splitting of this extension amounts to a splitting $P' \cong U_0 \rtimes P$. But 3.21 shows that whenever the abelian variety part of the unipotent fibre is nontrivial, then one can get ample line bundles only with extensions that do not split. Thus we needed a more sophisticated description of relatively ample line bundles than in [AMRT]. A geometric illustration of this will be given in 9.39 (b).

9. Algebraization of the toroidal compactification

To finish the algebraization of $M^{K_f}(P,\mathfrak{X},\mathcal{S})(\mathbb{C})$ we construct cone decompositions that satisfy the requirements of the preceding chapter. This construction is an analog of [AMRT] ch.II §§5.3-4. Under a mild condition on $(P,\mathfrak{X})$, we begin (9.1-5) with the study of a certain set of splittings that satisfy the condition 8.13 (b). Our cone decompositions will be described in terms of "locally polyhedral" subsets $\Sigma \subset C^*(\mathfrak{X}'^{\circ},P_1')$, defined in 9.8. Their properties (9.9-11) lead to cone decompositions that satisfy the desired conditions (9.12-13), and it remains to construct compatible systems of such subsets Σ. These constructions occupy most of the rest of this chapter.

First we study the behavior of Σ with respect to different boundary components (9.14-15). This leads to our first existence theorem 9.16: asserting that some such systems of Σ's exists. Next we show how to obtain refinements with specific properties (9.18-20), in particular concerning smoothness and the extendability of the maps 3.4. In 9.21-23 we extend this to arbitrary $(P,\mathfrak{X})$. These results show that $M^{K_f}(P,\mathfrak{X})(\mathbb{C})$, and certain $M^{K_f}(P,\mathfrak{X},\mathcal{S})(\mathbb{C})$, possess canonical algebraic structures (9.24-26). In 9.28-33, we extend the existence of an algebraic structure to all $M^{K_f}(P,\mathfrak{X},\mathcal{S})(\mathbb{C})$ for which, loosely speaking, $\mathcal{S}$ is sufficiently fine or K_f sufficiently small. This implies that there always exists a canonical structure of algebraic space over $\mathbb{C}$; which, however, is not always a scheme (9.34-35).

Having constructed an algebraic structure on most $M^{K_f}(P,\mathfrak{X},\mathcal{S})(\mathbb{C})$, it follows from the construction that the maps 6.25 are algebraic, and most properties carry over to the algebraic side (9.25). In 9.36-38 we prove the same for the stratifications described in chapter 7, in particular that 7.17 (b) gives an isomorphism of the formal completions. The chapter concludes with an example (9.39).

<u>9.1. Proposition</u>: Let $(P,\mathfrak{X})$ be mixed Shimura data, and $(P,\mathfrak{X}) \to (\mathbb{G}_{m,\mathbb{Q}},\mathcal{H}_0)$ a morphism. As in 8.7 every connected component $\mathfrak{X}'^{o}$ determines an isomorphism $\lambda: U_0(\mathbb{R})(-1) \to U_0(\mathbb{R}) = \mathbb{R}$ and a convex rational polyhedral cone $\sigma_o := \lambda^{-1}(\mathbb{R}^{\geq 0}) \subset U_0(\mathbb{R})(-1)$.

(a) Suppose that $(P,\mathfrak{X})$ is irreducible. Then there exists a $((P_0,\mathfrak{X}_0) \to (\mathbb{G}_{m,\mathbb{Q}},\mathcal{H}_0))$-torsor $(P',\mathfrak{X}') \to (P,\mathfrak{X})$, and a splitting $\mu: U' \to U_0$, such that if $\Psi: V \times V \to U'$ is the pairing induced by the commutator in W', then $\lambda \circ \mu \circ \Psi$ defines a polarization of Hodge structures on V_{2g} for every point in $\mathfrak{X}'^{o}$.

(b) Let $(P',\mathfrak{X}') \to (P,\mathfrak{X})$ be as in (a). Then there exists pure Shimura data $(\tilde{P}_*,\tilde{\mathfrak{X}}_*)$, a rational boundary component $(\tilde{P},\tilde{\mathfrak{X}})$ of $(\tilde{P}_*,\tilde{\mathfrak{X}}_*)$, and an embedding $(P',\mathfrak{X}') \hookrightarrow (\tilde{P},\tilde{\mathfrak{X}})$, subject to the following condition. For every connected component $\mathfrak{X}'^{o}$ of $\mathfrak{X}'$ let $\tilde{\mathfrak{X}}^{o}$ and $\tilde{\mathfrak{X}}^{o}_*$ be the corresponding connected components of $\tilde{\mathfrak{X}}$, resp. of $\tilde{\mathfrak{X}}_*$. Then the image of σ_o in $\tilde{U}(\mathbb{R})(-1)$ is contained in $\overline{C(\tilde{\mathfrak{X}}^{o}_*,\tilde{P})}$.

<u>Proof</u>: (a) If $V=1$, let $(P',\mathfrak{X}') := (P_0,\mathfrak{X}_0) \times_{(\mathbb{G}_{m,\mathbb{Q}},\mathcal{H}_0)} (P,\mathfrak{X})$, then any splitting is okay. Otherwise it suffices to take $(P',\mathfrak{X}')$ and μ be as in the proof of 2.26 (b).

(b) As in the proof of 2.26 (b) the set of all μ that satisfy (a) above is an open subset of the set of all splittings. This also holds if $V=1$, since then all splittings are okay. Let $2g := \dim(V)$, then every such splitting defines a morphism $(P',\mathfrak{X}') \to (P_{2g},\mathfrak{X}_{2g})$. As in 2.26 any set of splittings in the above open set, that generates the $\mathbb{Q}$-vector space $\mathrm{Hom}(U',U_0)$, determines an embedding $(P',\mathfrak{X}') \hookrightarrow (P,\mathfrak{X})/W \times (P_{2g},\mathfrak{X}_{2g})^r$ for some r. By 4.25 $(P_{2g},\mathfrak{X}_{2g})$ can be interpreted as a rational boundary component of $(CSp_{2(g+1),\mathbb{Q}},\mathcal{H}_{2(g+1)})$. So let $(\tilde{P}_*,\tilde{\mathfrak{X}}_*) := (P,\mathfrak{X})/W \times (CSp_{2(g+1),\mathbb{Q}},\mathcal{H}_{2(g+1)})^r$, and let $(\tilde{P},\tilde{\mathfrak{X}})$ be the (unique) irreducible component of $(P,\mathfrak{X})/W \times (P_{2g},\mathfrak{X}_{2g})^r$ that contains the image of $(P',\mathfrak{X}')$. We are done if we can choose the splittings such that σ_o maps to $\overline{C(\mathcal{H}^o_{2(g+1)},P_{2g})}$ under each of the morphisms $(P',\mathfrak{X}')\to(P_{2g},\mathfrak{X}_{2g})$.

If $V=1$, then this is easy to achieve, since $\overline{C(\mathcal{H}^o_{2(g+1)},P_{2g})}$ is a half-line in a vector space of dimension 1, and we may replace any splitting μ by $-\mu$. Otherwise by 8.11 the fact that $\lambda\circ\mu\circ\Psi$ defines a polarization of Hodge structures on V means that $\lambda\circ\mu$ is strictly positive on $B(\mathfrak{X}'^o,P')$, or equivalently that $\mu(B(\mathfrak{X}'^o,P'))$ is contained in $(\sigma_o)^o$. By functoriality (8.11) it follows that $B(\mathfrak{X}^o_{2g})\cap(\sigma_o)^o\neq\emptyset$, where we denote the image of σ_o again by σ_o. But by 8.12 $B(\mathfrak{X}^o_{2g}) = B(\mathcal{H}^o_{2(g+1)},P_{2g})\subset\overline{C(\mathcal{H}^o_{2(g+1)},P_{2g})}\setminus\{0\}$, and both σ_o and $\overline{C(\mathcal{H}^o_{2(g+1)},P_{2g})}$ are half-lines in a one dimensional vector space, so $\sigma_o=\overline{C(\mathcal{H}^o_{2(g+1)},P_{2g})}$, as desired. q.e.d.

9.2. A certain set of splittings: Let $(P,\mathfrak{X})$ be mixed Shimura data, and $\pi_o: (P',\mathfrak{X}')\to(P,\mathfrak{X})$ a $((P_0,\mathfrak{X}_0)\to(\mathbb{G}_{m,\mathbb{Q}},\mathcal{H}_0))$-torsor. As in the assertion of 9.1 (a) assume that there exists a (not necessarily P-equivariant) splitting $\mu: U'\to U_0$, such that if $\Psi: V\times V\to U'$ is the pairing induced by the commutator in W', then $\lambda\circ\mu\circ\Psi$ defines a polarization of Hodge structures on V_{2g} for every point in $\mathfrak{X}'^o$. Under this assumption, for every irreducible component $(P_{(o)},\mathfrak{X}_{(o)})$ of $(P,\mathfrak{X})$, the assertion in 9.1 (b) holds. We fix such data for most of this chapter. We shall eventually construct cone decompositions for the pair $(P',\mathfrak{X}')\to(P,\mathfrak{X})$, that satisfy all requirements of 8.13. For this we first concentrate on condition 8.13 (b). The main problem is to produce a sufficient supply of splittings that satisfy this condition.

Consider a rational boundary component $(P_1',\mathfrak{X}_1')$ of $(P',\mathfrak{X}')$, let $(P_1,\mathfrak{X}_1)\cong(P_1',\mathfrak{X}_1')/U_0$ be the corresponding rational boundary component of $(P,\mathfrak{X})$, and Q' the associated parabolic subgroup of P'. Let $\mathfrak{X}'^o$ be a connected component of $\mathfrak{X}'$ that is mapped to $\mathfrak{X}_1'$, and let $\mathfrak{X}^o$ be the corresponding connected component of $\mathfrak{X}$. Let $(P'_{(o)},\mathfrak{X}'_{(o)})$ be the unique irreducible component of $(P',\mathfrak{X}')$ with $\mathfrak{X}'^{o}\subset\mathfrak{X}'_{(o)}$.

Consider $(\tilde{P},\tilde{\mathfrak{X}})$, $(\tilde{P}_*,\tilde{\mathfrak{X}}_*)$, and an embedding $(P'_{(o)},\mathfrak{X}'_{(o)})\hookrightarrow(\tilde{P},\tilde{\mathfrak{X}})$ as in 9.1 (b), then by functoriality we get a rational boundary component $(\tilde{P}_1,\tilde{\mathfrak{X}}_1)$ of $(\tilde{P},\tilde{\mathfrak{X}})$, and connected components $\tilde{\mathfrak{X}}_1^o$, $\tilde{\mathfrak{X}}^o$, and $\tilde{\mathfrak{X}}_*^o$. Consider a homomorphism $\tilde{\mu}\colon \tilde{U}_1\to U_0$ such that $\lambda\circ\tilde{\mu}$ is strictly positive on $\overline{C(\tilde{\mathfrak{X}}_*^o,\tilde{P}_1)}\setminus\{0\}$, and which restricts to the identity on $U_0\subset U'\hookrightarrow\tilde{U}_1$. Then its restriction to U_1' is a splitting of the short exact sequence $0\to U_0\to U_1'\to U_1\to 0$. The corresponding splitting map $e\colon U_1\to U_1'$ is determined by the equation $\tilde{\mu}|_{U_1'}=\mathrm{id}-e\circ\pi_0$.

Let $\mathcal{E}(\mathfrak{X}'^{o},P_1')$ be the set of all splittings $e\colon U_1\to U_1'$ that arise in this way for some choice of $(\tilde{P},\tilde{\mathfrak{X}})$ and $(\tilde{P}_*,\tilde{\mathfrak{X}}_*)$. Strictly speaking this is an abuse of notation, since this set depends not only on $\mathfrak{X}'^{o}$ and P_1', but also on the torsor structure on $(P',\mathfrak{X}')\to(P,\mathfrak{X})$. But since this is fixed, there should be no confusion. Clearly the definition of $\mathcal{E}(\mathfrak{X}'^{o},P_1')$ is invariant under $P'(\mathbb{Q})$. In particular the group $\mathrm{Stab}_{Q'(\mathbb{Q})}(\mathfrak{X}'^{o})$ acts on $\mathcal{E}(\mathfrak{X}'^{o},P_1')$. We shall denote this operation by $e\mapsto e^q:=(u\mapsto q^{-1}\cdot e(q\cdot u))$ for any $q\in\mathrm{Stab}_{Q'(\mathbb{Q})}(\mathfrak{X}'^{o})$. Note that $\mathrm{Stab}_{P'(\mathbb{Q})}(\mathfrak{X}'^{o})$ acts on U_0 through positive scalars, since we have a morphism $(P',\mathfrak{X}')\to(P_0,\mathfrak{X}_0)$. This implies that for every $u'\in U'(\mathbb{R})(-1)$

$$\lambda\circ(\mathrm{id}-e^q\circ\pi_0)(u') = \lambda(q^{-1}\cdot(\mathrm{id}-e\circ\pi_0)(q\cdot u'))$$

is a positive multiple of $\lambda\circ(\mathrm{id}-e\circ\pi_0)(q\cdot u')$.

<u>9.3. Proposition</u>: (a) $\mathcal{E}(\mathfrak{X}'^{o},P_1')$ is a nonempty open subset of the space of all splittings $U_1\to U_1'$ (open with respect to the archimedean topology on the hyperplane of all splittings in $\mathrm{Hom}(U_1,U_1')$).

(b) For every $e \in \mathcal{E}(\mathcal{X}'^{o},P_1')$ the linear map $\lambda\circ(\mathrm{id}-e\circ\pi_o)$ is strictly positive on $B(\mathcal{X}'^{o},P_1')$, i.e. satisfies condition 8.13 (b).

Proof: (a) Since by 4.15 (d) $C(\tilde{\mathcal{X}}^{o}_{*},\tilde{P}_1)$ is a nondegenerate selfadjoint cone, the set of all $\lambda\circ\tilde{\mu}$ that are strictly positive on $\overline{C(\tilde{\mathcal{X}}^{o}_{*},\tilde{P}_1)}\setminus\{0\}$ is just the set of rational points in the interior of the dual cone, which is a nonempty open subset of the dual space of $\tilde{U}_1(\mathbb{Q})(-1)$. Since by assumption the image of σ_o lies in $\overline{C(\tilde{\mathcal{X}}^{o}_{*},\tilde{P}_1)}$, and by injectivity is not zero, the restriction of $\tilde{\mu}$ to σ_o is the multiplication by some positive scalar $\alpha\in\mathbb{Q}^{>0}$. Thus $\alpha^{-1}\cdot\tilde{\mu}$ satisfies the requirements of of 9.2, and runs through a nonempty open subset of splittings $\tilde{U}_1\to U_0$. Since the restriction map $\mathrm{Hom}(\tilde{U}_1,U_0)\to\mathrm{Hom}(U_1',U_0)$ is surjective, the assertion follows.

(b) By the functoriality 8.11, $B(\mathcal{X}'^{o},P_1')$ is mapped to $B(\tilde{\mathcal{X}}^{o},\tilde{P}_1) = B(\tilde{\mathcal{X}}^{o}_{*},\tilde{P}_1)$. By 8.12, the latter is contained in $\overline{C(\tilde{\mathcal{X}}^{o}_{*},\tilde{P}_1)}\setminus\{0\}$. Since by assumption $\lambda\circ\tilde{\mu}$ is strictly positive on $\overline{C(\tilde{\mathcal{X}}^{o}_{*},\tilde{P}_1)}\setminus\{0\}$, it follows that $\lambda\circ\tilde{\mu}|_{U_1'}=\lambda\circ(\mathrm{id}-e\circ\pi_o)$ is strictly positive on $B(\mathcal{X}'^{o},P_1')$, as desired. q.e.d.

We shall need the following information about the behavior of $\mathcal{E}(\mathcal{X}'^{o},P_1')$ under restriction to another rational boundary component.

9.4. Proposition: Let $(P_2',\mathcal{X}_2')$ be a rational boundary component between $(P_1',\mathcal{X}_1')$ and $(P',\mathcal{X}')$.

(a) The restriction map $\mathrm{Hom}(U_1',U_0)\to\mathrm{Hom}(U_2',U_0)$ induces a surjection $\mathcal{E}(\mathcal{X}'^{o},P_1')\to\mathcal{E}(\mathcal{X}'^{o},P_2')$.

(b) Let $e\in\mathcal{E}(\mathcal{X}'^{o},P_1')$, $A\subset C^*(\mathcal{X}'^{o},P_1')\setminus C^*(\mathcal{X}'^{o},P_2')$ a finite subset, and $\alpha\in\mathbb{Q}$. Then there exists $e'\in\mathcal{E}(\mathcal{X}'^{o},P_1')$ such that $e|_{U_2}=e'|_{U_2}$, $\lambda\circ(\mathrm{id}-e'\circ\pi_o)\geq\lambda\circ(\mathrm{id}-e\circ\pi_o)$ on $C^*(\mathcal{X}'^{o},P_1')$, and $\lambda\circ(\mathrm{id}-e'\circ\pi_o)(u')\geq\alpha$ for every $u'\in A$.

Proof: (a) We first show that every $e\in\mathcal{E}(\mathcal{X}'^{o},P_1')$ restricts to an element of $\mathcal{E}(\mathcal{X}'^{o},P_2')$. Suppose that e comes from a splitting $\tilde{\mu}:\tilde{U}_1\to U_0$, then $e|_{U_2}$ comes from the restriction of $\tilde{\mu}$ to $\tilde{U}_2$. Since by 4.21 (b) $\overline{C(\tilde{X}^o_*,\tilde{P}_2)}$ is contained in $\overline{C(\tilde{X}^o_*,\tilde{P}_1)}$, the map $\tilde{\mu}|_{U_2'}$ satisfies the assumptions in 9.2, as desired. For the surjectivity we have to extend a homomorphism $\tilde{\mu}_2$: $\tilde{U}_2 \to U_0$, such that $\lambda \circ \tilde{\mu}_2$ is strictly positive on $\overline{C(\tilde{X}^o_*,\tilde{P}_2)} \setminus \{0\}$, to a homomorphism $\tilde{\mu}$: $\tilde{U}_1 \to U_0$ such that $\lambda \circ \tilde{\mu}$ is strictly positive on $\overline{C(\tilde{X}^o_*,\tilde{P}_1)} \setminus \{0\}$. This is possible by the following lemma 9.5, applied to $(\tilde{P}_*,\tilde{X}_*)$ in place of (P,X), $(\tilde{P}_1,\tilde{X}_1)$ in place of (P_1,X_1), the analogous rational boundary component in place of (P_2,X_2), and $\lambda \circ \tilde{\mu}_2$, $\lambda \circ \tilde{\mu}$ in place of μ_2, resp. μ.

(b) Since $C(\tilde{X}^o_2/\tilde{W}_2,\tilde{P}_1/(\tilde{P}_1 \cap \tilde{W}_2))$ is a nondegenerate selfadjoint cone, there exists a homomorphism $\overline{\nu}$: $\tilde{U}_1/\tilde{U}_1 \cap \tilde{W}_2 \to U_0$ such that $\lambda \circ \overline{\nu}$ is strictly positive on $\overline{C(\tilde{X}^o_2/\tilde{W}_2,\tilde{P}_1/(\tilde{P}_1 \cap \tilde{W}_2))} \setminus \{0\}$. Note that by 4.21 (a) $C(\tilde{X}^o_2/\tilde{W}_2,\tilde{P}_1/(\tilde{P}_1 \cap \tilde{W}_2))$ is the image of $C(\tilde{X}^o_*,\tilde{P}_1)$, or also of $C(\tilde{X}^o,\tilde{P}_1)$, under the canonical projection $\tilde{U}_1 \twoheadrightarrow \tilde{U}_1/\tilde{U}_1 \cap \tilde{W}_2$. Thus, lifting ν to a homomorphism $\tilde{\nu}$: $\tilde{U}_1 \to U_0$, it follows that $\lambda \circ \tilde{\nu}$ is nonnegative on $\overline{C(\tilde{X}^o_*,\tilde{P}_1)}$, and strictly positive on $C^*(\tilde{X}^o,\tilde{P}_1) \setminus (\tilde{U}_1 \cap \tilde{W}_2)(\mathbb{R})(-1)$. Now suppose that e comes from a splitting $\tilde{\mu}$: $\tilde{U}_1 \to U_0$ as in 9.2. Then for every $\beta \in \mathbb{Q}^{\geq 0}$ the splitting $\tilde{\mu}+\beta \cdot \tilde{\nu}$ also satisfies the requirements of 9.2, and the splitting e': $U_1' \to U_0$ associated to $\tilde{\mu}+\beta \cdot \tilde{\nu}$ has the first two of the three desired properties. By 4.23, the image of A is contained in $C^*(\tilde{X}^o,\tilde{P}_1) \setminus C^*(\tilde{X}^o,\tilde{P}_2)$. By 4.22 (c), this set is equal to $C^*(\tilde{X}^o,\tilde{P}_1) \setminus (\tilde{U}_1 \cap \tilde{W}_2)(\mathbb{R})(-1)$, hence $\lambda \circ \tilde{\nu}$ is strictly positive on the image of A. Thus the third property also holds, provided that β is sufficiently large. q.e.d.

9.5. Lemma: Let (P,X) be pure Shimura data, (P_2,X_2) a rational boundary component of (P,X), and (P_1,X_1) a rational boundary component of (P_2,X_2). Let X^o be a connected component of X that is mapped to X_1. Then for every linear form μ_2: $U_2(\mathbb{Q})(-1) \to \mathbb{Q}$, which is strictly positive on $\overline{C(X^o,P_2)} \setminus \{0\}$, there exists a linear form μ: $U_1(\mathbb{Q})(-1)\to\mathbb{Q}$, which is strictly positive on $\overline{C(X^o,P_1)} \setminus \{0\}$, such that $\mu_2 = \mu|_{U_2(\mathbb{Q})(-1)}$.

Proof: Since the positivity is an open condition, it is easy to see that it suffices to prove the same assertion with linear forms defined over $\mathbb{R}$. Next note that since by 4.21 (b) $\overline{C(X^o,P_2)}$ is contained in $\overline{C(X^o,P_1)}$, the restriction of any μ: $U_1(\mathbb{R})(-1)\to\mathbb{R}$, that is strictly positive on $\overline{C(X^o,P_1)} \setminus \{0\}$, is strictly positive on $\overline{C(X^o,P_2)} \setminus \{0\}$. Thus the desired assertion holds for some μ_2. Now the set of μ_2's in question is the interior of the dual cone of $C(X^o,P_2)$. Let Q_i be the parabolic subgroup of P associated to (P_i,X_i) for $i=1, 2$. We are going to show that $(Q_1\cap Q_2)(\mathbb{R})^o$ acts transitively on $C(X^o,P_2)$. The same group then also acts transitively on the set of μ_2's, and since the restriction map $\mu\mapsto\mu|_{U_2(\mathbb{R})(-1)}$ is equivariant, the desired assertion holds for every μ_2.

Since by 2.14 (a) P_i acts through a nontrivial scalar character on U_i, the image of $P_i(\mathbb{R})^o$ in $GL(U_i)(\mathbb{R})$ consists of all positive scalars. Clearly $P_1\subset Q_1\cap P_2\subset Q_1\cap Q_2$, so the image of $(Q_1\cap Q_2)(\mathbb{R})^o$ in $GL(U_2)(\mathbb{R})$ contains all positive scalars. On the other hand, by 4.19 (b) we have $Q_2 = (Q_1\cap Q_2)\cdot P_2$. Since $P_2(\mathbb{R})^o$ acts through positive scalars on $U_2(\mathbb{R})$, the image of $(Q_1\cap Q_2)(\mathbb{R})^o$ in $GL(U_2)(\mathbb{R})$ coincides with that of $Q_2(\mathbb{R})^o$. Now by 4.15 (c) $C(X^o,P_2)$ is an orbit under $Q_2(\mathbb{R})^o$, so the same follows for the group $(Q_1\cap Q_2)(\mathbb{R})^o$, as desired. q.e.d.

9.6. An almost arithmetic subgroup: Let $\bar{\rho}$: $Q'\to PGL(U_1')$ be the homomorphism induced by the natural projective representation on U_1'. We shall be concerned with an arbitrary subgroup

$$\Gamma \subset \mathrm{Stab}_{Q'(\mathbb{Q})}(X'^o)$$

whose image in $\bar{\rho}(Q')(\mathbb{Q})$ is an arithmetic subgroup. For example if K'_f is an open compact subgroup of $P'(\mathbb{A}_f)$, then

$$\Gamma := \mathrm{Stab}_{Q'(\mathbb{Q})}(\mathcal{X}'^o) \cap K'_f \cdot P'_1(\mathbb{A}_f)$$

is such a group. Indeed, this is a subgroup since Q' normalizes P'_1, and its image in $\bar{\rho}(Q')(\mathbb{Q})$ is an arithmetic subgroup, since by 2.14 (a) P'_1 acts through scalars on U'_1. We can also take any arithmetic subgroup of $\mathrm{Stab}_{Q'(\mathbb{Q})}(\mathcal{X}'^o)$. Moreover let $(P'_{(o)},\mathcal{X}'_{(o)})$ be the unique irreducible component of $(P',\mathcal{X}')$ with $\mathcal{X}'^o \subset \mathcal{X}'_{(o)}$. Then by the condition 2.1 (viii) every sufficiently small arithmetic subgroup of $P'(\mathbb{Q})$ is contained in $P'_{(o)}(\mathbb{Q})\cdot Z(P')(\mathbb{Q})$. Since $Z(P')$ acts trivially on U'_1, we can also take any arithmetic subgroup of $\mathrm{Stab}_{(P'_{(o)}\cap Q')(\mathbb{Q})}(\mathcal{X}'^o)$.

Recall (9.2) that $\mathrm{Stab}_{P'(\mathbb{Q})}(\mathcal{X}'^o)$ acts through positive scalars on U_0. Thus whenever an element $\gamma\in\Gamma$ acts through a scalar on U'_1, this scalar must be positive, since it is so on U_0. Thus for all purposes, or objects, which are invariant under positive scalars, we can treat Γ like an arithmetic subgroup of $\rho(Q')(\mathbb{Q})$. Among such objects are in particular all convex rational polyhedral cones. Our aim is to construct Γ-invariant rational partial cone decompositions of $C^*(\mathcal{X}'^o,P'_1)$ and of $C^*(\mathcal{X}^o,P_1)$, that satisfy certain additional conditions.

<u>9.7. Proposition</u>: Let $e\in\mathcal{E}(\mathcal{X}'^o,P'_1)$ and $u' \in C^*(\mathcal{X}'^o,P'_1)\cap U'_1(\mathbb{Q})(-1)$. Then the set

$$\{\gamma\in\Gamma \mid \lambda\circ(\mathrm{id}-e^{\gamma}\circ\pi_o)(u')\leq 0\}$$

is a finite union of $\mathrm{Stab}_\Gamma(\mathbb{R}^{\geq 0}\cdot u')$-cosets.

<u>Proof</u>: By the remarks in 9.6 we may without loss of generality assume that Γ is an arithmetic subgroup of $\mathrm{Stab}_{(P'_{(o)}\cap Q')(\mathbb{Q})}(\mathcal{X}'^o)$. Recall that by the remark at the end of 9.2 we may write $\lambda\circ(\mathrm{id}-e\circ\pi_o)(\gamma\cdot u')$

instead of $\lambda\circ(\mathrm{id}-e^{\gamma}\circ\pi_o)(u')$. Suppose that e comes from a splitting $\tilde{\mu}: \tilde{U}_1 \to U_0$ as in 9.2, and let $\tilde{u}$ be the image of u' in $C^*(\tilde{X}^o,\tilde{P}_1) \cap \tilde{U}_1(\mathbb{Q})(-1)$. By the injectivity of $U_1' \hookrightarrow \tilde{U}_1$ we have to show that the set

$$\{\gamma\cdot\tilde{u} \mid \gamma\in\Gamma \text{ such that } \lambda\circ\tilde{\mu}(\gamma\cdot\tilde{u})\leq 0\}$$

is finite. We shall a fortiori show that for all $\tilde{u} \in C^*(\tilde{X}^o,\tilde{P}_1)\cap\tilde{U}_1(\mathbb{Q})(-1)$ and $\alpha\in\mathbb{Q}$ the set

$$\{\gamma\cdot\tilde{u} \mid \gamma\in\Gamma \text{ such that } \lambda\circ\tilde{\mu}(\gamma\cdot\tilde{u})\leq \alpha\}$$

is finite.

Since Γ acts on $\tilde{U}_1$ through $\tilde{P}(\mathbb{Q})$, by 2.14 (a) it acts through scalars on $\tilde{U}$. But by 9.2 it acts through positive scalars on U_0, so since it is an arithmetic subgroup, it acts trivially on $\tilde{U}$. This shows that the assertion is invariant under shifting $\tilde{u}$ by elements of $\tilde{U}(\mathbb{Q})(-1)$, if at the same time α is changed by a constant. Since by 4.22 (b) $C^*(\tilde{X}^o,\tilde{P}_1) = C^*(\tilde{X}^o_*,\tilde{P}_1)+\tilde{U}(\mathbb{R})(-1)$, we may therefore assume that $\tilde{u} \in C^*(\tilde{X}^o_*,\tilde{P}_1)\cap \tilde{U}_1(\mathbb{Q})(-1)$. Fix a Γ-invariant lattice $\tilde{U}_1(\mathbb{Z})(-1)$ that contains $\tilde{u}$. Since the set $\{\tilde{u}\in\overline{C(\tilde{X}^o_*,\tilde{P}_1)} \mid \tilde{\mu}(\tilde{u})\leq\alpha\}$ is compact (cf. [AMRT] ch.II §5.3 p.137), the set $\{\gamma\cdot\tilde{u} \mid \gamma\in\Gamma,\ \lambda\circ\tilde{\mu}(\gamma\cdot\tilde{u})\leq\alpha\}$ is a bounded subset of a lattice. It is therefore finite, as desired. <u>q.e.d.</u>

<u>9.8. Definition of certain "locally polyhedral" subsets:</u> Consider a nonempty collection $\{e_1,\dots,e_m\}$ of splittings in $\mathcal{E}(X'^o,P_1')$. Define

$$\Sigma := \bigcap_{\gamma\in\Gamma} \bigcap_{i=1}^{m} \sigma_o + e_i^{\gamma}(C^*(X^o,P_1))$$
$$= \{u'\in C^*(X'^o,P_1') \mid \forall\gamma\in\Gamma\ \forall 1\leq i\leq m\ \ \lambda\circ(\mathrm{id}-e_i^{\gamma}\circ\pi_o)(u')\geq 0\}.$$

Clearly Σ is a Γ-invariant convex cone, with $\Sigma=\Sigma+\sigma_o$, and relatively closed inside $C^*(X'^o,P_1')$ (<u>Caution</u>: in general it is not closed in $U_1'(\mathbb{R})(-1)$). In the following we shall always <u>assume</u>:

Σ contains no nontrivial linear subspace.

Note that every nontrivial linear subspace of $U_1'(\mathbb{R})(-1)$ that is contained in $C^*(\mathcal{X}'^{o},P_1')$, already lies in $U'(\mathbb{R})(-1)$. By definition

$$U'(\mathbb{R})(-1) \cap \Sigma \subset \{u' \in U'(\mathbb{R})(-1) \mid \forall 1 \le i \le m \ \ \lambda \circ (id - e_i \circ \pi_0)(u') \ge 0\},$$

so the assumption certainly holds if the $id - e_i \circ \pi_0$ generate the vector space $Hom(U',U_0)$. Since these are splittings, the differences $e_i - e_j$ factor through homomorphisms $U \to U_0$, and the assumption is verified if the $e_i - e_j$ generate the $\mathbb{Q}$-vector space $Hom(U,U_0)$.

We shall show (in 9.12) that every such Σ gives rise to rational partial polyhedral decompositions satisfying the conditions 8.13 (a) and (b), and the condition (*) in 8.5 except for the integrality of the splittings. The set Σ plays the role that Γ-polyhedral cocores play in [AMRT] ch.II §5.3. The term "locally polyhedral" is justified by the following proposition.

<u>9.9. Proposition</u>: For any convex rational polyhedral cone $\sigma' \subset C^*(\mathcal{X}'^{o},P_1')$, the intersection $\sigma' \cap \Sigma$ is again a convex rational polyhedral cone.

<u>Proof</u>: (Compare [AMRT] ch.II §5.4 p.140f) Every convex rational polyhedral cone is contained in a finite union of simplicial convex rational polyhedral cones of top dimension. Thus we may fix $u_1',\dots,u_n' \in C^*(\mathcal{X}'^{o},P_1') \cap U_1'(\mathbb{Q})(-1)$ that form a basis of $U_1'(\mathbb{Q})(-1)$, and assume that $\sigma' = \{\sum_{j=1}^{n} \alpha_j \cdot u_j' \mid \forall 1 \le j \le n \ \alpha_j \in \mathbb{R}^{\ge 0}\}$. Choose a lattice $U_1'(\mathbb{Z})(-1)$ that contains all u_j'. Let $\Gamma' \subset \Gamma$ be the subgroup of all elements that stabilize this lattice, then by 9.6 Γ' is, up to scalars, of finite index in Γ. Since $e_i^{\gamma} = e_i$ for every scalar γ, the set $\{e_i^{\gamma} \mid 1 \le i \le m, \gamma \in \Gamma\}$ is the union of finitely many Γ'-orbits. Thus there exists a positive integer N so that

$$\lambda\circ(\mathrm{id}-e_i^\gamma\circ\pi_o)(U_1'(\mathbb{Z})(-1)) \subset \frac{1}{N}\cdot\mathbb{Z}$$

for all $1\le i\le m$ and $\gamma\in\Gamma$. Now for all i and j, proposition 9.7 implies that the set $\{\lambda\circ(\mathrm{id}-e_i^\gamma\circ\pi_o)(u_j')|\gamma\in\Gamma\}$ is bounded below. Thus there exists an integer M so that for all $1\le i\le m$, $1\le j\le n$, and $\gamma\in\Gamma$

$$a_{i,j,\gamma} := N\cdot\lambda\circ(\mathrm{id}-e_i^\gamma\circ\pi_o)(u_j') \in M+\mathbb{Z}^{\ge 0}.$$

By the definitions of σ' and of Σ we get

$$\sigma'\cap\Sigma = \{\textstyle\sum_{j=1}^n \alpha_j\cdot u_j' \,|\, \forall 1\le j\le n\ \alpha_j\in\mathbb{R}^{\ge 0} \text{ and } \forall\gamma\in\Gamma\ \forall 1\le i\le m\ \sum_{j=1}^n a_{i,j,\gamma}\cdot\alpha_j\ge 0\}.$$

Let S be the set of all n-tuples $(a_{i,j,\gamma})_{1\le j\le n}$ for $1\le i\le m$ and $\gamma\in\Gamma$, then we can also write

$$\sigma'\cap\Sigma = \{\textstyle\sum_{j=1}^n \alpha_j\cdot u_j' \,|\, \forall 1\le j\le n\ \alpha_j\in\mathbb{R}^{\ge 0} \text{ and } \forall \underline{s}\in S\ \sum_{j=1}^n s_j\cdot\alpha_j\ge 0\}.$$

Now S is a subset of $(M+\mathbb{Z}^{\ge 0})^n$, so by the lemma [AMRT] ch.II §5.4 p.140 there exists a finite subset $S_o\subset S$, such that

$$\forall \underline{s}\in S\ \exists \underline{s}'\in S \text{ so that } \forall 1\le j\le n\ \ s_j\ge s_j'.$$

This implies that

$$\sigma'\cap\Sigma = \{\textstyle\sum_{j=1}^n \alpha_j\cdot u_j' \,|\, \forall 1\le j\le n\ \alpha_j\in\mathbb{R}^{\ge 0} \text{ and } \forall \underline{s}\in S_o\ \sum_{j=1}^n s_j\cdot\alpha_j\ge 0\},$$

which is a convex rational polyhedral cone, as desired. q.e.d.

9.10. Proposition: There exist finitely many $u_1',\ldots,u_n' \in C^*(X'^o,P_1')\cap U_1'(\mathbb{Q})(-1)$ such that Σ is the convex closure of the set

$$\textstyle\bigcup_{\gamma\in\Gamma}\bigcup_{j=1}^n \mathbb{R}^{\ge 0}\cdot\gamma\cdot u_j'.$$

Proof: By 6.19 (b) there exists a convex rational polyhedral cone $\sigma' \subset C^*(X'^o,P_1')$ so that $\Gamma\cdot\sigma' = C^*(X'^o,P_1')$. Choose the u_j' such that $\sigma'\cap\Sigma$ is the convex closure of $\bigcup_{j=1}^n \mathbb{R}^{\ge 0}\cdot u_j'$, this is possible by 9.9. Then $\Sigma =$

$\Gamma\cdot(\sigma'\cap\Sigma)$ is contained in the convex closure of $\bigcup_{\gamma\in\Gamma}\bigcup_{j=1}^{n}\mathbb{R}^{\geq 0}\cdot\gamma\cdot u_j'$. But Σ itself is convex, and contains all $\mathbb{R}^{\geq 0}\cdot\gamma\cdot u_j'$, so we have equality. q.e.d.

9.11. Proposition: For all $1\leq i\leq m$ and $\gamma\in\Gamma$, the subset $e_i^{\gamma}(U_1(\mathbb{R})(-1))\cap\Sigma$ is a convex rational polyhedral cone.

Proof: By Γ-invariance we may assume $\gamma=1$. Then the set in question can be written as $\{u'\in\Sigma|\lambda\circ(id-e_i\circ\pi_o)(u')=0\}$. Let $u_1',\ldots,u_n'$ be as in 9.10, then $\lambda\circ(id-e_i\circ\pi_o)(\gamma'\cdot u_j')\geq 0$ for all $1\leq j\leq n$ and $\gamma'\in\Gamma$. Thus 9.10 implies that $e_i(U_1(\mathbb{R})(-1))\cap\Sigma$ is the convex closure of the union of all $\mathbb{R}^{\geq 0}\cdot\gamma'\cdot u_j'$ with $\lambda\circ(id-e_i\circ\pi_o)(\gamma'\cdot u_j')=0$. Since

$$\lambda\circ(id-e_i\circ\pi_o)(\gamma'\cdot u_j') = \lambda(\gamma'\cdot(id-e_i^{\gamma'}\circ\pi_o)(u_j')),$$

this equation is equivalent to $\lambda\circ(id-e_i^{\gamma'}\circ\pi_o)(u_j')=0$. But by 9.7 there are, modulo $Stab_\Gamma(\mathbb{R}^{\geq 0}\cdot u_j')$, only finitely many pairs (γ',j) with this property. Thus our set is the convex closure of the union of finitely many $\mathbb{R}^{\geq 0}\cdot\gamma'\cdot u_j'$, as desired. q.e.d.

9.12. Proposition: Let e_i and Σ be as in 9.8. Let $\mathcal{S}_\Sigma^+$ be the set of all faces of all cones $e_i^{\gamma}(U_1(\mathbb{R})(-1))\cap\Sigma$. Let $\mathcal{S}_\Sigma := \{\pi_o(\sigma')|\sigma'\in\mathcal{S}_\Sigma^+\}$, and $\mathcal{S}_\Sigma' := \{\sigma',\sigma'+\sigma_o|\sigma'\in\mathcal{S}_\Sigma^+\}$. Then $\mathcal{S}_\Sigma^+$ and $\mathcal{S}_\Sigma'$ are rational partial polyhedral decompositions of $C^*(X'^o,P_1')$, and $\mathcal{S}_\Sigma$ is a complete rational polyhedral decomposition of $C^*(X^o,P_1)$. All of them are Γ-invariant and finite modulo Γ. They satisfy the line bundle condition of 5.11 (except for the integrality), i.e. $\mathcal{S}_\Sigma^+=\{e_\sigma(\sigma)|\sigma\in\mathcal{S}_\Sigma\}$ for certain splittings $e_\sigma\colon U_1\to U_1'$. Moreover they satisfy conditions 8.13 (a) and (b).

Proof: We first show that $\mathcal{S}_\Sigma^+$ is a rational partial cone decomposition. By 9.11 $\mathcal{S}_\Sigma^+$ is a nonempty set of convex rational polyhedral cones. Of the conditions 5.1 (i)-(iii) the first holds by definition of $\mathcal{S}_\Sigma^+$. The last one holds since Σ does not contain a nontrivial linear subspace. For the second one let $\sigma := e_i^{\gamma}(U_1(\mathbb{R})(-1))\cap\Sigma$, $\sigma' := e_{i'}^{\gamma'}(U_1(\mathbb{R})(-1))\cap\Sigma$, τ a face of σ, and τ' a face of σ'. We can write $\tau=\{u'\in\sigma \mid \forall\nu\ \ell_\nu(u')=0\}$ with finitely many linear forms $\ell_\nu: U'(\mathbb{Q})(-1)\to\mathbb{Q}$ that are nonnegative on σ, and likewise $\tau'=\{u'\in\sigma' \mid \forall\nu'\ \ell'_{\nu'}(u')=0\}$. Then

$$\tau\cap\tau' = \{u'\in\sigma\cap\sigma' \mid \forall\nu\ \ell_\nu(u')=0 \text{ and } \forall\nu'\ \ell'_{\nu'}(u')=0\},$$

which by definition is a face of $\sigma\cap\sigma'$. But this cone can be written as $\{u'\in\sigma \mid \lambda\circ(id-e_{i'}^{\gamma'}\circ\pi_o)(u')=0\}$, and $\lambda\circ(id-e_{i'}^{\gamma'}\circ\pi_o)$ is nonnegative on σ, so $\sigma\cap\sigma'$ is a face of σ. Thus $\tau\cap\tau'$ is a face of σ, as desired. This proves that $\mathcal{S}_\Sigma^+$ is a rational partial cone decomposition of $C^*(\mathfrak{X}'^o,P_1')$.

From this it easily follows that $\mathcal{S}_\Sigma$ and $\mathcal{S}'_\Sigma$ are rational partial cone decompositions. For all γ and i we have

$$(e_i^{\gamma}(U_1(\mathbb{R})(-1))\cap\Sigma) + \sigma_o =$$
$$= \{u'\in\Sigma \mid \forall\gamma'\in\Gamma\ \forall 1\le i'\le m\ \ \lambda\circ(id-e_{i'}^{\gamma'}\circ\pi_o)(u') \ge \lambda\circ(id-e_i^{\gamma}\circ\pi_o)(u')\}.$$

so 9.7 implies that Σ is the union of all these, i.e. that $|\mathcal{S}'_\Sigma| = \Sigma$. Again by 9.7 we have $\Sigma + U_0(\mathbb{R})(-1) = C^*(\mathfrak{X}'^o,P_1')$, whence $\pi_o(\Sigma) = C^*(\mathfrak{X}^o,P_1)$, and $\mathcal{S}_\Sigma$ is a complete rational polyhedral decomposition of $C^*(\mathfrak{X}^o,P_1)$. The Γ-invariance and finiteness modulo Γ, as well as the line bundle condition of 5.11 and conditions 8.13 (a) and (b) are clear from the definitions. q.e.d.

9.13. Corollary: Let $(P',\mathfrak{X}')\to(P,\mathfrak{X})$ be as in 9.2, $K_f'\subset P'(\mathbb{A}_f)$ an open compact subgroup, and $K_f:=\pi_o(K_f')$. Let $\mathcal{S}'$ be a K_f'-admissible partial cone decomposition for $(P',\mathfrak{X}')$, and $\mathcal{S}$ a K_f-admissible partial cone decomposition for $(P,\mathfrak{X})$. Assume that for every rational boundary component $(P_1',\mathfrak{X}_1')$ of $(P',\mathfrak{X}')$, every connected component $\mathfrak{X}'^o$ of $\mathfrak{X}'$ that maps into $\mathfrak{X}_1'$, and every $p_f'\in P'(\mathbb{A}_f)$, we have $\mathcal{S}'(\mathfrak{X}'^o,P_1',p_f') = \mathcal{S}'_\Sigma$

and $\mathfrak{S}(\mathfrak{X}^o,P_1,\pi_o(p_f')) = \mathfrak{S}_\Sigma$ for some $\Sigma \subset C^*(\mathfrak{X}'^o,P_1')$ as in 9.8. Then after possibly replacing K_f' by $K_f'\cdot K_f^U$ for some open compact subgroup K_f^U of $U_0(\mathbb{A}_f)$, the conditions 8.5 (*) and 8.13 (a) and (b) hold. In particular $M^{K_f}(P,\mathfrak{X},\mathfrak{S})(\mathbb{C})$ is projective.

<u>Proof</u>: Modulo the left action of $P'(\mathbb{Q})$ and the right action of K_f' there are only finitely many $(\mathfrak{X}'^o,P_1',p_f')$. Since by 9.12 $\mathfrak{S}_\Sigma'$ is finite modulo Γ, it follows that $\mathfrak{S}'$ is finite. The same follows for $\mathfrak{S}$, and since by 9.12 $|\mathfrak{S}_\Sigma|=C^*(\mathfrak{X}^o,P_1)$, it follows that $\mathfrak{S}$ is complete. Moreover by 9.12 the pair $(\mathfrak{S}',\mathfrak{S})$ satisfies the conditions 8.13 (a) and (b), plus the line bundle condition (*) of 8.5 except possibly for the integrality of the splitting of the short exact sequence $0 \to U_0(\mathbb{Q})\cap\Gamma_{U'} \to \Gamma_{U'} \to \Gamma_U \to 0$. But as in the proof of 8.8, if we replace K_f' by $K_f'\cdot K_f^U$ for a large open compact subgroup K_f^U of $U_0(\mathbb{A}_f)$, then $U_0(\mathbb{Q})\cap\Gamma_{U'}$ is replaced by a larger lattice. Thus for any fixed $\sigma\in\mathfrak{S}$ we can choose K_f^U such that the corresponding splitting becomes integral. By the finiteness of $\mathfrak{S}$, and since this integrality condition is invariant under the operations 6.4 (ii) and (iv), we can even choose K_f^U so that every such splitting becomes integral. If we now replace K_f' by $K_f'\cdot K_f^U$, then the other conditions still hold, and all splittings are integral, as desired. <u>q.e.d.</u>

The following fact was implicit in the assumptions of the preceding corollary.

<u>9.14. Proposition:</u> Let Γ be as in 9.6, Σ as in 9.8, and $\mathfrak{S}_\Sigma$, $\mathfrak{S}_\Sigma^+$, $\mathfrak{S}_\Sigma'$ as in 9.12. Let $(P_2',\mathfrak{X}_2')$ be a rational boundary component between $(P_1',\mathfrak{X}_1')$ and $(P',\mathfrak{X}')$, and $\Gamma_2 := \mathrm{Stab}_\Gamma(C^*(\mathfrak{X}'^o,P_2'))$. Then $\Sigma_2 := C^*(\mathfrak{X}'^o,P_2')\cap\Sigma$ can be described as in 9.8, with $(P_2',\mathfrak{X}_2')$ in place of $(P_1',\mathfrak{X}_1')$, and Γ_2 in place of Γ. Moreover $\mathfrak{S}_{\Sigma_2}^+ = \{\sigma'\in\mathfrak{S}_\Sigma^+ \,|\, \sigma'\subset C^*(\mathfrak{X}'^o,P_2')\}$, and likewise for $\mathfrak{S}_{\Sigma_2}$ and $\mathfrak{S}_{\Sigma_2}'$.

<u>Proof</u>: Let us first show that Γ_2 is of the form described in 9.6 with (P_2', X_2') in place of (P_1', X_1'). Let Q_2' be the parabolic subgroup of P' associated to (P_2', X_2'), then $\Gamma_2 = Q_2'(\mathbb{Q}) \cap \Gamma \subset \mathrm{Stab}_{Q_2'(\mathbb{Q})}(X'^{o})$. By 9.6 the image of Γ_2 in $GL(U_2')$ is up to scalars commensurable with the image of an arithmetic subgroup of $(Q_1' \cap Q_2')(\mathbb{Q})$. But by 4.19 (b) $Q_2' = (Q' \cap Q_2') \cdot P_2'$, so since P_2' acts through scalars on U_2', the image of Γ_2 in $GL(U_2')$ is up to scalars commensurable with the image of an arithmetic subgroup of $Q_2'(\mathbb{Q})$, as desired.

According to 6.19 (b) choose a convex rational polyhedral cone $\sigma \subset C^*(X'^{o}, P_2')$ such that $\Gamma_2 \cdot \sigma = C^*(X'^{o}, P_2')$, such a cone must contain $U'(\mathbb{R})(-1)$. By 9.9 $\sigma \cap \Sigma$ is again a convex rational polyhedral cone. Since by definition

$$\sigma \cap \Sigma = \{u' \in \sigma \mid \forall \gamma \in \Gamma \ \forall 1 \le i \le m \ \ \lambda \circ (\mathrm{id} - e_i^{\gamma} \circ \pi_o)(u') \ge 0\},$$

there exists a finite set S of pairs (γ, i) such that

$$\sigma \cap \Sigma = \{u' \in \sigma \mid \forall (\gamma, i) \in S \ \ \lambda \circ (\mathrm{id} - e_i^{\gamma} \circ \pi_o)(u') \ge 0\}.$$

Write $\{e_1', \ldots, e_n'\} := \{e_i^{\gamma}|_{U_2'} \mid (\gamma, i) \in S\}$. We have

$$\begin{aligned} \sigma \cap \Sigma &= \{u' \in \sigma \mid \forall 1 \le j \le n \ \ \lambda \circ (\mathrm{id} - e_j' \circ \pi_o)(u') \ge 0\} \\ &= \{u' \in \sigma \mid \forall 1 \le j \le n \ \forall \gamma \in \Gamma_2 \ \ \lambda \circ (\mathrm{id} - e_j'^{\gamma} \circ \pi_o)(u') \ge 0\}, \end{aligned}$$

hence

$$\Sigma_2 = \Gamma_2 \cdot (\sigma \cap \Sigma) = \{u' \in C^*(X'^{o}, P_2') \mid \forall 1 \le j \le n \ \forall \gamma \in \Gamma_2 \ \ \lambda \circ (\mathrm{id} - e_j'^{\gamma} \circ \pi_o)(u') \ge 0\},$$

as in 9.8. The assertions about $\mathcal{S}_{\Sigma_2}$, $\mathcal{S}^+_{\Sigma_2}$, and $\mathcal{S}'_{\Sigma_2}$ are obvious. <u>q.e.d.</u>

The following construction is a kind of inverse to 9.14, and will be needed in the construction of cone decompositions as in 9.13.

<u>9.15. Proposition</u>: Let Γ be as in 9.6, and $\Delta \subset C^*(X'^o,P_1') \setminus C(X'^o,P_1')$ a subset, such that for every rational boundary component (P_2',X_2') between (P_1',X_1') and (P',X'), different from (P_1',X_1'), the intersection $C^*(X'^o,P_2') \cap \Delta$ is of the form described in 9.8 with (P_2',X_2') in place of (P_1',X_1') and some group Γ_2. Then there exists a Γ-invariant subset $\Sigma \subset C^*(X'^o,P_1')$ as in 9.8, such that $\Sigma \setminus C(X'^o,P_1') = \Delta$.

<u>Proof</u>: In the case $(P_1',X_1')=(P',X')$ the assertion is just the existence of some Σ as in 9.8. But by 9.3 (a) there exist splittings e_i satisfying the condition in 9.8, so this is clear. Now assume that (P_1',X_1') is a proper rational boundary component of (P',X'). There are only finitely many Γ-conjugacy classes of rational boundary components between (P_1',X_1') and (P',X'). Let (P_ν',X_ν') be representatives for these conjugacy classes. Let $\Gamma_\nu := \mathrm{Stab}_\Gamma(C^*(X'^o,P_\nu'))$, by the proof of 9.14 this group is a valid choice as in 9.6 for (P_ν',X_ν') in place of (P_1',X_1'). Thus by assumption there exist finitely many $e_{\nu,i} \in \mathcal{E}(X'^o,P_\nu')$ such that

$$C^*(X'^o,P_\nu') \cap \Delta = \{u' \in C^*(X'^o,P_\nu') \mid \forall \gamma \in \Gamma_\nu\ \forall i\ \ \lambda \circ (id - e_{\nu,i}{}^\gamma \circ \pi_o)(u') \geq 0\}.$$

Also by 9.10 there exist finitely many $u_{\nu,j}' \in C^*(X'^o,P_\nu') \cap U_\nu'(\mathbb{Q})(-1)$ such that $\Delta \cap C^*(X'^o,P_\nu')$ is the convex closure of the set

$$\bigcup_{\gamma \in \Gamma_\nu} \bigcup_j \mathbb{R}^{\geq 0} \cdot \gamma \cdot u_{\nu,j}'.$$

Let us fix ν and i for a moment. By 9.4 (a) there exists $\tilde{e}_{\nu,i} \in \mathcal{E}(X'^o,P_1')$ such that $e_{\nu,i} = \tilde{e}_{\nu,i}|U_\nu$. We would be happy if we had

$$\lambda \circ (id - \tilde{e}_{\nu,i}{}^\gamma \circ \pi_o)(u_{\mu,j}') \geq 0$$

for all μ, j, and all $\gamma \in \Gamma$. Since we have only finitely many $u_{\mu,j}'$, by 9.7 this inequality fails at most if $\gamma \cdot u_{\mu,j}'$ is contained in one of finitely many half-lines. Since $u_{\mu,j}' \in \Delta$, these points cannot lie in $C^*(X'^o,P_\nu')$. Thus by 9.4 (b) there exists $\tilde{\tilde{e}}_{\nu,i} \in \mathcal{E}(X'^o,P_1')$ such that $e_{\nu,i} = \tilde{\tilde{e}}_{\nu,i}|U_\nu$, that $\lambda \circ (id - \tilde{\tilde{e}}_{\nu,i} \circ \pi_o) \geq \lambda \circ (id - \tilde{e}_{\nu,i} \circ \pi_o)$ on $C^*(X'^o,P_1')$, and that $\lambda \circ (id - \tilde{\tilde{e}}_{\nu,i} \circ \pi_o)$ is nonnegative on each of these finitely many half-lines. Hence the desired inequalities hold for $\tilde{\tilde{e}}_{\nu,i}$ in place of $\tilde{e}_{\nu,i}$.

Now we define Σ as in 9.8 using the splittings $\tilde{\tilde{e}}_{\nu,i}$. To prove $\Sigma \smallsetminus C(\mathcal{X}'^{o},P_1') = \Delta$, by Γ-invariance it suffices to prove $C^*(\mathcal{X}'^{o},P_\nu') \cap \Sigma = C^*(\mathcal{X}'^{o},P_\nu') \cap \Delta$ for every ν. But by definition

$$\begin{aligned} C^*(\mathcal{X}'^{o},P_\nu') \cap \Sigma &= \{u' \in C^*(\mathcal{X}'^{o},P_\nu') \mid \forall \gamma \in \Gamma\ \forall \mu\ \forall i\ \ \lambda \circ (id - e_{\mu,i}{}^{\gamma} \circ \pi_o)(u') \geq 0\} \\ &\subset \{u' \in C^*(\mathcal{X}'^{o},P_\nu') \mid \forall \gamma \in \Gamma_\nu\ \forall i\ \ \lambda \circ (id - e_{\nu,i}{}^{\gamma} \circ \pi_o)(u') \geq 0\} \\ &= C^*(\mathcal{X}'^{o},P_\nu') \cap \Delta, \end{aligned}$$

so one inclusion is clear. For the other inclusion it suffices to prove that all $u_{\nu,j}' \in \Sigma$, and this follows from the choice of the $\tilde{\tilde{e}}_{\mu,i}$. Finally by assumption $(P',\mathcal{X}')$ is one of the $(P_\nu',\mathcal{X}_\nu')$, so the equality $U'(\mathbb{R})(-1) \cap \Sigma = C^*(\mathcal{X}'^{o},P') \cap \Sigma = C^*(\mathcal{X}'^{o},P') \cap \Delta$ implies in particular that Σ, like Δ, contains no nontrivial linear subspace. q.e.d.

9.16. Proposition: Let $(P',\mathcal{X}') \to (P,\mathcal{X})$ be as in 9.2, $K_f' \subset P'(\mathbb{A}_f)$ an open compact subgroup, and $K_f := \pi_o(K_f')$. Then there exist admissible cone decompositions $\mathcal{S}'$ and $\mathcal{S}$ as in 9.13.

Proof: By induction we shall construct admissible cone decompositions $\mathcal{S}_1' \subsetneq \mathcal{S}_2' \subsetneq \ldots$ and $\mathcal{S}_1 \subsetneq \mathcal{S}_2 \subsetneq \ldots$ with the following property: For every rational boundary component $(P_1',\mathcal{X}_1')$ of $(P',\mathcal{X}')$, every connected component $\mathcal{X}'^{o}$ of $\mathcal{X}'$, and every $p_f' \in P'(\mathbb{A}_f)$ either $|\mathcal{S}_i'(\mathcal{X}'^{o},P_1',p_f')| \subset C^*(\mathcal{X}'^{o},P_1') \smallsetminus C(\mathcal{X}'^{o},P_1')$ and $|\mathcal{S}_i(\mathcal{X}^{o},P_1,\pi_o(p_f'))| \subset C^*(\mathcal{X}^{o},P_1) \smallsetminus C(\mathcal{X}^{o},P_1)$, or there exists Σ as in 9.8 with $\mathcal{S}_i'(\mathcal{X}'^{o},P_1',p_f') = \mathcal{S}_\Sigma'$ and $\mathcal{S}_i(\mathcal{X}^{o},P_1,\pi_o(p_f')) = \mathcal{S}_\Sigma$. Since modulo the left action of $P'(\mathbb{Q})$ and the right action of K_f' there are only finitely many triples $(\mathcal{X}'^{o},P_1',p_f')$, this sequence stops at some place, at which $\mathcal{S}_i'$ and $\mathcal{S}_i$ have the desired properties.

Suppose that either $i=0$, or $\mathcal{S}'_i$ and $\mathcal{S}_i$ are already constructed. We want to construct $\mathcal{S}'_{i+1}$ and $\mathcal{S}_{i+1}$. Choose $(\mathfrak{X}'^{o},P'_1,p'_f)$ such that $\Delta := |\mathcal{S}'_i(\mathfrak{X}'^{o},P'_1,p'_f)|$ has the property required in 9.15. Let

$$\Gamma := \mathrm{Stab}_{Q'(\mathbb{Q})}(\mathfrak{X}'^{o}) \cap p'_f\cdot K'_f\cdot p'^{-1}_f\cdot P'_1(\mathbb{A}_f),$$

then by the K'_f-admissibility of $\mathcal{S}'_i$ the set Δ is invariant under Γ. If we let Σ be as in 9.15, then by 9.14 the cone decomposition $\mathcal{S}'_{i+1}(\mathfrak{X}'^{o},P'_1,p'_f) := \mathcal{S}'_\Sigma$ contains $\mathcal{S}'_i(\mathfrak{X}'^{o},P'_1,p'_f)$, and $\mathcal{S}_{i+1}(\mathfrak{X}^{o},P_1,\pi_o(p'_f)) := \mathcal{S}_\Sigma$ contains $\mathcal{S}_i(\mathfrak{X}^{o},P_1,\pi_o(p'_f))$. By the following lemma, $\mathcal{S}'_{i+1}(\mathfrak{X}'^{o},P'_1,p'_f) \cup \mathcal{S}'_i$ and $\mathcal{S}_{i+1}(\mathfrak{X}^{o},P_1,\pi_o(p'_f)) \cup \mathcal{S}_i$ extend to a K'_f-, resp. K_f-admissible cone decompositions. By construction they have the desired properties. q.e.d.

9.17. Lemma: Let $(P,\mathfrak{X})$ be mixed Shimura data, $K_f \subset P(\mathbb{A}_f)$ an open compact subgroup, and $\mathcal{S}$ a K_f-admissible partial cone decomposition for $(P,\mathfrak{X})$. Let $(P_1,\mathfrak{X}_1)$ be a rational boundary component of $(P,\mathfrak{X})$, $\mathfrak{X}^o$ a connected component of $\mathfrak{X}$ that maps to $\mathfrak{X}_1$, and $p_f \in P(\mathbb{A}_f)$. Let Q be the parabolic subgroup of P associated to $(P_1,\mathfrak{X}_1)$, and

$$\Gamma_1 := \mathrm{Stab}_{Q(\mathbb{Q})}(\mathfrak{X}^o) \cap p_f\cdot K_f\cdot p_f^{-1}\cdot P_1(\mathbb{A}_f).$$

Let $\mathcal{S}_o$ be a rational partial polyhedral decomposition of $C^*(\mathfrak{X}^o,P_1)$, containing $\mathcal{S}(\mathfrak{X}^o,P_1,p_f)$, such that $\sigma^o \subset C(\mathfrak{X}^o,P_1)$ for every $\sigma \in \mathcal{S}_o \smallsetminus \mathcal{S}(\mathfrak{X}^o,P_1,p_f)$. Let $\mathcal{T}(\mathfrak{X}^o,P_1,p_f) := \mathcal{S}(\mathfrak{X}^o,P_1,p_f) \cup \mathcal{S}_o$. Then $\mathcal{S} \cup \mathcal{T}(\mathfrak{X}^o,P_1,p_f)$ extends to a K_f-admissible partial cone decomposition for $(P,\mathfrak{X})$ if and only if $\mathcal{S}_o$ is Γ_1-invariant.

Proof: Since the conditions 6.4 (ii)-(iv) already hold for $\mathcal{S}$, it is easily checked that they hold for $\mathcal{T}$ if and only if for all $p, p' \in P(\mathbb{Q})$, $p_{1,f}, p'_{1,f} \in P_1(\mathbb{A}_f)$, $k_f, k'_f \in K_f$, we have $p\cdot p_{1,f}\cdot\mathcal{S}_o\cdot k_f = p'\cdot p'_{1,f}\cdot\mathcal{S}_o\cdot k'_f$, if $p\cdot\mathfrak{X}^o = p'\cdot\mathfrak{X}^o$, $p\cdot p_{1,f}\cdot p_f\cdot k_f = p'\cdot p'_{1,f}\cdot p_f\cdot k'_f$, and $\mathrm{int}(p)(P_1) = \mathrm{int}(p')(P_1)$. Here

$\mathcal{S}_o$ is identified with the corresponding decomposition of $C^*(\mathcal{X}^o,P_1)\times\{p_f\}$. In terms of the original decomposition of $C^*(\mathcal{X}^o,P_1)$ this means that $p\cdot\mathcal{S}_o=p'\cdot\mathcal{S}_o$ for all $p, p' \in P(\mathbb{Q})$ such that $p'^{-1}\cdot p\cdot\mathcal{X}^o=\mathcal{X}^o$, $p'^{-1}\cdot p \in P_1(\mathbb{A}_f)\cdot p_f\cdot K_f\cdot p_f^{-1}\cdot P_1(\mathbb{A}_f)$, and $\mathrm{int}(p'^{-1}\cdot p)(P_1)=P_1$. Writing $q=p'^{-1}\cdot p$ this is equivalent to $q\cdot\mathcal{S}_o=\mathcal{S}_o$ for all $q \in \mathrm{Stab}_{Q(\mathbb{Q})}(\mathcal{X}^o) \cap P_1(\mathbb{A}_f)\cdot p_f\cdot K_f\cdot p_f^{-1}\cdot P_1(\mathbb{A}_f)$. But this set is just Γ_1, since $Q(\mathbb{Q})$ normalizes $P_1(\mathbb{A}_f)$. q.e.d.

The following analog of 5.22 permits to construct refinements of cone decompositions $\mathcal{S}$ as in 9.13.

9.18. Proposition: Let Γ be as in 9.6, and Σ as in 9.8. Consider a Γ-invariant function $f: C^*(\mathcal{X}^o,P_1) \to U_0(\mathbb{R})(-1)$, such that the restriction of $\lambda\circ f$ to every $\sigma\in\mathcal{S}_\Sigma$ is a piecewise linear convex rational function, with respect to the rational structure given by $U_1(\mathbb{Q})(-1)$. For every $\varepsilon\in\mathbb{Q}^{>0}$ define

$$T := \{u'\in C^*(\mathcal{X}'^o,P_1')\,|\,u'+\varepsilon\cdot f\circ\pi_o(u')\in\Sigma\}$$
$$= \{u'+u_0\,|\,u'\in\Sigma,\ u_0\in U_0(\mathbb{R})(-1) \text{ with } \lambda(u_0+\varepsilon\cdot f\circ\pi_o(u'))\geq 0\}.$$

If ε is sufficiently small, then T is again as in 9.8, and

$$\mathcal{S}_T = \bigcup_{\sigma\in\mathcal{S}_\Sigma} \mathcal{S}(\lambda\circ f|_\sigma).$$

Proof: Choose representatives σ_i for the finitely many Γ-conjugacy classes in $\mathcal{S}_\Sigma$ with $\mathbb{R}\cdot\sigma_i=U_1(\mathbb{R})(-1)$, and let $e_i \in \mathcal{E}(\mathcal{X}'^o,P_1')$ be the associated splittings. Let $\tau_{i,\mu}$ be the cones in $\mathcal{S}(\lambda\circ f|_{\sigma_i})$ with $\mathbb{R}\cdot\tau_{i,\mu}=U_1(\mathbb{R})(-1)$, and $\ell_{i,\mu}: U_1\to U_0$ the (unique) linear maps such that $f=\ell_{i,\mu}$ on $\tau_{i,\mu}$. Then by definition

$$T \cap \pi_o^{-1}(\tau_{i,\mu}) = \{u'\in\pi_o^{-1}(\tau_{i,\mu})\,|\,\lambda\circ(\mathrm{id}-(e_i-\varepsilon\cdot\ell_{i,\mu})\circ\pi_o)(u')\geq 0\}.$$

If ε is sufficiently small, then by 9.3 (a) $e_i-\varepsilon\cdot\ell_{i,\mu} \in \mathcal{E}(X'^o,P_1')$. To finish we need that $\lambda\circ(id-(e_i-\varepsilon\cdot\ell_{i,\mu})\circ\pi_o)$ is strictly positive on $T\setminus\pi_o^{-1}(\tau_{i,\mu})$. In fact, it then follows that T is as in 9.8 with respect to all $e_i-\varepsilon\cdot\ell_{i,\mu}$, and all $\tau_{i,\mu}$ are in $\mathcal{S}_T$. This implies that all $\mathcal{S}(\lambda\circ f|_{\sigma_i})$ are contained in $\mathcal{S}_T$, and since both $\mathcal{S}_T$ and $\bigcup_{\sigma\in\mathcal{S}_\Sigma}\mathcal{S}(\lambda\circ f|_\sigma)$ are complete rational polyhedral decompositions of $C^*(X^o,P_1)$, the desired equality follows.

Let $\mathbb{R}^{\geq 0}\cdot u_{j,\nu}$ be all one dimensional cones in $\mathcal{S}(\lambda\circ f|_{\sigma_j})$. Since

$$T\cap\pi_o^{-1}(\mathbb{R}^{\geq 0}\cdot u_{j,\nu}) = \sigma_o + \mathbb{R}^{\geq 0}\cdot(e_j-\varepsilon\cdot f)(u_{j,\nu}),$$

we have to show that for all $\gamma\in\Gamma$

$$\lambda\circ(id-(e_i-\varepsilon\cdot\ell_{i,\mu})\circ\pi_o)(\gamma\cdot(e_j-\varepsilon\cdot f)(u_{j,\nu})) > 0,$$

unless $\gamma\cdot u_{j,\nu} \in \tau_{i,\mu}$. Fix i, j, μ, ν, and $\varepsilon_o>0$ such that $e_{\tau_i}-\varepsilon\cdot\ell_{i,\nu} \in \mathcal{E}(X'^o,P_1')$ for all $\varepsilon_o>\varepsilon\in\mathbb{Q}^{>0}$ then by 9.7 each of the four inequalities

$$\lambda\circ(id-(e_i-\varepsilon_1\cdot\ell_{i,\mu})\circ\pi_o)(\gamma\cdot(e_j-\varepsilon_2\cdot f)(u_{j,\nu})) > 0$$

with $\varepsilon_1, \varepsilon_2 \in \{0,\varepsilon_o\}$ fails at most if $\gamma\cdot u_{j,\nu}$ lies in one of finitely many half-lines. This shows that there exists a finite number of $\gamma_k\in\Gamma$, such that for every $\varepsilon_o>\varepsilon\in\mathbb{Q}^{>0}$ the inequality above fails at most if $\gamma\cdot u_{j,\nu} \in \mathbb{R}^{\geq 0}\cdot\gamma_k\cdot u_{j,\nu}$ for some k. Thus we are reduced to showing the inequality for some fixed $\gamma\cdot u_{j,\nu}\notin\tau_{i,\mu}$. If $\gamma\cdot u_{j,\nu}\notin\sigma_i$, then

$$\lambda\circ(id-e_i\circ\pi_o)(\gamma\cdot e_j(u_{j,\nu})) > 0$$

by the definition of $\mathcal{S}_\Sigma$, so the inequality certainly holds if ε is sufficiently small. Otherwise $\gamma\cdot u_{j,\nu}\in\sigma_i$ implies that $\mathbb{R}^{\geq 0}\cdot\gamma\cdot u_{j,\nu} \in \mathcal{S}(\lambda\circ f|_{\sigma_i})$, so we may assume that $j=i$ and $\gamma=1$. Then we have

$$\lambda\circ(id-(e_i-\varepsilon\cdot\ell_{i,\mu})\circ\pi_o)\circ(e_i-\varepsilon\cdot f)(u_{i,\nu}) = \varepsilon\cdot\lambda\circ(\ell_{i,\mu}-f)(u_{i,\nu}),$$

so the assertion is equivalent to $\lambda\circ(\ell_{i,\mu}-f)(u_{i,\nu}) > 0$. But by the definition of $\tau_{i,\mu}$ and $\ell_{i,\mu}$ this inequality holds on all of $\sigma_i-\tau_{i,\mu}$, as desired. q.e.d.

<u>9.19. Proposition</u>: Let $(P',\mathfrak{X}')\to(P,\mathfrak{X})$, K_f', K_f, $\mathcal{S}'$, and $\mathcal{S}$ be as in 9.13. Let $\tilde{\mathcal{S}}$ be a refinement of $\mathcal{S}$. Then there exist $\mathcal{T}'$ and $\mathcal{T}$ satisfying the same conditions of 9.13, such that $\mathcal{T}$ is a refinement of $\tilde{\mathcal{S}}$.

<u>Proof</u>: The essential points of the proof are 5.21 and 9.18. To be able to use 9.18 we need a function $f\colon \mathfrak{C}(P,\mathfrak{X})\times P(\mathbb{A}_f)\to U_0(\mathbb{R})(-1)$ with the following properties:

(i) The restriction of $\lambda\circ f$ to every $\sigma\in\mathcal{S}$ is a piecewise linear convex rational function.

(ii) For every $\sigma\in\mathcal{S}$, $\mathcal{S}(\lambda\circ f|_\sigma)$ is a refinement of $\{\tau\in\tilde{\mathcal{S}}\,|\,\tau\subset\sigma\}$.

(iii) f is invariant under the three actions 6.4 (ii)-(iv).

Suppose that such a function is given. For every $(\mathfrak{X}'^{o},P_1',p_f')$ let $\Sigma := |\mathcal{S}'(\mathfrak{X}'^{o},P_1',p_f')|$, define T as in 9.18, and let $\mathcal{T}'(\mathfrak{X}'^{o},P_1',p_f') := \mathcal{S}_T'$ and $\mathcal{T}(\mathfrak{X}^{o},P_1,\pi_o(p_f')) := \mathcal{S}_T$. By 9.18 we can make ε so small that any finite number of $\mathcal{T}'(\mathfrak{X}'^{o},P_1',p_f')$ and $\mathcal{T}(\mathfrak{X}^{o},P_1,\pi_o(p_f'))$ satisfy all our requirements. Since by assumption this construction of $\mathcal{T}'$ and $\mathcal{T}$ is invariant under the actions in 6.4 (ii)-(iv), and there are only finitely many triples $(\mathfrak{X}'^{o},P_1',p_f')$ modulo this action, there exists ε so that this works for all such triples, as desired.

We construct the desired function by induction over the classes of cones in $\mathcal{S}$ modulo the three operations 6.4 (ii)-(iv). If it has not yet been defined on all of these, choose $(\mathfrak{X}'^{o},P_1',p_f')$ and $\sigma\in\mathcal{S}(\mathfrak{X}^{o},P_1,\pi_o(p_f'))$ with $\sigma^{o}\subset C(\mathfrak{X}^{o},P_1)$, such that f is already defined on every proper face of σ, but not on σ itself. By 5.21 there exists a piecewise linear convex rational function $g\colon\sigma\to\mathbb{R}$, which coincides with $\lambda\circ f$ on every proper face of σ, such that $\mathcal{S}(g)$ is a refinement of $\{\tau\in\tilde{\mathcal{S}}\,|\,\tau\subset\sigma\}$.

Let us check what the invariance conditions say about the desired extension of f. Let $\Gamma_1 := \mathrm{Stab}_{Q'(\mathbb{Q})}(\mathfrak{X}'^{o}) \cap P_1'(\mathbb{A}_f)\cdot p_f'\cdot K_f'\cdot p_f'^{-1}$. It is easy to see that any extension of f to σ can be extended invariantly to the whole conjugacy class of σ if and only if $f|_\sigma$ is invariant under the action of $\mathrm{Stab}_{\Gamma_1}(\sigma)$. Let a_γ be the scalar through which an element $\gamma\in\Gamma_1$ acts on U_0, by 9.2 this scalar is positive. Since $\lambda\circ f|_\sigma$ is to be a piecewise linear convex rational function, we have $\lambda\circ f^\gamma(u) = \lambda(a_\gamma^{-1}\cdot f(\gamma\cdot u)) = \lambda\circ f(a_\gamma^{-1}\cdot\gamma\cdot u)$ for all $u\in\sigma$, so we want $\lambda\circ f|_\sigma$ to be invariant under the twisted action $(\gamma,u)\mapsto a_\gamma^{-1}\cdot\gamma\cdot u$ of $\mathrm{Stab}_{\Gamma_1}(\sigma)$. Since P_1' acts through scalars on U_1', it acts trivially under this twisted action. But from 6.19 (a) it follows that $\mathrm{Stab}_{\Gamma_1}(\sigma)$ is finite modulo $P_1'(\mathbb{Q})\cap\Gamma_1$, so all we need is that $\lambda\circ f|_\sigma$ is invariant under a finite group of automorphisms of $U_1(\mathbb{Q})(-1)$ that fixes σ. Now the function $g\colon\sigma\to\mathbb{R}$ above is not necessarily invariant. But by assumption it coincides with $\lambda\circ f$ on $\sigma\setminus\sigma^0$, so at least on this subset it is invariant. Thus if we replace g by the average of all its conjugates under that finite group, then it is invariant while still satisfying the other conditions. If we now define f to be equal to $\lambda^{-1}\circ g$ on σ, then we can extend it to the whole conjugacy class of σ, as desired. q.e.d.

9.20. Proposition: In 9.19 we can achieve in addition that $\mathcal{S}$ is smooth with respect to K_f, and that the condition 7.12 (*) is satisfied. Then in particular $M^{K_f}(P,\mathfrak{X},\mathcal{S})(\mathbb{C})$ is smooth, projective, and the complement $M^{K_f}(P,\mathfrak{X},\mathcal{S})(\mathbb{C})\setminus M^{K_f}(P,\mathfrak{X})(\mathbb{C})$ is a union of smooth divisors with only normal crossings.

Proof: (For the smoothness compare [N] thm. 7.20, thm. 7.26.) If in the proof of 9.19 we construct f using the strengthening 5.23 of 5.21, then the resulting cone decomposition is smooth. To prove that in addition the condition 7.12 (*) can be satisfied, we may therefore assume that $\tilde{\mathcal{S}}=\mathcal{S}$, and is already smooth. We want to apply the same procedure as in the proof of 9.19. Using 5.25 we can choose the new function f such that the associated refinement $\mathcal{T}$ of $\mathcal{S}$ is the barycentric subdivision of

$\mathcal{S}$ with respect to the lattices given in 6.4. By 5.24 $\mathcal{T}$ is again smooth, and it remains to show that it satisfies the condition 7.12 (*).

For this choose $\tau \in \mathcal{T}(\mathfrak{X}^o, P_2, p_f)$ as in 7.12, let τ' be a face of τ that lies in $\mathcal{T}(\mathfrak{X}^o, P_1, p_f)$, and suppose that $p \cdot p_{1,f} \cdot \tau' \cdot k_f$ is also a face of τ. We have to show that $p \cdot p_{1,f} \cdot \tau' \cdot k_f = \tau'$. Choose $\sigma \in \mathcal{S}(\mathfrak{X}^o, P_2, p_f)$ so that $\tau^o \subset \sigma^o$, and $\sigma' \in \mathcal{S}(\mathfrak{X}^o, P_1, p_f)$ so that $\tau'^o \subset \sigma'^o$. Then by assumption $(p \cdot p_{1,f} \cdot \sigma' \cdot k_f)^o \supset (p \cdot p_{1,f} \cdot \tau' \cdot k_f)^o \subset \tau \subset \sigma$, which implies $(p \cdot p_{1,f} \cdot \sigma' \cdot k_f)^o \cap \sigma = \emptyset$, so $p \cdot p_{1,f} \cdot \sigma' \cdot k_f$ is a face of σ. Now we have an isomorphism $p \cdot p_{1,f} \cdot \sigma' \cdot k_f \cong \sigma'$, and by assumption both $p \cdot p_{1,f} \cdot \tau' \cdot k_f$ and τ' are faces of τ, which occurs in the barycentric subdivision of σ. As explained in 5.24 it follows that $p \cdot p_{1,f} \cdot \tau' \cdot k_f = \tau'$, as desired. q.e.d.

9.21. Theorem: Let $(P, \mathfrak{X})$ be arbitrary mixed Shimura data, and $K_f \subset P(\mathbb{A}_f)$ a neat open compact subgroup. There exists a K_f-admissible complete cone decomposition $\mathcal{S}$ for $(P, \mathfrak{X})$ such that $M^{K_f}(P, \mathfrak{X}, \mathcal{S})(\mathbb{C})$ is smooth, projective, and the complement $M^{K_f}(P, \mathfrak{X}, \mathcal{S})(\mathbb{C}) \setminus M^{K_f}(P, \mathfrak{X})(\mathbb{C})$ is a union of smooth divisors with only normal crossings. Moreover if $\mathcal{T}$ is any K_f-admissible complete cone decomposition for $(P, \mathfrak{X})$, then $\mathcal{S}$ can be chosen to be a refinement of $\mathcal{T}$. Some ample line bundle on $M^{K_f}(P, \mathfrak{X}, \mathcal{S})(\mathbb{C})$ can be described by algebraic constructions involving ω[dlog] and line bundles as in 8.13.

Proof: We shall reduce the assertion to 9.20. The problem is to relate $(P, \mathfrak{X})$ with mixed Shimura data as in 9.2. First let $(\tilde{P}, \tilde{\mathfrak{X}}) := (P, \mathfrak{X}) \times (\mathbb{G}_{m,\mathbb{Q}}, \mathcal{H}_0)$. Consider the open compact subgroup $(\hat{\mathbb{Z}})^\times$ of $\mathbb{G}_m(\mathbb{A}_f)$, then

$$M^{(\hat{\mathbb{Z}})^\times}(\mathbb{G}_{m,\mathbb{Q}}, \mathcal{H}_0)(\mathbb{C}) = \mathbb{Q}^\times \backslash \mathcal{H}_0 \times (\mathbb{G}_m(\mathbb{A}_f)/(\hat{\mathbb{Z}})^\times) \cong \{\pm 1\} \backslash \mathcal{H}_0$$

is just a single point. Let $\tilde{K}_f := K_f \times (\hat{\mathbb{Z}})^\times$, then there is an obvious one-to-one relation between all K_f-admissible partial cone decompositions $\mathcal{S}$ for

$(P,\mathfrak{X})$ and all $\tilde{K}_f$-admissible partial cone decompositions $\tilde{\mathcal{S}}$ for $(\tilde{P},\tilde{\mathfrak{X}})$, such that the projection $(\tilde{P},\tilde{\mathfrak{X}}) \to (P,\mathfrak{X})$ induces isomorphisms

$$M^{\tilde{K}_f}(\tilde{P},\tilde{\mathfrak{X}},\tilde{\mathcal{S}})(\mathbb{C}) \cong M^{K_f}(P,\mathfrak{X},\mathcal{S})(\mathbb{C}) \times M^{(\hat{\mathbb{Z}})^\times}(\mathbb{G}_{m,\mathbb{Q}},\mathcal{H}_0)(\mathbb{C}) \xrightarrow{\sim} M^{K_f}(P,\mathfrak{X},\mathcal{S})(\mathbb{C}).$$

This reduces the assertion to that for $(\tilde{P},\tilde{\mathfrak{X}})$ and $\tilde{K}_f$.

Next let $(\tilde{P}_{(0)},\tilde{\mathfrak{X}}_{(0)}) \hookrightarrow (\tilde{P},\tilde{\mathfrak{X}})$ be an irreducible component. Since the other projection defines a morphism $(\tilde{P}_{(0)},\tilde{\mathfrak{X}}_{(0)}) \to (\mathbb{G}_{m,\mathbb{Q}},\mathcal{H}_0)$, we can find a $((P_0,\mathfrak{X}_0)\to(\mathbb{G}_{m,\mathbb{Q}},\mathcal{H}_0))$-torsor $(\tilde{P}'_{(0)},\tilde{\mathfrak{X}}'_{(0)})\to(\tilde{P}_{(0)},\tilde{\mathfrak{X}}_{(0)})$ as in 9.1 (a). Let $P_* := \tilde{P}_{(0)}\cdot Z(\tilde{P})^0 \subset \tilde{P}$, and $(P_*,\mathfrak{X}_*) \hookrightarrow (\tilde{P},\tilde{\mathfrak{X}})$ be the corresponding embedding as in 2.13. Since the extension class of the torsor $(\tilde{P}'_{(0)},\tilde{\mathfrak{X}}'_{(0)}) \to(\tilde{P}_{(0)},\tilde{\mathfrak{X}}_{(0)})$ is fixed by $Z(\tilde{P})$, we can extend it to a torsor $(P'_*,\mathfrak{X}'_*)\to (P_*,\mathfrak{X}_*)$. This torsor satisfies the requirements of 9.2. Now by 7.10 we have an isomorphism

$$\coprod_\nu \Delta_\nu\backslash M^{K_f^\nu}(P_*,\mathfrak{X}_*,\mathcal{S}_\nu)(\mathbb{C}) \xrightarrow{\coprod[\,\cdot p_f^\nu]\cdot[\varphi]} M^{\tilde{K}_f}(\tilde{P},\tilde{\mathfrak{X}},\mathcal{S})(\mathbb{C})$$

with finitely many $p_f^\nu \in \tilde{P}(\mathbb{A}_f)$, $K_f^\nu := P_*(\mathbb{A}_f)\cap p_f^\nu\cdot K_f\cdot(p_f^\nu)^{-1}$, $\mathcal{S}_\nu := ([\cdot p_f^\nu]^*\tilde{\mathcal{S}})|_{(P_*,\mathfrak{X}_*)}$, and

$$\Delta_\nu := (\mathrm{Stab}_{\tilde{P}(\mathbb{Q})}(\mathfrak{X}_*) \cap (P_*(\mathbb{A}_f)\cdot p_f^\nu\cdot \tilde{K}_f\cdot(p_f^\nu)^{-1}))/P_*(\mathbb{Q}).$$

We claim that in our situation we have $\Delta_\nu=1$. In fact, since $\tilde{P}=P\times\mathbb{G}_{m,\mathbb{Q}}$, we have $\{1\}\times\mathbb{G}_{m,\mathbb{Q}}\subset Z(\tilde{P})^0\subset P_*$. Thus the image of $P_*(\mathbb{A}_f)\cdot p_f^\nu\cdot\tilde{K}_f\cdot(p_f^\nu)^{-1}$ in $(P/P_*)(\mathbb{A}_f)$ is equal to that of $K_f \subset \tilde{K}_f = K_f\times(\hat{\mathbb{Z}})^\times$, which by 0.6 is neat. As in the proof of 7.10 it follows that Δ_ν is contained in a neat arithmetic subgroup of $(P/P_*)(\mathbb{Q})$, which by 2.1 (viii) must be trivial.

Now for any fixed collection of p_f^ν, as before there is an obvious one-to-one relation between $\tilde{K}_f$-admissible cone decompositions $\tilde{\mathcal{S}}$ for $(\tilde{P},\tilde{\mathfrak{X}})$, and collections of K_f^ν-admissible partial cone decomposition $\mathcal{S}_\nu$

for $(P_*, \mathfrak{X}_*)$. Thus the assertion is reduced to $(P_*, \mathfrak{X}_*)$. But in that case it follows from 9.20. q.e.d.

9.22. Functoriality of cone decompositions: Consider a map $[\varphi]$: $M^{K_f^1}(P_1,\mathfrak{X}_1)(\mathbb{C}) \to M^{K_f^1}(P_2,\mathfrak{X}_2)(\mathbb{C})$ of 3.4 (b). Let $\mathcal{S}_i$ be a K_f^i-admissible cone decomposition for every $(P_i,\mathfrak{X}_i)$. Then we define another K_f^1-admissible partial cone decomposition $\mathcal{T}_1$ for $(P_1,\mathfrak{X}_1)$ as follows. Every rational boundary component $(P_{11},\mathfrak{X}_{11})$ of $(P_1,\mathfrak{X}_1)$ determines a rational boundary component $(P_{21},\mathfrak{X}_{21})$ of $(P_2,\mathfrak{X}_2)$, and every connected component $\mathfrak{X}_1^0$ of $\mathfrak{X}_1$ maps to a unique connected component $\mathfrak{X}_2^0$ of $\mathfrak{X}_2$. The homomorphism $\varphi: U_{11} \to U_{21}$ then maps $C^*(\mathfrak{X}_1^0,P_{11})$ to $C^*(\mathfrak{X}_2^0,P_{21})$. For all such $(P_1,\mathfrak{X}_1)$ and $\mathfrak{X}_1^0$, and every $p_f \in P_1(\mathbb{A}_f)$, we define

$$\mathcal{T}_1(\mathfrak{X}_1^0,P_{11},p_{1,f}) := \{\sigma_1 \cap \varphi^{-1}(\sigma_2) \,|\, \sigma_1 \in \mathcal{S}_1(\mathfrak{X}_1^0,P_{11},p_f),\ \sigma_2 \in \mathcal{S}_2(\mathfrak{X}_2^0,P_{21},\varphi(p_f))\}.$$

It is clear from the definition that $\mathcal{T}_1$ is a K_f^1-admissible partial cone decomposition for $(P_1,\mathfrak{X}_1)$. If $\mathcal{S}_1$ and $\mathcal{S}_2$ are finite or complete, then $\mathcal{T}_1$ has the same property. Moreover $\mathcal{T}_1 = \mathcal{S}_1$ if and only if the condition in 6.25 (b) holds for the pair $(\mathcal{S}_1,\mathcal{S}_2)$. In general by 6.25 (b) we have the following maps

$$M^{K_f^1}(P_1,\mathfrak{X}_1,\mathcal{S}_1)(\mathbb{C}) \longleftarrow M^{K_f^1}(P_1,\mathfrak{X}_1,\mathcal{T}_1)(\mathbb{C}) \longrightarrow M^{K_f^2}(P_2,\mathfrak{X}_2,\mathcal{S}_2)(\mathbb{C}).$$

A particular case is the identity map on $(P_1,\mathfrak{X}_1)$, where $\mathcal{T}_1$ is the coarsest common refinement of $\mathcal{S}_1$ and $\mathcal{S}_2$. If $(P_2,\mathfrak{X}_2)=(P_1,\mathfrak{X}_1)$ and $p_f \in P_1(\mathbb{A}_f)$, then applying this to $\mathcal{S}_1$ and $[\cdot p_f]^*\mathcal{S}_2$ yields the analogous statements for the morphism $[\cdot p_f]$ of 3.4 (a). The same arguments work for any finite number of morphisms of 3.4 (a) and (b).

9.23. Corollary: In 9.21 suppose that in addition a finite number of maps $M^{K_f}(P,\mathfrak{X})(\mathbb{C}) \to M^{K_f^i}(P_i,\mathfrak{X}_i)(\mathbb{C})$ of 3.4 (a) and (b) are given, and for every i a complete K_f^i-admissible cone decomposition $\mathcal{S}_i$ for $(P_i,\mathfrak{X}_i)$. Then one can choose $\mathcal{S}$ such that all the maps $M^{K_f}(P,\mathfrak{X},\mathcal{S})(\mathbb{C}) \to M^{K_f^i}(P_i,\mathfrak{X}_i,\mathcal{S}_i)(\mathbb{C})$ (of 6.25 (a) and (b)) are defined.

Proof: By 9.22 there exists a complete K_f-admissible cone decomposition $\mathcal{T}$ for $(P,\mathfrak{X})$ such that the conditions in 6.25 (a) and (b) are verified for all the morphisms $M^{K_f}(P,\mathfrak{X},\mathcal{T})(\mathbb{C}) \to M^{K_f^i}(P_i,\mathfrak{X}_i,\mathcal{S}_i)(\mathbb{C})$. If in 9.21 $\mathcal{S}$ is a refinement of $\mathcal{T}$, then the same holds for $\mathcal{S}$ in place of $\mathcal{T}$, as desired. q.e.d.

9.24. Proposition: For every mixed Shimura data $(P,\mathfrak{X})$, and every open compact subgroup $K_f \subset P(\mathbb{A}_f)$, the mixed Shimura variety $M^{K_f}(P,\mathfrak{X})(\mathbb{C})$ possesses a canonical structure of a normal quasiprojective algebraic variety over $\mathbb{C}$, such that all the maps of 3.4 become algebraic morphisms. Whenever $\mathcal{S}$ is a complete K_f-admissible cone decomposition for $(P,\mathfrak{X})$ such that $M^{K_f}(P,\mathfrak{X},\mathcal{S})(\mathbb{C})$ is projective, then the open embedding $M^{K_f}(P,\mathfrak{X})(\mathbb{C}) \hookrightarrow M^{K_f}(P,\mathfrak{X},\mathcal{S})(\mathbb{C})$ is algebraic.

Proof: Let $\mathcal{S}$ be a complete K_f-admissible cone decomposition for $(P,\mathfrak{X})$ such that $M^{K_f}(P,\mathfrak{X},\mathcal{S})(\mathbb{C})$ is projective. Since by 6.25 (a) the complement of $M^{K_f}(P,\mathfrak{X})(\mathbb{C})$ in $M^{K_f}(P,\mathfrak{X},\mathcal{S})(\mathbb{C})$ is a closed analytic set, by Chow's theorem (see [S] §19 prop.13) it is a Zariski-closed subset. Thus its complement $M^{K_f}(P,\mathfrak{X})(\mathbb{C})$ inherits the structure of a quasiprojective variety over $\mathbb{C}$.

To prove the functoriality of this algebraic structure consider other mixed Shimura data $(P^*,\mathfrak{X}^*)$, an open compact subgroup $K_f^* \subset P^*(\mathbb{A}_f)$, and a complete K_f^*-admissible cone decomposition for $(P^*,\mathfrak{X}^*)$, such that $M^{K_f^*}(P^*,\mathfrak{X}^*,\mathcal{S}^*)(\mathbb{C})$ is projective. Consider a map $M^{K_f}(P,\mathfrak{X})(\mathbb{C}) \to M^{K_f^*}(P^*,\mathfrak{X}^*)(\mathbb{C})$ of 3.4 (a) or (b). Choose a neat open normal subgroup $K_f^+ \subset K_f$, then by 9.23 there exists a refinement $\mathcal{T}$ of $\mathcal{S}$, such that the map $M^{K_f^+}(P,X,\mathcal{T})(\mathbb{C}) \to M^{K_f^*}(P^*,X^*,\mathcal{S}^*)(\mathbb{C})$ of 6.25 (a) or (b) is defined. Then the maps

$$M^{K_f}(P,X,\mathcal{S})(\mathbb{C}) \xleftarrow{[\mathrm{id}]} M^{K_f^+}(P,X,\mathcal{T})(\mathbb{C}) \longrightarrow M^{K_f^*}(P^*,X^*,\mathcal{S}^*)(\mathbb{C})$$

are holomorphic maps between projective complex spaces, hence algebraic. In particular the maps

$$M^{K_f}(P,X)(\mathbb{C}) \xleftarrow{[\mathrm{id}]} M^{K_f^+}(P,X)(\mathbb{C}) \longrightarrow M^{K_f^*}(P^*,X^*)(\mathbb{C})$$

are algebraic with respect to the induced algebraic structures. Recall that by 6.25 (a) $M^{K_f}(P,X)(\mathbb{C})$ is as complex space the quotient of $M^{K_f^+}(P,X)(\mathbb{C})$ by the finite group K_f/K_f^+. Since the map on the right hand side factors through $M^{K_f}(P,X)(\mathbb{C})$, the map $M^{K_f}(P,X)(\mathbb{C}) \to M^{K_f^*}(P^*,X^*)(\mathbb{C})$ is also algebraic, as desired.

A special case of the functoriality is that the algebraic structure on $M^{K_f}(P,X)(\mathbb{C})$ is independent of the choice of $\mathcal{S}$. Thus the assertions are already proved whenever there exists a complete K_f-admissible cone decomposition for (P,X) such that $M^{K_f}(P,X,\mathcal{S})(\mathbb{C})$ is projective. By 9.21 this is the case if K_f is neat. For arbitrary K_f choose a neat open normal subgroup $K_f^+ \subset K_f$. By the functoriality for $M^{K_f^+}(P,X)(\mathbb{C})$ the action of K_f/K_f^+ respects the canonical algebraic structure on $M^{K_f^+}(P,X)(\mathbb{C})$. Since this is a quasiprojective variety, and by [S⌐A1] exp.VIII cor. 7.7 the quotient of every quasiprojective algebraic variety by a finite group exists and is again quasiprojective, we get a canonical quasiprojective algebraic structure on $M^{K_f}(P,X)(\mathbb{C})$. The functoriality for this algebraic structure follows by the same arguments as above, in particular it does not depend on the choice of K_f^+. q.e.d.

9.25. Definition: For every mixed Shimura data (P,X) and every open compact subgroup $K_f \subset P(\mathbb{A}_f)$ let

$$M^{K_f}_{\mathbb{C}}(P,\mathfrak{X})$$

be the normal algebraic variety over $\mathbb{C}$ associated by 9.24 to the normal complex space $M^{K_f}(P,\mathfrak{X})(\mathbb{C})$. Let $\mathcal{S}$ be a K_f-admissible partial cone decomposition for $(P,\mathfrak{X})$. If there exists an algebraic structure on $M^{K_f}(P,\mathfrak{X},\mathcal{S})(\mathbb{C})$ extending the above algebraic structure on $M^{K_f}(P,\mathfrak{X})(\mathbb{C})$, we let

$$M^{K_f}_{\mathbb{C}}(P,\mathfrak{X},\mathcal{S})$$

be the associated normal algebraic variety over $\mathbb{C}$. Furthermore if P is reductive, then we denote the normal projective variety corresponding to the Baily-Borel compactification $M^{K_f}(P,\mathfrak{X})^*(\mathbb{C})$ by

$$M^{K_f}_{\mathbb{C}}(P,\mathfrak{X})^*.$$

In all these cases the variety is, strictly speaking, defined by a universal property, so it really consists of a complex variety X together with an isomorphism $X(\mathbb{C})\cong M^{K_f}(P,\mathfrak{X})(\mathbb{C})$, resp. $X(\mathbb{C})\cong M^{K_f}(P,\mathfrak{X},\mathcal{S})(\mathbb{C})$, resp. $X(\mathbb{C})\cong M^{K_f}(P,\mathfrak{X})^*(\mathbb{C})$. We have to prove that it is unique up to canonical isomorphism. For $M^{K_f}_{\mathbb{C}}(P,\mathfrak{X})$ this follows from 9.24, for $M^{K_f}_{\mathbb{C}}(P,\mathfrak{X})^*$ by projectivity. For $M^{K_f}_{\mathbb{C}}(P,\mathfrak{X},\mathcal{S})$ recall that $M^{K_f}(P,\mathfrak{X})(\mathbb{C})$ is dense in $M^{K_f}(P,\mathfrak{X},\mathcal{S})(\mathbb{C})$, hence Zariski-dense with respect to any algebraic structure. It is easily seen that an analytic map between two normal complex varieties is an algebraic morphism if it is an algebraic morphism on a Zariski-open dense subset. Thus $M^{K_f}_{\mathbb{C}}(P,\mathfrak{X},\mathcal{S})$ is well-defined whenever such an algebraic structure exists.

By 9.24 we already know that all the maps 3.4 (a) and (b) correspond to unique algebraic morphisms between different $M^{K_f}_{\mathbb{C}}(P,\mathfrak{X})^*$. By the argument above, the same holds for the maps 6.25 (a) and (b), if $M^{K_f}_{\mathbb{C}}(P,\mathfrak{X},\mathcal{S})$ etc. exists. For the Baily-Borel compactification the same follows by projectivity. Moreover for arbitrary P, by the argument above the map $[\pi]^*$ defined in 6.24 corresponds to a morphism

$M_{\mathbb{C}}^{K_f}(P,\mathfrak{X},\mathfrak{S}) \to M_{\mathbb{C}}^{\pi(K_f)}((P,\mathfrak{X})/W)^*$ whenever $M_{\mathbb{C}}^{K_f}(P,\mathfrak{X},\mathfrak{S})$ exists. In all these cases we shall use the same symbols to denote the corresponding algebraic morphisms.

The assertions of 6.25 (functoriality, open embeddings, isomorphisms, quotients by finite groups), 6.26 (smoothness), 8.6 (line bundle structure), 8.13-14 (ampleness) are equally valid for these algebraic varieties, since these are properties that translate directly between the algebraic and the analytic category, once all the objects and maps are algebraic.

9.26. Remark: From the proof of 9.24 it follows that the system of algebraic structures on all $M^{K_f}(P,\mathfrak{X})(\mathbb{C})$ is uniquely determined by the properties expressed in 9.24. One may ask whether these algebraic structures can be characterized uniquely without reference to toroidal compactifications. As it turns out, they are already characterized uniquely by the functoriality with respect to all the maps 3.4.

Let us briefly indicate why. If P is reductive and possesses no almost direct factor of type $SL_{2,\mathbb{Q}}$, then the boundary of $M^{K_f}(P,\mathfrak{X})(\mathbb{C})$ in the Baily-Borel compactification has codimension ≥ 2, so it possesses at most one algebraic structure. For $P=GL_{2,\mathbb{Q}}$ the uniqueness follows via an embedding $(GL_{2,\mathbb{Q}},\mathcal{H}_2) \hookrightarrow (CSp_{4,\mathbb{Q}},\mathcal{H}_4)$, as in the proof of 8.4. Thus we have uniqueness whenever P is reductive. Next, if $U=1$, then $M^{K_f}(P,\mathfrak{X})(\mathbb{C}) \to M^{\pi(K_f)}((P,\mathfrak{X})/W)(\mathbb{C})$ is proper, so again the uniqueness follows by functoriality. Finally observe that every torus possesses a unique algebraic structure. Thus for arbitrary P, and sufficiently small K_f, we may use the torsor structure on $M^{K_f}(P,\mathfrak{X})(\mathbb{C}) \to M^{\pi'(K_f)}((P,\mathfrak{X})/U)(\mathbb{C})$ to get the result.

<u>9.27. Remarks about the existence of $M_{\mathbb{C}}^{K_f}(P,X,\mathcal{S})$</u>: (a) By the definition in 9.25, $M_{\mathbb{C}}^{K_f}(P,X,\mathcal{S})$ exists whenever $M^{K_f}(P,X,\mathcal{S})(\mathbb{C})$ is projective, in particular in the situations of 8.13, 9.13, or 9.21. If $M_{\mathbb{C}}^{K_f}(P,X,\mathcal{S})$ exists, then the same follows for every K_f-admissible partial cone decomposition contained in $\mathcal{S}$. If $\mathcal{S}$ is the union of any (finite or infinite) collection $\mathcal{S}_i$, such that $M_{\mathbb{C}}^{K_f}(P,X,\mathcal{S}_i)$ exists, then by gluing it follows that $M_{\mathbb{C}}^{K_f}(P,X,\mathcal{S})$ exists. These three facts, together with 9.16-21, show that $M_{\mathbb{C}}^{K_f}(P,X,\mathcal{S})$ exists for a large class of cone decompositions.

(b) In the situation of 8.13 (e.g. 9.13) not only $M_{\mathbb{C}}^{K_f}(P,X,\mathcal{S})$, but also the "line bundle" $M_{\mathbb{C}}^{K_f'}(P',X',\mathcal{S}')$ exists as an algebraic variety and is quasiprojective. In fact, by ampleness one can construct a canonical ample line bundle on the associated $\mathbb{P}^1(\mathbb{C})$-bundle obtained by adding the ∞-section. This, too, can be expressed in the formalism of chapter 8.

(c) The knowledge so far about the existence of $M_{\mathbb{C}}^{K_f}(P,X,\mathcal{S})$ is probably sufficient for most purposes. Nevertheless it seems somewhat unsatisfactory to stop here and not try to find out more about the algebraicity of our toroidal compactifications. Therefore we include more results in this direction, although the proof of 9.29 is rather technical. See also 9.35 below.

First we show (using 9.20) that $M_{\mathbb{C}}^{K_f}(P,X,\mathcal{S})$ exists whenever $\mathcal{S}$ is sufficiently "fine".

<u>9.28. Proposition</u>: Let $(P',X')\to(P,X)$, K_f', K_f, $\mathcal{S}'$, and $\mathcal{S}$ be as in 9.13. Assume that the condition 7.12 (*) is satisfied for $\mathcal{S}$. Let $\mathcal{T}$ be a K_f-admissible partial cone decomposition for (P,X) such that every $\tau\in\mathcal{T}$ is contained in some $\sigma\in\mathcal{S}$. Then $M_{\mathbb{C}}^{K_f}(P,X,\mathcal{T})$ exists. In fact there exists a collection of $\overline{\mathcal{T}}_i'$ and $\overline{\mathcal{T}}_i$ as in 9.13, and $\mathcal{T}_i\subset\overline{\mathcal{T}}_i$, such that $\mathcal{T}$ is the union of all $\mathcal{T}_i$.

Proof: It suffices to prove that for every convex rational polyhedral cone τ contained in some $\sigma\in\mathcal{S}$ there exist $\bar{\mathcal{T}}'$ and $\bar{\mathcal{T}}$ as in 9.13 such that $\tau\in\bar{\mathcal{T}}$. Using the same procedure as in the proof of 9.19 it suffices to construct the function f such that $\tau\in\mathcal{S}(\lambda\circ f|_\sigma)$. To do this choose a piecewise linear convex rational function $g\colon\sigma\to\mathbb{R}$ such that $\tau\in\mathcal{S}(g)$. To show that such a function exists, take any linear rational function $\ell\colon\sigma\to\mathbb{R}$ that is strictly positive on $\sigma\setminus\{0\}$. Then let g be the smallest nonnegative convex function $\sigma\to\mathbb{R}^{\geq 0}$ which is $\geq\ell$ on τ. It is clear that $g=\ell$ on τ, and $g>\ell$ on $\sigma\setminus\tau$, hence $\tau\in\mathcal{S}(g)$, as desired.

Now define $f|_\sigma := \lambda^{-1}\circ g\colon \sigma\to U_0(\mathbb{R})(-1)$. The condition 7.12 (*) implies that no two different faces of σ are equivalent under the actions 6.4 (ii)-(iv). This in turn implies that whenever these actions induce an automorphism on a face ρ of σ, then this automorphism maps every face of ρ to itself. But by 6.19 (a) and 9.6 some power of this automorphism acts like a scalar on ρ. This shows that the automorphism acts by the same scalar on every one dimensional face of ρ, hence it acts like a scalar on ρ. Thus there is no obstruction to extending $f|_\sigma$ invariantly to all conjugates of σ under the actions 6.4 (ii)-(iv). Finally by the proof of 9.19 (with $\tilde{\mathcal{S}}=\mathcal{S}$) it can be extended to all of $\mathfrak{C}(P,\mathfrak{X})\times P(\mathbb{A}_f)\to U_0(\mathbb{R})(-1)$ in the desired way. q.e.d.

9.29. Proposition: Let $(P',\mathfrak{X}')\to(P,\mathfrak{X})$ be as in 9.2, $(P_1,\mathfrak{X}_1)$ a rational boundary component of $(P,\mathfrak{X})$, and $\mathfrak{X}^0$ a connected component of $\mathfrak{X}$ that maps into $\mathfrak{X}_1$. Let $p_f\in P(\mathbb{A}_f)$, and σ a convex rational polyhedral cone in $C(\mathfrak{X}^0,P_1)$ that does not contain a nontrivial linear subspace. Then there exists an open compact subgroup $K_f'\subset P'(\mathbb{A}_f)$ and $\mathcal{S}$ and $\mathcal{S}'$ as in 9.13, such that $\sigma\in\mathcal{S}(\mathfrak{X}^0,P_1,p_f)$.

Remark: Incidentally, this shows that the cones in a complete admissible decomposition can be arbitrarily large.

Proof: Using the operation 6.5 (a) we may assume that $p_f=1$. Without loss of generality we may assume that $\mathbb{R}\cdot\sigma=U_1(\mathbb{R})(-1)$, since otherwise we may (repeatedly) replace σ by $\sigma+\mathbb{R}^{\geq 0}\cdot u'$ for arbitrary $u'\in C^*(\mathcal{X}^o,P_1)\cap U_1(\mathbb{Q})(-1)\setminus\mathbb{R}\cdot\sigma$. Let $(P_1',\mathcal{X}_1')$ be the rational boundary component of $(P',\mathcal{X}')$ with $(P_1,\mathcal{X}_1)=(P_1',\mathcal{X}_1')/U_0$, and $\mathcal{X}'^o$ the connected component of $\mathcal{X}'$ corresponding to $\mathcal{X}^o$. We want to apply the same inductive procedure as in the proof of 9.16. Suppose that for some K_f' we have constructed a K_f'-admissible partial cone decomposition $\mathcal{S}'$ for $(P',\mathcal{X}')$, such that for some $\Sigma\subset C^*(\mathcal{X}'^o,P_1')$ as in 9.8 $\mathcal{S}'(\mathcal{X}'^o,P_1',1)=\mathcal{S}_\Sigma'$ and $\sigma\in\mathcal{S}_\Sigma$. Then as in the proof of 9.16 it can be extended to the desired $\mathcal{S}'$. We therefore have to construct $\Sigma\subset C^*(\mathcal{X}'^o,P_1')$ as in 9.8 such that $\sigma\in\mathcal{S}_\Sigma$, and $\mathcal{S}'(\mathcal{X}'^o,P_1',1)=\mathcal{S}_\Sigma'$ extends to a K_f'-admissible partial cone decomposition. While the first property is not difficult to achieve, the second one makes the proof complicated.

Fix an arbitrary $e_1\in\mathcal{E}(\mathcal{X}'^o,P_1')$. By 9.3 (a) we may fix $e_2,\ldots,e_m\in\mathcal{E}(\mathcal{X}'^o,P_1')$ such that the $\lambda\circ(e_1-e_i)\in\mathrm{Hom}(U_1(\mathbb{R})(-1),\mathbb{R})$ span the dual cone $\check{\sigma}$, in other words

$$\sigma=\{u\in C^*(\mathcal{X}^o,P_1)\,|\,\forall 2\leq i\leq m\ \ \lambda\circ(e_1-e_i)(u)\geq 0\}.$$

Let $u_1,\ldots,u_n\in C^*(\mathcal{X}^o,P_1)\cap U_1(\mathbb{Q})(-1)$ such that $\sigma=\sum_{j=1}^n\mathbb{R}^{\geq 0}\cdot u_j$. Suppose that K_f' has been chosen, and let

$$\Gamma_1:=\mathrm{Stab}_{Q'(\mathbb{Q})}(\mathcal{X}'^o)\cap K_f'\cdot P_1'(\mathbb{A}_f).$$

Define Σ_0 as in 9.8 using this group and the sections $e_1,\ldots,e_m$. Since σ does not contain a nontrivial linear subspace, its dual cone generates the dual of $U(\mathbb{R})(-1)$, so by the choice of the e_i the assumption in 9.8 holds. Suppose that all $e_1(u_j)$ lie in Σ_0. Then $e_1(\sigma)\subset\Sigma_0$, and the choice of $e_2,\ldots,e_m$ implies that $\lambda\circ(\mathrm{id}-e_1\circ\pi_o)$ is strictly positive on $\Sigma_0\setminus e_1(\sigma)$. Thus $\sigma\in\mathcal{S}_{\Sigma_0}$, which is one of the desired properties. But Σ_0 is not yet the right subset, since $\mathcal{S}_{\Sigma_0}'$ does not necessarily extend to a K_f'-admissible partial cone decomposition. To remedy this we shall construct the "right" Σ as a Γ_1-invariant subset of Σ_0 that contains all the $e_1(u_j)$. By the same argument it then follows that $\sigma \in \mathcal{S}_\Sigma$, and it remains to choose Σ such that $\mathcal{S}'_\Sigma$ extends.

Let us check what this extendability condition amounts to. By lemma 9.32 below $\mathcal{S}'_\Sigma$ extends to a K'_f-admissible partial cone decomposition if and only if for every rational boundary component (P'_2,X'_2) between (P'_1,X'_1) and (P',X'), and every $q' \in \mathrm{Stab}_{Q'(\mathbb{Q})}(X'^{o}) \cap K'_f \cdot P'_2(\mathbb{A}_f)$, we have

$$\{q' \cdot \tau \,|\, \tau \in \mathcal{S}'_\Sigma,\ \tau \subset C^*(X'^{o},P'_2)\} = \{\tau \in \mathcal{S}'_\Sigma \,|\, \tau \subset q' \cdot C^*(X'^{o},P'_2)\}.$$

By the definition of $\mathcal{S}'_\Sigma$ this can be translated into the following equality in terms of Σ:

$$q' \cdot (\Sigma \cap C^*(X'^{o},P'_2)) = \Sigma \cap q' \cdot C^*(X'^{o},P'_2).$$

Together with the condition $e_1(u_j) \in \Sigma$ this has the following consequence. Every u_j lies in $C(X^{o},P_j)$ for a unique rational boundary component (P'_j,X'_j) between (P'_1,X'_1) and (P',X'). Thus $e_1(u_j) \in \Sigma$ implies $q' \cdot e_1(u_j) \in \Sigma$ for all j and $q' \in \mathrm{Stab}_{Q'(\mathbb{Q})}(X'^{o}) \cap K'_f \cdot P'_j(\mathbb{A}_f)$.

We shall construct Σ by induction over the (P'_2,X'_2) for which the above equality holds. Consider an increasing sequence $\emptyset = \mathfrak{Q}_0 \subsetneq \mathfrak{Q}_1 \subsetneq \ldots$ of Γ_1-invariant collections of rational boundary components between (P'_1,X'_1) and (P',X'), such that whenever $(P'_2,X'_2) \in \mathfrak{Q}_\nu$, then so is every rational boundary component between (P'_2,X'_2) and (P',X'). We shall construct these $\mathfrak{Q}_\nu$ and a sequence of Γ_1-invariant subsets $\Sigma_0 \supset \Sigma_1 \supset \ldots$ as in 9.8, such that the above equality holds for all Σ_ν and $(P'_2,X'_2) \in \mathfrak{Q}_\nu$. Furthermore we require that $q' \cdot e_1(u_j) \in \Sigma_\nu$ for all ν, j, and $q' \in \mathrm{Stab}_{Q'(\mathbb{Q})}(X'^{o}) \cap K'_f \cdot P'_j(\mathbb{A}_f)$. Since modulo Γ_1 there are only finitely many rational boundary components between (P'_1,X'_1) and (P',X'), this sequence stops at some point, where $\Sigma := \Sigma_\nu$ has all the desired properties.

Of course this works only if K_f' is chosen in advance. So we first show that it can be chosen so that the conditions hold for Σ_0. Since $\mathfrak{Q}_0 = \emptyset$, only the condition $q' \cdot e_1(u_j) \in \Sigma_0$ matters. By the definition of Σ_0 this is equivalent to

$$\lambda \circ (id - e_i^{\gamma} \circ \pi_o)(q' \cdot e_1(u_j)) \geq 0$$

for all i, j, $\gamma \in \Gamma_1$, and $q' \in Stab_{Q'(\mathbb{Q})}(\mathcal{X}'^o) \cap K_f' \cdot P_j'(\mathbb{A}_f)$. By the remark in 9.2 this is equivalent to $\lambda \circ (id - e_i \circ \pi_o)(\gamma \cdot q' \cdot e_1(u_j)) \geq 0$, so by lemma 9.30 below we are reduced to $\gamma = 1$. Since there are only finitely many i and j, the desired K_f' exists by lemma 9.31 below. We fix such K_f' once and for all.

Next assume that Σ_ν has been constructed. Fix a rational boundary component $(P_2', \mathcal{X}_2')$ between $(P_1', \mathcal{X}_1')$ and $(P', \mathcal{X}')$, such that $(P_2', \mathcal{X}_2') \notin \mathfrak{Q}_\nu$, but $\mathfrak{Q}_\nu$ contains every rational boundary component between $(P_2', \mathcal{X}_2')$ and $(P', \mathcal{X}')$ that is different from $(P_2', \mathcal{X}_2')$. Let $\mathfrak{Q}_{\nu+1}$ be the union of $\mathfrak{Q}_\nu$ with the set of all Γ_1-conjugates of $(P_2', \mathcal{X}_2')$. Let Q_2' be the parabolic subgroup of P' associated to $(P_2', \mathcal{X}_2')$, and define $\Gamma_2 := Stab_{(Q' \cap Q_2')(\mathbb{Q})}(\mathcal{X}'^o) \cap K_f' \cdot P_2'(\mathbb{A}_f)$, this group contains $Stab_{\Gamma_1}(C^*(\mathcal{X}'^o, P_2'))$ as a subgroup of finite index. Thus the set

$$\Sigma' := \bigcap_{\gamma \in \Gamma_2} \gamma \cdot (\Sigma_\nu \cap C^*(\mathcal{X}'^o, P_2'))$$

is of the form of 9.8 for $(P_2', \mathcal{X}_2')$ in place of $(P_1', \mathcal{X}_1')$. We shall construct a Γ_1-invariant subset $\Sigma_{\nu+1} \subset \Sigma_\nu$ as in 9.8, such that

(a) $q' \cdot e_1(u_j) \in \Sigma_{\nu+1}$ for all j and all $q' \in Stab_{Q'(\mathbb{Q})}(\mathcal{X}'^o) \cap K_f' \cdot P_j'(\mathbb{A}_f)$,
(b) $\Sigma_{\nu+1} \cap C^*(\mathcal{X}'^o, P_3') = \Sigma_\nu \cap C^*(\mathcal{X}'^o, P_3')$ for every $(P_3', \mathcal{X}_3') \in \mathfrak{Q}_\nu$, and
(c) $\Sigma_{\nu+1} \cap q' \cdot C^*(\mathcal{X}'^o, P_2') = q' \cdot \Sigma'$ for every $q' \in Stab_{Q'(\mathbb{Q})}(\mathcal{X}'^o) \cap K_f' \cdot P_2'(\mathbb{A}_f)$.

Then by (c) and by Γ_1-invariance the extendability condition holds for all Γ_1-conjugates of $(P_2', \mathcal{X}_2')$, and by (b) and the assumption about Σ_ν it holds for all $(P_3', \mathcal{X}_3') \in \mathfrak{Q}_\nu$, so it holds for all $(P_3', \mathcal{X}_3') \in \mathfrak{Q}_{\nu+1}$, as desired.

Using the fact that $\Sigma_{\nu+1}$ is Γ_1-invariant we can reduce this infinity of conditions to a finite number. First by lemma 9.30 below it suffices to require (c) only for a finite subset $\mathfrak{R} \subset \mathrm{Stab}_{Q'(\mathbb{Q})}(\mathfrak{X}'^o) \cap K_f' \cdot P_2'(A_f)$. For the same reason it suffices to require (a) only for a finite subset $\mathfrak{R}_j \subset \mathrm{Stab}_{Q'(\mathbb{Q})}(\mathfrak{X}'^o) \cap K_f' \cdot P_j'(A_f)$. For (b) we are reduced to a finite number of $(P_3', \mathfrak{X}_3') \in \mathfrak{a}_\nu$. By 9.14 every $\Sigma_\nu \cap C^*(\mathfrak{X}'^o, P_3')$ is of the form in 9.8 with respect to the group $\mathrm{Stab}_{\Gamma_1}(C^*(\mathfrak{X}'^o, P_3'))$, so by 9.10 this set is the convex closure of the union of all $\mathrm{Stab}_{\Gamma_1}(C^*(\mathfrak{X}'^o, P_3'))$-orbits of finitely many rational half-lines. Therefore by $\mathrm{Stab}_{\Gamma_1}(C^*(\mathfrak{X}'^o, P_3'))$-invariance the condition (b) is equivalent to $\mathfrak{U} \subset \Sigma_{\nu+1}$ for some finite subset

$$\mathfrak{U} \subset \Sigma_\nu \cap U_1'(\mathbb{Q})(-1) \cap \bigcup_{(P_3', \mathfrak{X}_3') \in \mathfrak{a}_\nu} C^*(\mathfrak{X}'^o, P_3').$$

Now for every $q' \in \mathfrak{R}$ let $\Gamma_{2,q'} := \mathrm{Stab}_{\Gamma_1}(C^*(\mathfrak{X}'^o, q' \cdot P_2' \cdot q'^{-1}))$, and choose finitely many $\bar{e}_{q',i} \in \mathfrak{E}(\mathfrak{X}'^o, q' \cdot P_2' \cdot q'^{-1})$ so that $q' \cdot \Sigma'$ is of the form in 9.8 with this group and these splittings. Lift every $\bar{e}_{q',i}$ to an element $\tilde{e}_{q',i} \in \mathfrak{E}(\mathfrak{X}'^o, P_1')$, and define $\Sigma_{\nu+1}$ as in 9.8 with the group Γ_1, and as splittings these $\tilde{e}_{q',i}$ and those that define Σ_ν. It remains to show that the conditions (a)-(c) hold for a suitable choice of the liftings $\tilde{e}_{q',i}$. Let us rewrite these conditions in terms of the $\tilde{e}_{q',i}$. For (c) note that for every $q' \in \mathfrak{R}$ the inclusion $\Sigma_{\nu+1} \cap q' \cdot C^*(\mathfrak{X}'^o, P_2') \subset q' \cdot \Sigma'$ holds by construction. For the other inclusion we can as above write $q' \cdot \Sigma'$ as the convex closure of the union of all $\mathrm{Stab}_{\Gamma_1}(q' \cdot C^*(\mathfrak{X}'^o, P_2'))$-orbits of finitely many rational half-lines, so it suffices to show that $\mathfrak{V} \subset \Sigma_{\nu+1}$ for some finite subset

$$\mathfrak{V} \subset \bigcup_{q' \in \mathfrak{R}} q' \cdot \Sigma' \cap U_1'(\mathbb{Q})(-1).$$

Altogether it follows that the conditions (a)-(c) hold if and only if $\Sigma_{\nu+1}$ contains the finite subset

$$\mathfrak{W} := \{q' \cdot e_1(u_j) \mid 1 \leq j \leq n,\ q' \in \mathfrak{R}_j\} \cup \mathfrak{U} \cup \mathfrak{V} \subset \Sigma_\nu \cap U_1'(\mathbb{Q})(-1).$$

By the definition of $\Sigma_{\nu+1}$ this means that for all $q' \in \mathcal{R}$, i, $u' \in \mathcal{U}$, and $\gamma \in \Gamma_1$ we need

$$\lambda \circ (id - \tilde{e}_{q',i} \circ \pi_o)(\gamma \cdot u') \geq 0.$$

By the argument in the proof of 9.15 one can choose the liftings $\tilde{e}_{q',i}$ such that this inequality fails at most if $\gamma \cdot u' \in q' \cdot C^*(\mathcal{X}'^o, P_2')$. Thus it remains to prove $\Gamma_1 \cdot \mathcal{U} \cap q' \cdot C^*(\mathcal{X}'^o, P_2') \subset q' \cdot \Sigma'$ for every $q' \in \mathcal{R}$.

Translated back into the original formulation of these conditions we have to prove for all $q' \in Stab_{Q'(\mathbb{Q})}(\mathcal{X}'^o) \cap K_f' \cdot P_2'(\mathbb{A}_f)$:

(a') For all j and all $q'' \in Stab_{Q'(\mathbb{Q})}(\mathcal{X}'^o) \cap K_f' \cdot P_j'(\mathbb{A}_f)$:

$$q'' \cdot e_1(u_j) \in q' \cdot C^*(\mathcal{X}'^o, P_2') \Rightarrow q'' \cdot e_1(u_j) \in q' \cdot \Sigma'.$$

(b') $\Sigma_\nu \cap C^*(\mathcal{X}'^o, P_3') \cap q' \cdot C^*(\mathcal{X}'^o, P_2') \subset q' \cdot \Sigma'$ for every $(P_3', \mathcal{X}_3') \in \mathfrak{Q}_\nu$.

(c') $q'' \cdot \Sigma' \cap q' \cdot C^*(\mathcal{X}'^o, P_2') \subset q' \cdot \Sigma'$ for all $q'' \in Stab_{Q'(\mathbb{Q})}(\mathcal{X}'^o) \cap K_f' \cdot P_2'(\mathbb{A}_f)$.

Since $q' \cdot \gamma_2 \in Stab_{Q'(\mathbb{Q})}(\mathcal{X}'^o) \cap K_f' \cdot P_2'(\mathbb{A}_f)$ for every $\gamma_2 \in \Gamma_2$, by the definition of Σ' we may replace the inclusions $\subset q' \cdot \Sigma'$ by the inclusions $\subset q' \cdot \Sigma_\nu$. Now in (a') the assumption implies that $q'' \cdot C(\mathcal{X}'^o, P_j') \cap q' \cdot C^*(\mathcal{X}'^o, P_2') \neq \emptyset$, so $int(q')((P_2', \mathcal{X}_2'))$ is a rational boundary component of $int(q'')((P_j', \mathcal{X}_j'))$, and in particular $int(q''^{-1} \cdot q')((P_2', \mathcal{X}_2')) \subset (P_j', \mathcal{X}_j')$. Let $p_{2,f}' \in P_2'(\mathbb{A}_f)$ such that $q' \in K_f' \cdot p_{2,f}'$, then it follows that

$$\begin{aligned} q'^{-1} \cdot q'' &= p_{2,f}' \cdot q'^{-1} \cdot q'' \cdot int(q''^{-1} \cdot q')(p_{2,f}'^{-1}) \\ &\in K_f' \cdot K_f' \cdot P_j'(\mathbb{A}_f) \cdot int(q''^{-1} \cdot q')((P_2', \mathcal{X}_2')) = K_f' \cdot P_j'(\mathbb{A}_f), \end{aligned}$$

whence $q'^{-1} \cdot q'' \in Stab_{Q'(\mathbb{Q})}(\mathcal{X}'^o) \cap K_f' \cdot P_j'(\mathbb{A}_f)$. Thus without loss of generality we may assume $q' = 1$, so we are reduced to proving $q' \cdot e_1(u_j) \in \Sigma_\nu$ for all j, and $q' \in Stab_{Q'(\mathbb{Q})}(\mathcal{X}'^o) \cap K_f' \cdot P_j'(\mathbb{A}_f)$. But this is one of the assumptions about Σ_ν.

In (b') it suffices to consider only those $(P_3',\mathcal{X}_3')\in\mathfrak{Q}_v$, for which $\mathrm{int}(q')((P_2',\mathcal{X}_2'))$ is a rational boundary component of $(P_3',\mathcal{X}_3')$. As above it follows that $q'^{-1}\in \mathrm{Stab}_{Q'(\mathbb{Q})}(\mathcal{X}'^{o}) \cap K_f'\cdot P_3'(\mathbb{A}_f)$, and the inclusion

$$\Sigma_v \cap C^*(\mathcal{X}'^{o},P_3') \subset q'\cdot\Sigma_v$$
$$\Leftrightarrow\ q'^{-1}\cdot(\Sigma_v \cap C^*(\mathcal{X}'^{o},P_3')) \subset \Sigma_v$$

follows from the invariance condition for $(P_3',\mathcal{X}_3')$ and Σ_v. Finally for (c') the left hand side is always contained in $q''\cdot C^*(\mathcal{X}'^{o},P_3')$ for a rational boundary component $(P_3',\mathcal{X}_3')$ between $(P_2',\mathcal{X}_2')$ and $(P',\mathcal{X}')$, such that $\mathrm{int}(q')((P_2',\mathcal{X}_2'))$ is a rational boundary component of $\mathrm{int}(q'')((P_3',\mathcal{X}_3'))$. Thus as above we are reduced to proving the inclusion

$$q''\cdot(\Sigma'\cap C^*(\mathcal{X}'^{o},P_3')) \subset \Sigma_v$$

for all $q''\in \mathrm{Stab}_{Q'(\mathbb{Q})}(\mathcal{X}'^{o}) \cap K_f'\cdot P_3'(\mathbb{A}_f)$ such that $q''\cdot C^*(\mathcal{X}'^{o},P_3') \subset C^*(\mathcal{X}'^{o},P_2')$. In the case $(P_3',\mathcal{X}_3')=(P_2',\mathcal{X}_2')$ we must have $q''\in\Gamma_2$, so this follows from the definition of Σ'. Otherwise we have $(P_3',\mathcal{X}_3')\in\mathfrak{Q}_v$, so this follows from the invariance condition for $(P_3',\mathcal{X}_3')$ and Σ_v. Thus the conditions (a')-(c') are satisfied. q.e.d.

9.30. Lemma: Let $(P,\mathcal{X})$ be mixed Shimura data, $(P_2,\mathcal{X}_2)$ a rational boundary component of $(P,\mathcal{X})$, and $(P_1,\mathcal{X}_1)$ a rational boundary component of $(P_2,\mathcal{X}_2)$. Let $\mathcal{X}^o$ be a connected component of $\mathcal{X}$, Q the parabolic subgroup of P associated to $(P_1,\mathcal{X}_1)$, and K_f an open compact subgroup of $P(\mathbb{A}_f)$. Let

$$\Gamma_1 := \mathrm{Stab}_{Q(\mathbb{Q})}(\mathcal{X}^o) \cap K_f\cdot P_1(\mathbb{A}_f).$$

There exists a finite subset $\mathcal{R}\subset \mathrm{Stab}_{Q(\mathbb{Q})}(\mathcal{X}^o)\cap K_f\cdot P_2(\mathbb{A}_f)$ such that

$$\mathrm{Stab}_{Q(\mathbb{Q})}(\mathcal{X}^o)\cap K_f\cdot P_2(\mathbb{A}_f) = \Gamma_1\cdot\mathcal{R}\cdot\mathrm{Stab}_{(P_2\cap Q)(\mathbb{Q})}(\mathcal{X}^o).$$

<u>Proof</u>: For abbreviation write $A := \mathrm{Stab}_{Q(\mathbb{Q})}(\mathcal{X}^o) \cap K_f \cdot P_2(\mathbb{A}_f)$. Then we have $\Gamma_1 \cdot A = A$. Indeed, let $q \in A$ and $q' \in \Gamma_1$, then obviously $q' \cdot q \in \mathrm{Stab}_{Q(\mathbb{Q})}(\mathcal{X}^o)$, and

$$q' \cdot q \in K_f \cdot P_1(\mathbb{A}_f) \cdot q \subset K_f \cdot (K_f \cdot P_2(\mathbb{A}_f)) \cdot (q^{-1} \cdot P_1(\mathbb{A}_f) \cdot q) = K_f \cdot P_2(\mathbb{A}_f),$$

since q normalizes $P_1(\mathbb{A}_f)$. Clearly $\mathrm{Stab}_{(P_2 \cap Q)(\mathbb{Q})}(\mathcal{X}^o)$ acts on the right hand side on A, and we have to show that the double quotient $\Gamma_1 \backslash A / \mathrm{Stab}_{(P_2 \cap Q)(\mathbb{Q})}(\mathcal{X}^o)$ is finite.

Since $P_2 \cap Q$ is a parabolic subgroup of P_2, for every open compact subgroup $K_f^{P_2} \subset P_2(\mathbb{A}_f)$ the double quotient $K_f^{P_2} \backslash P_2(\mathbb{A}_f)/(P_2 \cap Q)(\mathbb{Q})$ is finite. In particular there exists a finite subset $\mathcal{R} \subset P_2(\mathbb{A}_f)$ such that

$$P_2(\mathbb{A}_f) = (P_2(\mathbb{A}_f) \cap K_f) \cdot \mathcal{R} \cdot \mathrm{Stab}_{(P_2 \cap Q)(\mathbb{Q})}(\mathcal{X}^o).$$

This implies

$$\begin{aligned} A &= \mathrm{Stab}_{Q(\mathbb{Q})}(\mathcal{X}^o) \cap K_f \cdot (P_2(\mathbb{A}_f) \cap K_f) \cdot \mathcal{R} \cdot \mathrm{Stab}_{(P_2 \cap Q)(\mathbb{Q})}(\mathcal{X}^o) \\ &= (\mathrm{Stab}_{Q(\mathbb{Q})}(\mathcal{X}^o) \cap K_f \cdot \mathcal{R}) \cdot \mathrm{Stab}_{(P_2 \cap Q)(\mathbb{Q})}(\mathcal{X}^o), \end{aligned}$$

and that

$$(\mathrm{Stab}_{Q(\mathbb{Q})}(\mathcal{X}^o) \cap K_f) \backslash (\mathrm{Stab}_{Q(\mathbb{Q})}(\mathcal{X}^o) \cap K_f \cdot \mathcal{R})$$

is finite. Since $\mathrm{Stab}_{Q(\mathbb{Q})}(\mathcal{X}^o) \cap K_f$ is contained in Γ_1, we are done. <u>q.e.d.</u>

<u>9.31. Lemma</u>: Consider the situation of 9.2. Let $(P_2', \mathcal{X}_2')$ be a rational boundary component between $(P_1', \mathcal{X}_1')$ and $(P', \mathcal{X}')$. Let $e \in \mathcal{E}(\mathcal{X}'^o, P_1')$ and $u' \in C(\mathcal{X}'^o, P_2') \cap U_2'(\mathbb{Q})(-1)$ such that $\lambda \circ (\mathrm{id} - e \circ \pi_o)(u') \geq 0$. There exists an open compact subgroup $K_f' \subset P'(\mathbb{A}_f)$ such that for all $q' \in \mathrm{Stab}_{Q'(\mathbb{Q})}(\mathcal{X}'^o) \cap K_f' \cdot P_2'(\mathbb{A}_f)$

$$\lambda \circ (\mathrm{id} - e \circ \pi_o)(q' \cdot u') \geq 0.$$

Proof: If $(P_2',\mathfrak{X}_2')=(P',\mathfrak{X}')$, then the inequalities always hold, since in this case $\mathrm{Stab}_{Q'(\mathbb{Q})}(\mathfrak{X}'^{o}) \cap K_f'\cdot P_2'(\mathbb{A}_f) = \mathrm{Stab}_{P'(\mathbb{Q})}(\mathfrak{X}'^{o})$ acts by positive scalars on U'. Otherwise fix an arbitrary K_f'. We first show that there are finitely many half-lines $\mathbb{R}^{\geq 0}\cdot u_i'$ such that the inequality fails at most if $\mathbb{R}^{\geq 0}\cdot q'\cdot u' = \mathbb{R}^{\geq 0}\cdot u_i'$ for some i. To prove this let

$$\Gamma_1 := \mathrm{Stab}_{Q'(\mathbb{Q})}(\mathfrak{X}'^{o}) \cap K_f'\cdot P_1'(\mathbb{A}_f),$$

and fix a finite subset $\mathfrak{R} = \{q_1',\dots,q_n'\} \subset \mathrm{Stab}_{Q(\mathbb{Q})}(\mathfrak{X}^{o})$ as in 9.30. Then for any $q' = \gamma\cdot q_j'\cdot p_2 \in \mathrm{Stab}_{Q'(\mathbb{Q})}(\mathfrak{X}'^{o})\cap K_f'\cdot P_2'(\mathbb{A}_f)$ with $p_2\in\mathrm{Stab}_{P_2'(\mathbb{Q})}(\mathfrak{X}'^{o})$ and $\gamma\in\Gamma_1$, the element $q'\cdot u'$ is a positive multiple of $\gamma\cdot q_j'\cdot u'$. By 9.7 there are at most finitely many half-lines $\mathbb{R}^{\geq 0}\cdot\gamma\cdot q_j'\cdot u'$, for which the inequality fails, as desired.

Next suppose that $\mathbb{R}\cdot u_i'=\mathbb{R}\cdot u'$ for some i. Then u_i' is a negative multiple of u', since by assumption the inequality fails for u_i'. Since both u' and u_i' lie in the convex set $C(\mathfrak{X}'^{o},P_2')$, this contains 0, in contradiction to the assumption $(P_2',\mathfrak{X}_2')\neq(P',\mathfrak{X}')$. Thus $\mathbb{R}\cdot u_i'\neq\mathbb{R}\cdot u'$ for all i. Now consider an open compact subgroup $K_f''\subset K_f'$. If the inequality fails for some $q' \in \mathrm{Stab}_{Q'(\mathbb{Q})}(\mathfrak{X}'^{o}) \cap K_f''\cdot P_2'(\mathbb{A}_f)$, then $\mathbb{R}^{\geq 0}\cdot q'\cdot u' = \mathbb{R}^{\geq 0}\cdot u_i'$ for some i. But for every i, by 7.14 this cannot happen if K_f'' is sufficiently small. Since there are only finitely many i, we are done. q.e.d.

9.32. Lemma: Let $(P,\mathfrak{X})$ be mixed Shimura data, $(P_1,\mathfrak{X}_1)$ a rational boundary component of $(P,\mathfrak{X})$, and $\mathfrak{X}^{o}$ a connected component of $\mathfrak{X}$ that maps to $\mathfrak{X}_1$. Let Q be the parabolic subgroup of P associated to $(P_1,\mathfrak{X}_1)$. Furthermore let $K_f\subset P(\mathbb{A}_f)$ be an open compact subgroup, and $\mathcal{S}_o$ a rational partial polyhedral decomposition of $C^*(\mathfrak{X}^{o},P_1)$. There exists a K_f-admissible partial cone decomposition $\mathcal{S}$ for $(P,\mathfrak{X})$ with $\mathcal{S}(\mathfrak{X}^{o},P_1,1)=\mathcal{S}_o$ if and only if the following condition holds:

For every rational boundary component $(P_2,\mathfrak{X}_2)$ between $(P_1,\mathfrak{X}_1)$ and $(P,\mathfrak{X})$, and every $q \in \mathrm{Stab}_{Q(\mathbb{Q})}(\mathfrak{X}^o) \cap K_f \cdot P_2(\mathbb{A}_f)$

$$\{q\cdot\sigma \mid \sigma\in\mathcal{S}_o,\ \sigma\subset C^*(\mathfrak{X}^o,P_2)\} = \{\sigma\in\mathcal{S}_o \mid \sigma\subset q\cdot C^*(\mathfrak{X}^o,P_2)\}.$$

Proof: If $\mathcal{S}$ is to exist, then for every $(P_2,\mathfrak{X}_2)$ above we must have $\mathcal{S}(\mathfrak{X}^o,P_2,1) := \{\sigma\in\mathcal{S}_o \mid \sigma\subset C^*(\mathfrak{X}^o,P_2)\}$. By the conditions 6.4 (ii)-(iv) we must also have

$$\mathcal{S}(p\cdot\mathfrak{X}^o, p\cdot P_2\cdot p^{-1}, p\cdot p_{2,f}\cdot k_f) = \{p\cdot\sigma \mid \sigma\in\mathcal{S}_o,\ \sigma\subset C^*(\mathfrak{X}^o,P_2)\}$$

for all $p\in P(\mathbb{Q})$, $p_{2,f}\in P_2(\mathbb{A}_f)$, and $k_f\in K_f$. Thus a necessary condition is that the right hand side does not change if p, $p_{2,f}$, k_f are replaced by other elements such that $p\cdot\mathfrak{X}^o$, $p\cdot P_2\cdot p^{-1}$, $p\cdot p_{2,f}\cdot k_f$ do not change. It is quite obvious that this condition is also sufficient. Now this condition holds if and only if for all $(P_2,\mathfrak{X}_2)$ and $p\in\mathrm{Stab}_{P(\mathbb{Q})}(\mathfrak{X}^o)\cap K_f\cdot P_2(\mathbb{A}_f)$, if $(P_1,\mathfrak{X}_1)$ is a boundary component of $\mathrm{int}(p)((P_2,\mathfrak{X}_2))$, then

$$\{p\cdot\sigma \mid \sigma\in\mathcal{S}_o,\ \sigma\subset C^*(\mathfrak{X}^o,P_2)\} = \{\sigma\in\mathcal{S}_o \mid \sigma\subset p\cdot C^*(\mathfrak{X}^o,P_2)\}.$$

For every such p, both $(P_1,\mathfrak{X}_1)$ and $\mathrm{int}(p^{-1})((P_1,\mathfrak{X}_1))$ are rational boundary components of $(P_2,\mathfrak{X}_2)$, so there exists $p_2\in\mathrm{Stab}_{P_2(\mathbb{Q})}(\mathfrak{X}^o)$ such that $\mathrm{int}(p^{-1})((P_1,\mathfrak{X}_1)) = \mathrm{int}(p_2^{-1})((P_1,\mathfrak{X}_1))$. In other words $q:=p\cdot p_2^{-1}$ normalizes $(P_1,\mathfrak{X}_1)$, hence lies in $Q(\mathbb{Q})$. But $p_2\cdot C^*(\mathfrak{X}^o,P_2) = C^*(\mathfrak{X}^o,P_2)$, and acts by scalars on $U_2(\mathbb{R})(-1)$. If $(P_2,\mathfrak{X}_2)$ is a proper boundary component, then this scalar is positive, and in the above equation we may replace p by q, as in the desired assertion. Otherwise we anyway have $Q=P$, and may even take $p_2=1$. q.e.d.

9.33. Corollary (of 9.29): Consider mixed Shimura data $(P,\mathfrak{X})$, an open compact subgroup $K_f\subset P(\mathbb{A}_f)$, and a finite K_f-admissible cone decomposition $\mathcal{S}$ for $(P,\mathfrak{X})$. Then $M_{\mathbb{C}}^{K_f^+}(P,\mathfrak{X},\mathcal{S})$ exists for every sufficiently small open compact subgroup $K_f^+\subset K_f$. In fact $M_{\mathbb{C}}^{K_f^+}(P,\mathfrak{X},\mathcal{S})$ is covered by quasiprojective $M_{\mathbb{C}}^{K_f^+}(P,\mathfrak{X},\mathfrak{S}_i)$, for which an ample line bundle can be described as in 8.14.

Remark: Except for the last sentence this can be proved with less pain than taken 9.29, if one disregards the invariance condition 6.4 (iv).

Proof: Since we may choose K_f^+ neat, as in the proof of 9.21 we can reduce to the case where $(P,\mathfrak{X})$ is as in 9.2. Let σ_i be (finitely many) representatives in $\mathfrak{S}$ for the two actions 6.4 (ii)-(iii) of $P(\mathbb{Q})$ and K_f. By 9.29 for every i there exists an open compact subgroup $K_f^i \subset P(\mathbb{A}_f)$, and a K_f^i-admissible partial cone decomposition $\mathfrak{S}_i$ for $(P,\mathfrak{X})$, that contains σ, and such that $M_{\mathbb{C}}^{K_f^i}(P,\mathfrak{X},\mathfrak{S}_i)$ exists. Since we have only finitely many i, we can find an open compact subgroup K_f^*, contained in the intersection of all K_f^i, that is a normal subgroup of K_f. Then every $M_{\mathbb{C}}^{K_f^*}(P,\mathfrak{X},\mathfrak{S}_i)$ exists, and so does $M_{\mathbb{C}}^{K_f^*}(P,\mathfrak{X},[\cdot k_f]^*\mathfrak{S}_i)$ for every $k_f \in K_f$ modulo K_f^*. Now we may replace every $\mathfrak{S}_i$ by the smallest K_f^*-admissible cone decomposition that contains σ_i. Then $\mathfrak{S}_i$ is contained in $\mathfrak{S}$, and by construction $\mathfrak{S}$ is the union of all $[\cdot k_f]^*\mathfrak{S}_i$. Thus by gluing $M_{\mathbb{C}}^{K_f^*}(P,\mathfrak{X},\mathfrak{S}_i)$ exists, as desired. q.e.d.

9.34. Corollary: (a) "$M_{\mathbb{C}}^{K_f}(P,\mathfrak{X},\mathfrak{S})$" always exists as an algebraic space.

(b) Suppose that $M_{\mathbb{C}}^{K_f}(P,\mathfrak{X},\mathfrak{S})$ exists. If $\mathfrak{S}$ is finite, resp. complete, then $M_{\mathbb{C}}^{K_f}(P,\mathfrak{X},\mathfrak{S})$ is of finite type, resp. proper, over $\mathbb{C}$.

Proof: (a) In the category of algebraic spaces every quotient of a variety by a finite group exists (see [K] introduction). Thus by 9.33, "$M_{\mathbb{C}}^{K_f}(P,\mathfrak{X},\mathfrak{S})$" exists as the quotient of $M_{\mathbb{C}}^{K_f^+}(P,\mathfrak{X},\mathfrak{S})$ by K_f/K_f^+, whenever $\mathfrak{S}$ is finite. By gluing along open subspaces of finite type, the same follows for arbitrary $\mathfrak{S}$.

(b) Both assertions are invariant under quotients by finite groups. Thus the first assertion follows from 9.34, and the second follows from this and the compactness result 6.27. q.e.d.

9.35. Remarks: (a) From now on we implicitly assume the existence of $M_{\mathbb{C}}^{K_f}(P,\mathfrak{X},\mathcal{S})$ as a scheme, whenever we write these symbols. Since by 9.28 and 9.33, it exists whenever $\mathcal{S}$ is sufficiently fine or K_f sufficiently small, this will not be a real restriction in the applications we have in mind. In any case, mutatis mutandum our assertions will also be valid in the category of algebraic spaces, without this restriction.

(b) Counterexample: $M_{\mathbb{C}}^{K_f}(P,\mathfrak{X},\mathcal{S})$ does not always exist. It is a nice exercise to realize the example [H2] app.B ex. 3.4.2 in the toroidal compactification for the mixed Shimura data $(P,\mathfrak{X})=(P_{2g},\mathfrak{X}_{2g})$ of 2.25 with $g=1$, or for $(P,\mathfrak{X}) = (CSp_{4,\mathbb{Q}},\mathcal{H}_4)$ of 2.7.

Finally, we consider the stratifications of chapter 7.

9.36. Proposition: The stratifications in 7.2, 7.3, and 7.10 are algebraic, and the assertion of 7.17 (a) holds algebraically.

Proof: By [S] §19 prop. 15 the map in 7.17 (a) is algebraic, since both sides are compact. This shows the second assertion. In particular this yields an algebraic morphism

$$\mathrm{Stab}_{\Delta_1}([\sigma])\backslash M_{\mathbb{C}}^{\pi[\sigma](K_f^1)}(P_{1,[\sigma]},\mathfrak{X}_{1,[\sigma]}) \longrightarrow M_{\mathbb{C}}^{K_f}(P,\mathfrak{X},\mathcal{S}),$$

which proves the other assertions whenever $\mathcal{S}$ is complete. By 9.33 the general case follows from this, since the assertions are invariant under replacing the open compact subgroup by a smaller one. q.e.d.

9.37. Proposition: The isomorphism of 7.17 (b) induces a canonical isomorphism between the formal completion of $M_{\mathbb{C}}^{K}f(P,\mathfrak{X},\mathcal{S})$ and of $\mathrm{Stab}_{\Delta_1}([\sigma])\backslash M_{\mathbb{C}}^{K_1^1}(P_1,\mathfrak{X}_1,\mathcal{S}_1)$ along

$$\mathrm{Stab}_{\Delta_1}([\sigma])\backslash M_{\mathbb{C}}^{\pi[\sigma](K_1^1)}(P_{1,[\sigma]},\mathfrak{X}_{1,[\sigma]},\mathcal{S}_{1,[\sigma]}).$$

Proof: This is a special case of the following lemma. q.e.d.

9.38. Lemma: Let X be a proper algebraic variety over $\mathbb{C}$, $Y\subset X$ a closed subvariety, and $\mathfrak{X}_Y$ the formal completion of X along Y. Let X_1 be another algebraic variety over $\mathbb{C}$, $Y_1\subset X_1$ a closed subvariety, and $\mathfrak{X}_{1,Y_1}$ the formal completion of X_1 along Y_1. Suppose that we are given open analytic neighborhoods $Y(\mathbb{C}) \subset U \subset X(\mathbb{C})$ and $Y_1(\mathbb{C}) \subset U_1 \subset X_1(\mathbb{C})$, and an analytic isomorphism $U\cong U_1$ that restricts to an isomorphism $Y(\mathbb{C})\cong Y_1(\mathbb{C})$:

$$\begin{array}{ccccc} Y(\mathbb{C}) & \subset & U & \subset & X(\mathbb{C}) \\ \downarrow\wr & & \downarrow\wr & & \\ Y_1(\mathbb{C}) & \subset & U_1 & \subset & X_1(\mathbb{C}) \end{array}$$

Then this isomorphism corresponds to a canonical isomorphism $\mathfrak{X}_Y \cong \mathfrak{X}_{1,Y_1}$:

$$\begin{array}{ccc} Y & \subset & \mathfrak{X}_Y \\ \downarrow\wr & & \downarrow\wr \\ Y_1 & \subset & \mathfrak{X}_{1,Y_1} \end{array}$$

Proof: The isomorphism $U\cong U_1$ induces an isomorphism between the nonreduced complex spaces $(Y(\mathbb{C}),\mathcal{O}_X^{an}/(\mathcal{J}_Y^{an})^n)$ and $(Y_1(\mathbb{C}),\mathcal{O}_{X_1}^{an}/(\mathcal{J}_{Y_1}^{an})^n)$. Since X is proper over $\mathbb{C}$, it follows that $Y(\mathbb{C})$, and hence $Y_1(\mathbb{C})$ is compact. Thus by [S] §19 prop. 15 the analytic isomorphism corresponds to a unique algebraic isomorphism between the infinitesimal neighborhoods $(Y, \mathcal{O}_X/(\mathcal{J}_Y)^n)$ and $(Y_1, \mathcal{O}_{X_1}/(\mathcal{J}_{Y_1})^n)$. By taking inverse limits the assertion follows. (Compare also [H1] ch.VI ex. 2.9) q.e.d.

9.39. Examples: (a) As pointed out in 9.8, our sets Σ play the role of the Γ-polyhedral cocores in [AMRT] ch.II §5.3. In the case where P is reductive, the precise correspondence is the following. As explained in 8.15 (a), we have a canonical splitting of U_1', whence a canonical decomposition $C^*(\mathcal{X}'^o, P_1') \cong \mathbb{R}\cdot\sigma_o \times C^*(\mathcal{X}^o, P_1)$. Fix some $u_o \in (\sigma_o)^o \cap U'(\mathbb{Q})(-1)$. The correspondence between Γ-polyhedral cocores $K \subset C^*(\mathcal{X}^o, P_1)$, and our sets Σ, can then be given by

$$K = \{u \in C^*(\mathcal{X}^o, P_1) \mid (-u_o, u) \in \Sigma\}, \text{ and}$$
$$\Sigma = \text{convex closure of } \sigma_o \cup \{-u_o\} \times K.$$

(b) We include an example not subsumed by (a), which at the same time illustrates the remark 8.15 (b). Let $(P', \mathcal{X}')$ be the mixed Shimura data $(P_{2g}, \mathcal{X}_{2g})$ defined in 2.25 for $g := 1$, and $(P, \mathcal{X}) := (P', \mathcal{X}')/U'$. Let $K_f' \subset P'(\mathbb{A}_f)$ be an open compact subgroup whose image in $(P'/W')(\mathbb{A}_f) \cong GL_2(\mathbb{A}_f)$ is conjugate to $GL_2(\hat{\mathbb{Z}})$. Then modulo the actions of K_f' and of $P'(\mathbb{Q})$, there is precisely one conjugacy class of proper rational boundary components. Describing it as in 4.27, we have $U_1' \cong (\mathbb{G}_{a,\mathbb{Q}})^{\oplus 3}$,

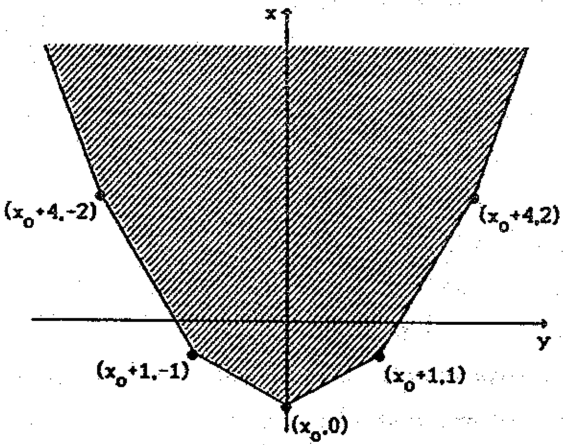
$$\lambda(C^*(\mathcal{X}'^o, P_1')) \cong \{(x,y,z) \in \mathbb{R}^3 \mid z > 0 \text{ or } y = z = 0\},$$

and $\Gamma_1 := \mathrm{Stab}_{Q'(\mathbb{Q})}(\mathcal{X}'^o) \cap K_f' \cdot P_1'(\mathbb{A}_f)$ acts on this through positive scalars and the unipotent transformations

$$(x,y,z) \mapsto (x + 2ny + n^2 z, y + nz, z)$$

for all $n \in \mathbb{Z}$.

A K_f'-admissible cone decomposition as in 9.13 now corresponds to a subset $\Sigma \subset C^*(\mathcal{X}'^o, P_1')$ as in 9.8 with respect to the above unipotent transformations. In terms of the above identification, 9.1 shows that $\sigma_o = \mathbb{R}^{\geq 0}\cdot(1,0,0)$. Clearly $\Sigma = \sigma_o \cup \mathbb{R}^{>0}\cdot\{(x,y,1)\in\Sigma\}$ for such Σ, so we can describe it in terms of its intersection with the plane $z=1$. This must be the convex closure of $\Gamma_1\cdot\{(x_i+\mathbb{R}^{\geq 0}, y_i, 1) \mid 1\leq i\leq n\}$ for some $x_i, y_i \in \mathbb{Q}$. In the special case $n=1$ and $y_1=0$, we get



A little calculation shows that the associated splittings lie in $\mathcal{E}(\mathcal{X}'^{o}, P_1')$ if and only if $x_1<0$. In this case the assumptions of 9.13 are verified.

Coming back to 8.15 (b), we now see a geometric reason for the necessity of quadratic terms in the action of Γ_1: the strict convexity condition. At the same time, it prohibits the existence of a splitting of the unipotent extension $(P',\mathcal{X}')\rightarrow(P,\mathcal{X})$.