For all neat open compact subgroups $K_f \subset P(\mathbb{A}_f)$ we have thus defined a model of $M_{\mathbb{C}}^{K_f}(P,\mathfrak{X})$ over E. Letting $\mathcal{K}$ be the set of these K_f, and $\mathbb{P}:=P(\mathbb{A}_f)$, it remains to verify the conditions 11.13. These follow easily from the construction, by the same arguments as in the proof of 11.13. (Condition 11.13 (a) is an application of descent of morphisms [SGA1] exp.VIII thm. 5.2) q.e.d.

12. Canonical model of the compactification

In this chapter we formulate and prove the main result of this thesis: the extension of the canonical model to both the Baily-Borel compactification and the toroidal compactifications, together with an explicit description of the model induced on the boundary. Before stating these theorems (12.3-5), we prove an assertion relating the reflex field of mixed Shimura data with that of a boundary component (12.1-2). For the structure of the proofs see 12.6-7; suffice it to say here that we are reaping the benefits of the consistent use of the adelic formalism, in particular in chapter 6. The proofs occupy 12.8-17. In 12.18-20, we apply the results to q-expansions of modular forms. We conclude with two examples (12.21-22).

<u>12.1. Proposition:</u> Let $(P_1,\mathfrak{X}_1)$ be a rational boundary component of some mixed Shimura data $(P,\mathfrak{X})$. Then

$$E(P_1,\mathfrak{X}_1) = E(P,\mathfrak{X}).$$

<u>Proof:</u> Let $x \in \mathfrak{X}$, and $x_1 \in \mathfrak{X}_1$ the corresponding point. We <u>claim:</u> $h_x \circ \mu$ is conjugate to $h_{x_1} \circ \mu$ under $P(\mathbb{C})$. To prove this, by the definition 4.11 of h_{x_1}, it suffices to show that the two cocharacters $h_o \circ \mu$ and $h_\infty \circ \mu$ of $H_{o,\mathbb{C}}$ are conjugate under $H_o(\mathbb{C})$, where H_o, h_o, h_∞ are as in 4.3. But it is easily checked that both of these are conjugate to the map

$$\mathbb{G}_m(\mathbb{C}) \to H_o(\mathbb{C}),\ z \mapsto (z, \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}).$$

Let $Q \subset P$ be the parabolic subgroup associated to $(P_1,\mathfrak{X}_1)$. By the following lemma 12.2, we have inclusions

$$Y^{u}(P_1) \hookrightarrow Y^{u}(Q) \supset Y^{u}(Q)^{+} \xrightarrow{\sim} Y^{u}(P).$$

By the claim above, and by 4.4, our conjugacy class $[h_x \circ \mu] = [h_{x_1} \circ \mu]$ lies in each of these subsets. By the definition of the reflex field, the assertion follows. q.e.d.

12.2. Lemma: For any linear algebraic group G denote by $Y^{\natural}(G)$ the (discrete) set of G-conjugacy classes of cocharacters of G.

(a) If U is the unipotent radical of G, then $Y^{\natural}(G) \cong Y^{\natural}(G/U)$.

(b) If H is a connected normal subgroup of G, then $Y^{\natural}(H) \hookrightarrow Y^{\natural}(G)$.

(c) If G is reductive, $P \subset G$ a parabolic subgroup, and U the unipotent radical of P, then the subset

$$Y^{\natural}(P)^{+} := \{\lambda \in Y^{\natural}(P) \mid \lambda \text{ has nonnegative weights on } U\}$$

maps isomorphically to $Y^{\natural}(G)$.

Proof: Assertion (a) follows from the conjugacy of all maximal tori. Assertion (b) holds, because G and H have the same image in $\mathrm{Aut}(H)$. For (c), fix a Borel subgroup B inside P. The fact that every weight is conjugate to a unique dominant weight, means that every element of $Y^{\natural}(G)$ possesses a unique representative in $Y^{\natural}(B)^{+}$. Applying this fact to $B \subset P$, the assertion follows. q.e.d.

We are now in the position to formulate our main results.

12.3. Main Theorem for the Baily-Borel compactification: Consider pure Shimura data $(P,\mathfrak{X})$ and an open compact subgroup $K_f \subset P(\mathbb{A}_f)$.

(a) The canonical model $M^{K_f}(P,\mathfrak{X})$ extends uniquely to a scheme $M^{K_f}(P,\mathfrak{X})^*$ over $E(P,\mathfrak{X})$, with an isomorphism

$$M^{K_f}_{\mathbb{C}}(P,\mathfrak{X})^* = M^{K_f}(P,\mathfrak{X})^* \times_{E(P,\mathfrak{X})} \mathbb{C}.$$

(b) All morphisms 3.4 correspond to morphisms of these schemes, defined over $E(P,\mathcal{X})$ in the case 3.4 (a), and over $E(P_1,\mathcal{X}_1)$ in the case 3.4 (b) (compare 6.2, 9.25). The functoriality of these morphisms is preserved, as well as the assertions of 8.2 and 8.4 (ampleness).

(c) The stratification in 6.3, and the maps in 7.6, are defined over the common reflex field $E(P,\mathcal{X})$.

<u>12.4. Main Theorem for the toroidal compactification</u>: Let $(P,\mathcal{X})$ be mixed Shimura data, $K_f \subset P(\mathbb{A}_f)$ an open compact subgroup, and $\mathcal{S}$ a K_f-admissible partial cone decomposition for $(P,\mathcal{X})$. We assume that $M_{\mathbb{C}}^{K_f}(P,\mathcal{X},\mathcal{S})$ exists and can be covered by quasiprojective $M_{\mathbb{C}}^{K_f}(P,\mathcal{X},\mathcal{S}_i)$ for $\mathcal{S}_i \subset \mathcal{S}$, such that, on each $M_{\mathbb{C}}^{K_f}(P,\mathcal{X},\mathcal{S}_i)$, some ample line bundle can be described as in terms of $\omega[\mathrm{dlog}]$ and line bundles as in 8.13 (e.g. in the situation of 9.21, 9.28, or 9.33).

(a) The canonical model $M^{K_f}(P,\mathcal{X})$ extends uniquely to a scheme $M^{K_f}(P,\mathcal{X},\mathcal{S})$ over $E(P,\mathcal{X})$, with an isomorphism

$$M_{\mathbb{C}}^{K_f}(P,\mathcal{X},\mathcal{S}) = M^{K_f}(P,\mathcal{X},\mathcal{S}) \times_{E(P,\mathcal{X})} \mathbb{C}.$$

(b) All morphisms 6.25 (a) ≅ 11.5 (a) correspond to morphisms of these schemes, defined over $E(P,\mathcal{X})$. The same holds for the morphism $[\pi]^*$ in 6.24. All morphisms 6.25 (b) ≅ 11.10 correspond to morphisms of these schemes, defined over $E(P_1,\mathcal{X}_1)$. The assertions of 6.25 (functoriality, open embeddings, isomorphisms, quotients by finite groups), 6.26 (smoothness), 8.6 (line bundle structure), and 8.13–14 (ampleness) are equally valid for these schemes.

(c) The stratifications in 7.2, 7.3, and 7.10 (see 9.36) are defined over the common reflex field $E(P,\mathcal{X})$, and the assertion of 7.17 (a) holds over $E(P,\mathcal{X})$. The isomorphism of 7.17 (b) (see 9.37) induces a canonical isomorphism between the formal completion of $M^{K_f}(P,\mathfrak{X},\mathcal{S})$ and of $\mathrm{Stab}_{\Delta_1}([\sigma])\backslash M^{K_f^1}(P_1,\mathfrak{X}_1,\mathcal{S}_1)$ along

$$\mathrm{Stab}_{\Delta_1}([\sigma])\backslash M^{\pi[\sigma](K_f^1)}(P_{1,[\sigma]},\mathfrak{X}_{1,[\sigma]},\mathcal{S}_{1,[\sigma]}).$$

12.5. Corollary: Without the assumption on $(P,\mathfrak{X})$, K_f, and $\mathcal{S}$, "$M^{K_f}(P,\mathfrak{X},\mathcal{S})$" always exists as an algebraic space. The remaining assertions of 12.4 carry over literally.

Proof: As in 9.34 (a), using 9.33 the assertion reduces to that in the theorem. q.e.d.

12.6. Beginning of the proof of 12.3 and 12.4: The central points in both theorems (which imply the others) are 12.3 (a), 12.4 (a): the existence of an extension of the canonical models; and the last assertion of 12.4 (c): the isomorphism of formal schemes at the boundary. The remaining assertions say that a certain morphism descends to the reflex field. For each such morphism, by the effectivity of descent of morphisms ([SGA1] exp.VIII thm. 5.2), it suffices to show that there is canonical descent data. Since all our schemes are normal, this is certainly so if the morphism descends on some open dense subscheme. Taking into account 9.25, 11.5 (a) and 11.10, this proves that part (b) of either theorem follows from part (a). In particular, this proves the uniqueness in (a). Likewise, in 12.4 (c) it suffices to prove the assertion for the formal neighborhood of the stratum $\mathrm{Stab}_{\Delta_1}([\sigma])\backslash M^{\pi[\sigma](K_f^1)}(P_{1,[\sigma]},\mathfrak{X}_{1,[\sigma]})$ instead of its closure. This assertion, together with 12.3 (a), also implies 12.3 (c). Indeed, fix any complete K_f-admissible cone decomposition $\mathcal{S}$ for $(P,\mathfrak{X})$, so that 12.4 holds for $M^{K_f}(P,\mathfrak{X},\mathcal{S})$. The map of 6.3 lies in the commutative diagram

$$\begin{array}{ccc}
\mathrm{Stab}_{\Delta_1}([\sigma])\backslash M_{\mathbb{C}}^{[\sigma](K_f^1)}(P_{1,[\sigma]},\mathfrak{X}_{1,[\sigma]}) & \hookrightarrow & M_{\mathbb{C}}^{K_f}(P,\mathfrak{X},\mathfrak{S}) \\
\downarrow & & \downarrow \\
\Delta_1\backslash M_{\mathbb{C}}^{K_f^1}((P_1,\mathfrak{X}_1)/W_1) & \hookrightarrow & M_{\mathbb{C}}^{K_f}(P,\mathfrak{X})^*
\end{array}$$

for some cone σ. By part (b) of both theorems, the two vertical maps descend to $E(P,\mathfrak{X})$. By assumption, the upper inclusion descends to E, so the same follows for the lower one, as desired.

By similar arguments, we may assume that K_f is arbitrarily small. In fact, either theorem reduces to the quasiprojective case, in which quotients by finite groups exist. Thus part (a) of either theorem reduces to all sufficiently small open compact subgroups, and the rest follows from the universal property of quotients by finite groups.

<u>12.7. Strategy of proof</u>: Of the statements that remain to be proved, the existence of $M^{K_f}(P,\mathfrak{X},\mathfrak{S})$ and the isomorphism of formal neighborhoods over the reflex field are intimately connected. In fact, we shall the construct $M^{K_f}(P,\mathfrak{X},\mathfrak{S})$ in such a way that 12.4 (c) holds automatically. The existence of $M^{K_f}(P,\mathfrak{X})^*$ for reductive P will be proved on the way. The intuitive idea is the following.

In a way, we can view $M^{K_f}(P,\mathfrak{X},\mathfrak{S})$ as being glued together from formal neighborhoods of torus embeddings along the unipotent fibre of other mixed Shimura varieties, much as in the construction of $M^{K_f}(P,\mathfrak{X},\mathfrak{S})(\mathbb{C})$ in chapter 6. Of course, one cannot glue formal schemes in the way that would be needed here, so we have to argue more indirectly. In any case, we must show that the "gluing isomorphism" given by 9.37 descends to the reflex field. To prove this we use an analog of "special points": namely certain embedded mixed Shimura varieties, for which the assertions can be shown directly. The descent is then effected by a density argument similar to 11.6–10.

This way of proving 12.4 (c) seems to assume 12.4 (a), but it proves this assertion as well. In fact, when we have not yet shown the existence of $M^{K_f}(P,\mathfrak{X},\mathcal{S})$, we can at least interpret the argument as proving that the descent data for $M_{\mathbb{C}}^{K_f}(P,\mathfrak{X})$ with respect to $E(P,\mathfrak{X})\subset\mathbb{C}$ extends to descent data for $M_{\mathbb{C}}^{K_f}(P,\mathfrak{X},\mathcal{S})$, and it remains to show that this descent is effective. Here our assumption about the ample line bundle comes in. Since it can be expressed purely in terms of the toroidal compactification of other mixed Shimura data, we get compatible descent data for this ample line bundle as well. In such a situation, descent is always effective, and we are done.

In detail, there are the following steps. First, we prove 12.4 in the case of torus embeddings along the unipotent fibre (12.8). Next, using modular interpretation we do the same for $(P,\mathfrak{X})=(GL_{2,\mathbb{Q}},\mathcal{H}_2)$ (see 12.9). Taken together, these special cases imply 12.4 for certain $(P,\mathfrak{X})$, and cone decompositions which are "supported" only on certain rational boundary components (see 12.10-11). In particular, if P is reductive, this holds for all boundary components of codimension 1 in the Baily-Borel compactification; and this is sufficient to prove 12.3 (a) (see 12.12). In 12.13, we prove that the embedded mixed Shimura varieties of the above special type satisfy the density condition alluded to before. Some descent questions are then dealt with in a general context (12.14-16), before (in 12.17) we put everything together to finish the proof.

<u>12.8. The case of a torus embedding along the unipotent fibre</u>: We first prove 12.4 in the situation of 6.8. Recall that $M_{\mathbb{C}}^{K_f}(P,\mathfrak{X}) \to M_{\mathbb{C}}^{\pi'(K_f)}((P,\mathfrak{X})/U)$ is a torsor under a (relative) torus, namely either $M_{\mathbb{C}}^{K_f^*}(P_*,\mathfrak{X}_*) \to M_{\mathbb{C}}^{\pi(K_f)}(\mathbb{G}_{m,\mathbb{Q}},\mathcal{H}_0)$, or the same with $\mathfrak{X}_*$, $\mathcal{H}_0$ replaced by $h(\mathfrak{X}_*)$, $h(\mathcal{H}_0)$ respectively. Moreover there exists a unique K_f^*-admissible cone decomposition $\mathcal{S}_*$ for $(P_*,\mathfrak{X}_*)$, resp. for $(P_*,h(\mathfrak{X}_*))$, such that

$M_{\mathbb{C}}^{K_f}(P,\mathfrak{X}) \hookrightarrow M_{\mathbb{C}}^{K_f}(P,\mathfrak{X},\mathcal{S})$ is the torus embedding associated to the torus embedding $M_{\mathbb{C}}^{K_f^*}(P_*,\mathfrak{X}_*) \hookrightarrow M_{\mathbb{C}}^{K_f^*}(P_*,\mathfrak{X}_*,\mathcal{S}_*)$ (resp. with $h(\mathfrak{X}_*)$ in place of $\mathfrak{X}_*$). By 11.2, the common reflex field of $(P_*,\mathfrak{X}_*)$, $(P_*,h(\mathfrak{X}_*))$, $(\mathbb{G}_{m,\mathbb{Q}},\mathcal{H}_0)$, and $(\mathbb{G}_{m,\mathbb{Q}},h(\mathcal{H}_0))$ is $\mathbb{Q}$, which is of course contained in $E(P,\mathfrak{X})=E(P',\mathfrak{X}')$. Thus, by 11.10, the torsor structure descends to the canonical model $M^{K_f}(P,\mathfrak{X})$. The assertion 12.4 (a) for $M^{K_f}(P,\mathfrak{X},\mathcal{S})$ now follows if we know it for $M^{K_f^*}(P_*,\mathfrak{X}_*,\mathcal{S}_*)$, resp. for $M^{K_f^*}(P_*,h(\mathfrak{X}_*),\mathcal{S}_*)$. The first sentence of (c) is then true by the definition of the stratification in terms of the canonical projection (see 5.2). The last assertion of (c) is a tautology.

It remains to prove 12.4 (a) for $(P_*,\mathfrak{X}_*)$ and $(P_*,h(\mathfrak{X}_*))$. We begin with $(P_*,\mathfrak{X}_*)$. Letting $\Gamma_U := U(\mathbb{Q})\cap K_f^U$, by 6.9 the map in 3.16 induces canonical isomorphisms

$$\begin{array}{ccccc} M_{\mathbb{C}}^{K_f^*}(P_*,\mathfrak{X}_*) & \xrightarrow{\sim} & \mathbb{G}_{m,\mathbb{C}}\otimes\Gamma_U & \times & M_{\mathbb{C}}^{\varphi(K_f)}(\mathbb{G}_{m,\mathbb{Q}},\mathcal{H}_0) \\ \cap & & \cap & & \| \\ M_{\mathbb{C}}^{K_f^*}(P_*,\mathfrak{X}_*,\mathcal{S}_*) & \xrightarrow{\sim} & (\mathbb{G}_{m,\mathbb{C}}\otimes\Gamma_U)_{\mathcal{S}_U} & \times & M_{\mathbb{C}}^{\varphi(K_f)}(\mathbb{G}_{m,\mathbb{Q}},\mathcal{H}_0). \end{array}$$

By 11.14, the first line descends to an isomorphism

$$M^{K_f^*}(P_*,\mathfrak{X}_*) \xrightarrow{\sim} \mathbb{G}_{m,\mathbb{Q}}\otimes\Gamma_U \times M^{\varphi(K_f)}(\mathbb{G}_{m,\mathbb{Q}},\mathcal{H}_0)$$

for the canonical models. Thus we may define $M^{K_f^*}(P_*,\mathfrak{X}_*,\mathcal{S}_*)$ as $(\mathbb{G}_{m,\mathbb{Q}}\otimes\Gamma_U)_{\mathcal{S}_U} \times M^{\varphi(K_f)}(\mathbb{G}_{m,\mathbb{Q}},\mathcal{H}_0)$.

Finally we show how this implies the assertion for $(P_*h(\mathfrak{X}_*))$. As in the proof of 11.15, let ι be the automorphism of $(P_*,\mathfrak{X}_*)$ that is the identity on P_* and on $h(\mathfrak{X}_*)$, but interchanges the two connected components of $\mathfrak{X}_*$. By assumption, this induces an involution $[\iota]$ on the canonical model $M^{K_f^*}(P_*,\mathfrak{X}_*,\mathcal{S}_*)$. We define $M^{K_f^*}(P_*,h(\mathfrak{X}_*),\mathcal{S}_*)$ as the quotient of $M^{K_f^*}(P_*,\mathfrak{X}_*,\mathcal{S}_*)$ by $[\iota]$; this quotient exists, since $M^{K_f^*}(P_*,\mathfrak{X}_*,\mathcal{S}_*)$ is covered by affine $M^{K_f^*}(P_*,\mathfrak{X}_*,\mathcal{S}_i)$, each of which is stable under $[\iota]$. Since $M^{K_f^*}(P_*,h(\mathfrak{X}_*))$ is the quotient of $M^{K_f^*}(P_*,\mathfrak{X}_*)$, and $M_{\mathbb{C}}^{K_f^*}(P_*,h(\mathcal{X}_*),\mathcal{S}_*)$ the quotient of $M_{\mathbb{C}}^{K_f^*}(P_*,\mathcal{X}_*,\mathcal{S}_*)$, our definition gives the desired extension of the canonical model.

<u>12.9. The case $P=GL_{2,\mathbb{Q}}$</u>: Next we prove 12.4 for $(P,\mathcal{X})=(GL_{2,\mathbb{Q}},\mathcal{H}_2)$, as defined in 2.7. In this case every K_f-admissible cone decomposition is contained in a unique complete cone decomposition $\mathcal{S}$, with $M_{\mathbb{C}}^{K_f}(P,\mathcal{X},\mathcal{S}) \cong M_{\mathbb{C}}^{K_f}(P,\mathcal{X})^*$. It suffices to consider this $\mathcal{S}$. By projectivity it suffices to prove the assertions for a cofinal system of open compact subgroups. So we may assume that $K_f=K_f(d)$ for $d\geq 3$, as in 10.19. Let $\overline{\mathfrak{M}}_d$ be the moduli scheme of generalized elliptic curves over $\mathbb{Q}$ with d-structure, and $\mathfrak{M}_d \subset \overline{\mathfrak{M}}_d$ the open subscheme parametrizing (honest) elliptic curves. By 10.20 and 11.16, we have compatible isomorphisms

$$M_{\mathbb{C}}^{K_f}(P,\mathcal{X},\mathcal{S}) \cong \overline{\mathfrak{M}}_{d,\mathbb{C}} \quad \text{and} \quad M^{K_f}(P,\mathcal{X}) \cong \mathfrak{M}_d.$$

To prove 12.4 (a), we just set $M^{K_f}(P,\mathcal{X},\mathcal{S}) := \overline{\mathfrak{M}}_d$.

In 12.4 (c), the last statement implies the earlier ones. Since in our case all the pairs $(p_f,(P_1,\mathcal{X}_1))$, consisting an element of $P(\mathbb{A}_f)$ and a proper rational boundary component of $(P,\mathcal{X})$, are conjugate under the left action of $GL_2(\mathbb{Q})$ and the right action of $GL_2(\hat{\mathbb{Z}})$ (which normalizes K_f), it suffices to prove that assertion in any one instance. This instance is provided by 10.22, taking into account 11.14 and 12.8.

Now we can prove the assertion for a certain type of boundary components.

<u>12.10. The case of boundary components "of codimension 1" for reductive P</u>: Consider pure Shimura data $(P,\mathcal{X})$ and a surjective homomorphism $P \twoheadrightarrow PGL_{2,\mathbb{Q}}$. The inverse image of any rational Borel subgroup of $PGL_{2,\mathbb{Q}}$ is an admissible $\mathbb{Q}$-parabolic subgroup of P. We shall

prove 12.4 in the case that $\mathcal{S}$ "is supported" only on those rational boundary components associated to such parabolic subgroups. If $(P_1,\mathfrak{X}_1)$ is such a rational boundary component, then by assumption U_1 has dimension 1, so, as in 12.9, we have to prove the assertion for just one $\mathcal{S}$, the maximal one with the above property. In other words, we want to prove the assertion for $M_{\mathbb{C}}^{K_f}(P,\mathfrak{X})^*$, as defined in 8.2.

Let us first assume that the surjection $P \twoheadrightarrow PGL_{2,\mathbb{Q}}$ lifts to a morphism $(P,\mathfrak{X}) \to (GL_{2,\mathbb{Q}},\mathcal{H}_2)$. Then $PGL_{2,\mathbb{Q}}$ lifts to an almost direct factor of P^{der}, isomorphic to $SL_{2,\mathbb{Q}}$. Let $(P',\mathfrak{X}') := (P,\mathfrak{X})/SL_{2,\mathbb{Q}}$ (see 2.9), then since $SL_2(\mathbb{R})$ is connected, we get an embedding $(P,\mathfrak{X}) \hookrightarrow (P',\mathfrak{X}') \times (GL_{2,\mathbb{Q}},\mathcal{H}_2)$. Since $E(GL_{2,\mathbb{Q}},\mathcal{H}_2)=\mathbb{Q}$, we clearly have $E(P,\mathfrak{X}) = E(P',\mathfrak{X}')$, which is also reflex field of this product. Using 12.9, the assertion of 12.4 follows for the product, with the product of the trivial cone decomposition for $(P',\mathfrak{X}')$, and the unique maximal one for $(GL_{2,\mathbb{Q}},\mathcal{H}_2)$. By descent of closed subschemes ([SGA1] exp.VIII cor. 1.9), resp. of morphisms, the assertions also follow for $(P,\mathfrak{X})$.

In the general case, let $\tilde{P}$ be the difference kernel of the two maps $P \times GL_{2,\mathbb{Q}} \rightrightarrows PGL_{2,\mathbb{Q}}$. The inclusion $\tilde{P} \hookrightarrow P \times GL_{2,\mathbb{Q}}$ corresponds to an embedding $(\tilde{P},\tilde{\mathfrak{X}}) \hookrightarrow (P,\mathfrak{X}) \times (GL_{2,\mathbb{Q}},\mathcal{H}_2)$ for some Shimura data $(\tilde{P},\tilde{\mathfrak{X}})$. The kernel of the projection $\tilde{P} \twoheadrightarrow P$ is isomorphic to $\mathbb{G}_{m,\mathbb{Q}}$, and $(P,\mathfrak{X}) \cong (\tilde{P},\tilde{\mathfrak{X}})/\mathbb{G}_{m,\mathbb{Q}}$. Clearly $E(\tilde{P},\tilde{\mathfrak{X}})=E(P,\mathfrak{X})$. By assumption, the desired assertions hold for $(\tilde{P},\tilde{\mathfrak{X}})$. Let $\tilde{\mathcal{S}}$, $\mathcal{S}$ be the respective partial cone decompositions satisfying the condition above. For all open compact subgroups $\tilde{K}_f$ mapping to K_f, the morphism $M^{\tilde{K}_f}(\tilde{P},\tilde{\mathfrak{X}},\tilde{\mathcal{S}})(\mathbb{C}) \to M^{K_f}(P,\mathfrak{X},\mathcal{S})(\mathbb{C})$ is finite; and even surjective, since $H^1(\mathbb{A},\mathbb{G}_m)=0$. Thus we may define $M^{K_f}(P,\mathfrak{X},\mathcal{S})$ as a finite quotient of $M^{\tilde{K}_f}(\tilde{P},\tilde{\mathfrak{X}},\tilde{\mathcal{S}})$, which exists by quasi-projectivity. This proves 12.4 (a), and the compatibilities (c) follow by descent of morphisms.

As the last special case, we prove 12.4 for certain unipotent extensions of the Shimura data of 12.10.

<u>12.11. Generalization of 12.10</u>: Let $(P,\mathfrak{X})$ and $\mathcal{S}$ be as in 12.10, only this time assume that $(P,\mathfrak{X})$ is irreducible. Let $(P',\mathfrak{X}')$ be a unipotent extension of $(P,\mathfrak{X})$ whose unipotent radical W' is pure of weight -2, i.e. $W'=U'$. Let $\mathcal{S}'$ be a finite admissible partial cone decomposition "supported" only on the rational boundary components $(P_1',\mathfrak{X}_1')$ associated to those $(P_1,\mathfrak{X}_1)$ in 12.10. As the last special case, we prove 12.4 for all such $\mathcal{S}'$ that are sufficiently fine, in the sense explained below, and a cofinal system of $K_f'\subset P'(\mathbb{A}_f)$.

With $(P_*,\mathfrak{X}_*)$ defined as in 12.8, $(P',\mathfrak{X}')$ is (non-canonically) isomorphic to the fibre product $(P_*,h(\mathfrak{X}_*))\times_{(P_0,h(\mathfrak{X}_0))}(P,\mathfrak{X})$ (see 2.20). Let $K_f^*=K_f^U\rtimes K_f^T\subset P_*(\mathbb{A}_f)$ be neat, $K_f\subset P(\mathbb{A}_f)$ mapping to K_f^T, and let $K_f'\subset P'(\mathbb{A}_f)$ be their fibre product. By 3.11,

$$M_{\mathbb{C}}^{K_f'}(P',\mathfrak{X}')\cong M_{\mathbb{C}}^{K_f^*}(P_*,h(\mathfrak{X}_*))\times_{M_{\mathbb{C}}^{K_f^T}(\mathbb{G}_{m,\mathbb{Q}},h(\mathcal{H}_0))}M_{\mathbb{C}}^{K_f}(P,\mathfrak{X}).$$

Fix any K_f^*-admissible complete cone decomposition $\mathcal{S}_*$ for $(P_*,h(\mathfrak{X}_*))$. By pullback, this $\mathcal{S}_*$, together with the partial cone decomposition $\mathcal{S}$ considered in 12.10 induce a finite K_f'-admissible partial cone decomposition $\mathcal{S}_0'$ for $(P',\mathfrak{X}')$. Clearly,

$$M_{\mathbb{C}}^{K_f'}(P',\mathfrak{X}',\mathcal{S}_0')\cong M_{\mathbb{C}}^{K_f^*}(P_*,h(\mathfrak{X}_*),\mathcal{S}_*)\times_{M_{\mathbb{C}}^{K_f^T}(\mathbb{G}_{m,\mathbb{Q}},h(\mathcal{H}_0))}M_{\mathbb{C}}^{K_f}(P,\mathfrak{X},\mathcal{S});$$

indeed, for any $(P_1',\mathfrak{X}_1')$ as above we have a corresponding isomorphism

$$M_{\mathbb{C}}^{K_{1f}'}(P_1',\mathfrak{X}_1',\mathcal{S}_{01}')\cong M_{\mathbb{C}}^{K_f^*}(P_*,h(\mathfrak{X}_*),\mathcal{S}_*)\times_{M_{\mathbb{C}}^{K_f^T}(\mathbb{G}_{m,\mathbb{Q}},h(\mathcal{H}_0))}M_{\mathbb{C}}^{K_{1f}}(P_1,\mathfrak{X}_1).$$

By 12.8 and 12.10 these fibre products descend to the canonical models over $E(P',\mathfrak{X}')=E(P,\mathfrak{X})$, so we may define

$$M^{K_f'}(P',\mathfrak{X}',\mathcal{S}_0'):=M^{K_f^*}(P_*,h(\mathfrak{X}_*),\mathcal{S}_*)\times_{M^{K_f^T}(\mathbb{G}_{m,\mathbb{Q}},h(\mathcal{H}_0))}M^{K_f}(P,\mathfrak{X},\mathcal{S}).$$

This proves all of 12.4 for the given $\mathcal{S}_0'$.

More generally, let $\mathcal{S}'$ be any finite partial refinement of $\mathcal{S}'_o$. Note that, since $\mathcal{S}'_o$ has the property that $|\mathcal{S}'_o(\mathfrak{X}'^o,P'_1,p'_f)| = C^*(\mathfrak{X}'^o,P'_1)$ for all p'_f and all $(P'_1,\mathfrak{X}'_1)$ as above, this includes any sufficiently fine decomposition "supported" only on these $(P'_1,\mathfrak{X}'_1)$. Then, both

$$M_{\mathbb{C}}^{K'_f}(P',\mathfrak{X}',\mathcal{S}') \to M_{\mathbb{C}}^{K'_f}(P',\mathfrak{X}',\mathcal{S}'_o)$$

and the various

$$M_{\mathbb{C}}^{K^1_f}(P'_1,\mathfrak{X}'_1,\mathcal{S}'_1) \to M_{\mathbb{C}}^{K^1_f}(P'_1,\mathfrak{X}'_1,\mathcal{S}'_{o1})$$

are allowable modifications in the sense of [KKMS] ch.II §2 p.87 def. 3. Since they are formally isomorphic, they correspond to the same "f.r.p.p. decomposition" (see [loc. cit.] p.86 def. 2). By the equivalence between allowable modifications and f.r.p.p. decompositions ([loc. cit.] p.90 thm. 6*), one of them descends to $E(P,\mathfrak{X})$ if and only the other does so. But by 12.8 the second one does descend, hence so does the first, and the compatibility is automatic.

12.12. <u>Proof of 12.3 (a)</u>: Consider pure Shimura data $(P,\mathfrak{X})$ and neat $K_f \subset P(\mathbb{A}_f)$. By 12.10, the canonical model extends to a model $M^{K_f}(P,\mathfrak{X})^+$ of $M_{\mathbb{C}}^{K_f}(P,\mathfrak{X})^+$, over $E:=E(P,\mathfrak{X})$. Define $M^{K_f}(P,\mathfrak{X})^*$ as the closure of $M^{K_f}(P,\mathfrak{X})^+$ inside

$$\mathbb{P}_E^N \cong \mathbb{P}\Gamma(M^{K_f}(P,\mathfrak{X})^+,\omega[\mathrm{dlog}]^{\otimes n})$$

for any sufficiently large n. By 8.2, this is a model of $M_{\mathbb{C}}^{K_f}(P,\mathfrak{X})^*$, as desired.

The following lemma implies that every boundary component can be deformed, along a given collection of tangent directions, into embedded mixed Shimura data of the special type considered in 12.11.

<u>12.13. Embedding Lemma</u>: Let $(P_1,\mathfrak{X}_1)$ be a proper rational boundary component of some mixed Shimura data $(P,\mathfrak{X})$. Let $U\subset U_1'\subset U_1$ with $\dim(U_1'/U)=1$, and such that, for some ($\Leftrightarrow$ for all) connected components $\mathfrak{X}^o$ of $\mathfrak{X}^+$ (see 4.11),

$$C(\mathfrak{X}^o,P_1)\cap U_1'(\mathbb{R})(-1)\neq\emptyset.$$

Then there exists irreducible mixed Shimura data $(P',\mathfrak{X}')$, an embedding $(P',\mathfrak{X}')\hookrightarrow(P,\mathfrak{X})$, and a rational boundary component $(P_1',\mathfrak{X}_1')$ of $(P',\mathfrak{X}')$, such that $(P_1,\mathfrak{X}_1)$ is the rational boundary component of $(P,\mathfrak{X})$ associated to $(P_1',\mathfrak{X}_1')$ by functoriality (4.16), and

(a) P' contains U, and P'/U is reductive,

(b) $P_1'\cap W_1=U_1'$ and $P_1'/U_1'\xrightarrow{\sim}P_1/W_1$, and

(c) $(P'/U)^{ad}\cong(P_1'/U_1')^{ad}\times PGL_{2,\mathbb{Q}}$.

<u>Proof</u>: Without loss of generality we may assume $U=1$. Since $C(\mathfrak{X}^o,P_1)$ is open, there exists an element $u_1\in C(\mathfrak{X}^o,P_1)\cap U_1'(\mathbb{Q})(-1)$. Take a point $x\in\mathfrak{X}^o$ that maps to a point $x_1\in\mathfrak{X}_1$ whose imaginary part is u_1. By definition (4.14), $\mathrm{int}(u_1^{-1})\circ h_{x_1}:\mathbb{S}\to P_{1,\mathbb{R}}$ is defined over $\mathbb{R}$. Fix a splitting $P_1=W_1\rtimes G_1$, defined over $\mathbb{Q}$. Then there exists a unique element $u_1'\in U_1(\mathbb{R})$ so that $\mathrm{int}(u_1')\circ\mathrm{int}(u_1^{-1})\circ h_{x_1}$ factors through $G_{1,\mathbb{R}}$. After replacing x by $u_1'\cdot x$, and x_1 by $u_1'\cdot x_1$, which has the same imaginary part, $\mathrm{int}(u_1^{-1})\circ h_{x_1}$ factors through $G_{1,\mathbb{R}}$.

Next, let ω_x be as in 4.6. Since $U=1$, it is defined over $\mathbb{R}$. Observe that, in the notations of 4.3, there exists a unique nonzero element $u_o\in U_o(\mathbb{R})(-1)$ such that $\mathrm{int}(u_o^{-1})\circ h_\infty$ is defined over $\mathbb{R}$. The identity $h_{x_1}=\omega_x\circ h_\infty$ implies $u_1=\omega_x(u_o)$. Since $\dim(U_1')=1$, it follows that $\omega_x(U_o)=U_{1,\mathbb{R}}'$. Now, by 4.9 (b),

$$\mathrm{int}(u_1)(G_{1,\mathbb{C}})\cdot\omega_x(H_o)_{\mathbb{C}}$$

is a subgroup of $P_{\mathbb{C}}$. Since $u_1\in\omega_x(H_o)(\mathbb{C})$, this is equal to $G_{1,\mathbb{C}}\cdot\omega_x(H_o)_{\mathbb{C}}$. As in the proof of 4.9 (a), this group has a parabolic subgroup

$$G_{1,\mathbb{C}}\cdot U'_{1,\mathbb{C}}\cdot(\omega_x\circ h_o\circ w(\mathbb{G}_{m,\mathbb{C}})),$$

which is contained in $(G_1\cdot U'_1\cdot Z(P))_{\mathbb{C}}$. Thus $(G_1\cdot U'_1\cdot Z(P))_{\mathbb{C}}$ is a parabolic subgroup of the group

$$P'_{\mathbb{C}} := G_{1,\mathbb{C}}\cdot \omega_x(H_o)_{\mathbb{C}}\cdot Z(P)_{\mathbb{C}}.$$

In particular, some parabolic subgroup of $P'_{\mathbb{C}}$ is defined over $\mathbb{Q}$. For the same reason as in the proof of 4.9 (a), $P'_{\mathbb{C}} = P'\times_{\mathbb{Q}}\mathbb{C}$ for a unique subgroup $P'\subset P$.

Now we are practically done. Defining $\mathfrak{X}'$ to be the $P'(\mathbb{R})$-orbit in $\mathfrak{X}$ generated by x, we get an embedding $(P',\mathfrak{X}') \hookrightarrow (P,\mathfrak{X})$. There is an obvious rational boundary component $(U'_1 \rtimes G_1, \mathfrak{X}'_1)$ of $(P',\mathfrak{X}')$. The assertions (a)-(c) hold by construction. If the $(P',\mathfrak{X}')$ so constructed is not irreducible, just replace it by an irreducible component. q.e.d.

Next we want to prove effectivity of descent in a certain general situation where the descent data involves formal completions.

12.14. An analog of Čech-resolution: Let $\overline{X}$ be a locally noetherian scheme, and Y a closed subscheme of $\overline{X}$. Let $\hat{X}$ denote the formal completion of $\overline{X}$ along Y. Denote the inclusion $X:=\overline{X}\setminus Y\hookrightarrow \overline{X}$ by j, the canonical morphism $\hat{X}\to\overline{X}$ by k. Since $\overline{X}$ is locally noetherian, [B-AC] ch.III §4 thm. 3 implies that $k_*\mathcal{O}_{\hat{X}}$ is flat over $\mathcal{O}_{\overline{X}}$. Since $j\amalg k$: $X\amalg\hat{X}\to\overline{X}$ is surjective, it is a faithfully flat morphism of formal schemes. Claim: There is a canonical exact sequence

$$0 \longrightarrow \mathcal{O}_{\overline{X}} \longrightarrow j_*\mathcal{O}_X\oplus k_*\mathcal{O}_{\hat{X}} \longrightarrow j_*\mathcal{O}_X\otimes_{\mathcal{O}_{\overline{X}}} k_*\mathcal{O}_{\hat{X}},$$

with the morphisms given as $s\mapsto(s,s)$ and $(s,t)\mapsto s\otimes 1-1\otimes t$ respectively. (If $j\amalg k$ were an open covering, this would be the Čech-resolution.)

Proof: The assertion is local in $\bar{X}$, so we may assume $\bar{X}$ to be affine, say $\bar{X}$=Spec(A). If $I \subset A$ is the ideal of Y, then $\hat{X}$ is the formal spectrum of $\hat{A}$, the I-adic completion of A. Fix an affine covering Spec($\tilde{A}$)→X, then by the sheaf property we have an exact sequence

$$0 \longrightarrow A^o := \Gamma(X,\mathcal{O}_X) \longrightarrow \tilde{A} \longrightarrow \tilde{A} \otimes_A \tilde{A},$$

where the homomorphism on the right is given as $\tilde{a} \mapsto \tilde{a}\otimes 1 - 1 \otimes \tilde{a}$. Since $\tilde{A}$ is flat over A, $\tilde{A}\oplus\hat{A}$ is faithfully flat, and we get an exact sequence

$$0 \longrightarrow A \longrightarrow \tilde{A}\oplus\hat{A} \longrightarrow (\tilde{A}\oplus\hat{A}) \otimes_A (\tilde{A}\oplus\hat{A}).$$

We shall prove the this sequence stays exact if the rightmost term is replaced by its quotient $\tilde{A} \otimes_A (\tilde{A}\oplus\hat{A})$. Contemplating the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \longrightarrow & \tilde{A}\oplus\hat{A} & \longrightarrow & (\tilde{A}\otimes_A\tilde{A}) \oplus (\tilde{A}\otimes_A\hat{A}) \\ & & \| & & \uparrow & & \uparrow \\ 0 & \longrightarrow & A & \longrightarrow & A^o\oplus\hat{A} & \longrightarrow & 0 \oplus A^o\otimes_A\hat{A} \end{array}$$

we find that this will imply the exactness of the second line, as desired.

To prove the exactness of the first line let K be the kernel of the map $A\to\tilde{A}$. Since supp(K)⊂Y and A is noetherian, it is annihilated by some power of I, whence $K \otimes_A \hat{A} \cong K$. Tensoring our sequence with the short exact sequence $0 \to K \to A \to A/K \to 0$, a little diagram chasing shows that it suffices to prove the assertion for A/K in place of A. In other words we may assume that the map $A\to\tilde{A}$ is injective. In this case, by the flatness of $\hat{A}$, the natural map $\hat{A}\otimes_A\hat{A} \to \tilde{A}\otimes_A\hat{A}\otimes_A\hat{A}$ is also injective. Now let $(\tilde{a},\hat{a})\in\tilde{A}\oplus\hat{A}$ so that $\tilde{a}\otimes 1 = 1\otimes\hat{a}$ in $\tilde{A}\otimes_A\hat{A}$. In $\tilde{A}\otimes_A\hat{A}\otimes_A\hat{A}$ we can calculate $1\otimes\hat{a}\otimes 1 = \tilde{a}\otimes 1\otimes 1 = 1\otimes 1\otimes\hat{a}$, whence by injectivity $\hat{a}\otimes 1 = 1\otimes\hat{a}$ in $\hat{A}\otimes_A\hat{A}$. This fact implies that our sequence stays exact when we drop the terms $\hat{A}\otimes_A(\tilde{A}\oplus\hat{A})$, as required. q.e.d.

<u>12.15. Functoriality</u>: Let us consider $j: X \to \overline{X} \supset Y$ and $k: \hat{X} \to \overline{X}$ as in 12.14, and another such situation with $j': X' \to \overline{X}' \supset Y'$ and $k': \hat{X}' \to \overline{X}'$. Any morphism $\overline{\varphi}: \overline{X}' \to \overline{X}$ with $\overline{\varphi}^{-1}(Y) = Y'$ induces compatible morphisms $\varphi: X' \to X$ and $\hat{\varphi}: \hat{X}' \to \hat{X}$. In particular we have a commutative diagram

$$\begin{array}{ccc} \mathcal{O}_{\overline{X}} & \hookrightarrow & j_*\mathcal{O}_X \oplus k_*\mathcal{O}_{\hat{X}} \\ \downarrow & & \downarrow \\ \overline{\varphi}_*\mathcal{O}_{\overline{X}'} & \hookrightarrow \ \overline{\varphi}_* j'_*\mathcal{O}_{X'} \oplus \overline{\varphi}_* k'_*\mathcal{O}_{\hat{X}'} \cong & j_*\varphi_*\mathcal{O}_{X'} \oplus k_*\hat{\varphi}_*\mathcal{O}_{\hat{X}'} \end{array}$$

<u>Claim</u>: If $\hat{\varphi}$ is scheme-theoretically dominant (cf. 5.6), this diagram is cartesian.

<u>Proof</u>: Contemplating the commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 \longrightarrow & \mathcal{O}_{\overline{X}} & \longrightarrow & j_*\mathcal{O}_X \oplus k_*\mathcal{O}_{\hat{X}} & \longrightarrow & j_*\mathcal{O}_X \otimes_{\mathcal{O}_{\overline{X}}} k_*\mathcal{O}_{\hat{X}} \\ & \downarrow & & \downarrow & & \downarrow \\ 0 \longrightarrow & \overline{\varphi}_*\mathcal{O}_{\overline{X}'} & \longrightarrow & \overline{\varphi}_* j'_*\mathcal{O}_{X'} \oplus \overline{\varphi}_* k'_*\mathcal{O}_{\hat{X}'} & \longrightarrow & \overline{\varphi}_*(j'_*\mathcal{O}_{X'} \otimes_{\mathcal{O}_{\overline{X}'}} k'_*\mathcal{O}_{\hat{X}'}) \end{array}$$

we see that it suffices to show that the vertical arrow on the right is injective. Again the assertion is local on $\overline{X}$, so let $A \supset I$, $\tilde{A}$, A^o, and $\hat{A}$ be as in 12.14. Let $\mathrm{Spec}(A') \to \overline{X}'$ be an affine covering of $\overline{X}'$, and define $\tilde{A}'$, $\hat{A}'$ likewise. Since $A^o \otimes_A \hat{A} \hookrightarrow \tilde{A} \otimes_A \hat{A}$, it suffices to prove the injectivity of the natural homomorphism

$$\tilde{A} \otimes_A \hat{A} \longrightarrow \tilde{A} \otimes_A \tilde{A}' \otimes_{A'} \hat{A}'.$$

By the assumption on $\hat{\varphi}$, we have $\hat{A} \hookrightarrow \hat{A}'$. Thus, by tensoring with the A-flat $\tilde{A}$, we get $\tilde{A} \otimes_A \hat{A} \hookrightarrow \tilde{A} \otimes_A \hat{A}' \cong (\tilde{A} \otimes_A A') \otimes_{A'} \hat{A}'$. On the other hand, $\tilde{A} \otimes_A A'$ injects into $\tilde{A} \otimes_A \tilde{A}'$ since

$$\begin{aligned} \mathrm{Spec}(\tilde{A} \otimes_A \tilde{A}') &\cong \mathrm{Spec}(\tilde{A}) \times_X \mathrm{Spec}(\tilde{A}') \\ &\longrightarrow \mathrm{Spec}(\tilde{A}) \times_X X' \cong \mathrm{Spec}(\tilde{A}) \times_{\overline{X}} \overline{X}' \cong \mathrm{Spec}(\tilde{A} \otimes_A A') \end{aligned}$$

is faithfully flat. Since $\hat{A}'$ is flat over A', the assertion follows. <u>q.e.d.</u>

<u>12.16. Descent Lemma</u>: Let $T \to S$ be a faithfully flat, quasi-compact morphism of locally noetherian schemes. Let $X_T \subset \overline{X}_T \supset Y_T$ and $\hat{X}_T$ be as in 12.14, this time as schemes over T. Let $\overline{L}_T$ be an invertible sheaf on $\overline{X}_T$, $L_T := \overline{L}_T|_{X_T}$, and let $\hat{L}_T$ be the associated invertible sheaf on $\hat{X}_T$. Assume that we are given models (X,L) of (X_T,L_T), and $(\mathfrak{X},\mathfrak{L})$ of $(\hat{X}_T,\hat{L}_T)$, over S. Consider another situation as in 12.14, this time with schemes $X' \subset \overline{X}' \supset Y'$, $\hat{X}'$ over S, and with compatible invertible sheaves $\overline{L}'$, L', $\hat{L}'$. Finally, suppose that we are given a morphism $\overline{\varphi}: \overline{X}'_T \to \overline{X}_T$ over T such that $\overline{\varphi}^{-1}(Y_T) = Y'_T$, and an isomorphism $\overline{\psi}: \overline{\varphi}^*\overline{L}_T \xrightarrow{\sim} \overline{L}'_T$. They induce morphisms $\varphi: X'_T \to X_T$ and $\hat{\varphi}: \hat{X}'_T \to \hat{X}_T \cong \mathfrak{X}_T$, and isomorphisms $\psi: \varphi^*(L_T) \xrightarrow{\sim} L'_T$ and $\hat{\psi}: \hat{\varphi}^*(\mathfrak{L}_T) \cong \hat{\varphi}^*(\hat{L}_T) \xrightarrow{\sim} \hat{L}'_T$. Let us assume that

- (i) (φ,ψ) descends to S.
- (ii) $(\hat{\varphi},\hat{\psi})$ descends to S.
- (iii) The morphism $\hat{\varphi}: \hat{X}'_T \to \mathfrak{X}_T$ is scheme-theoretically dominant.
- (iv) $\overline{L}_T$ is relatively ample with respect to $\overline{X}_T \to T$.

<u>Claim</u>: There exists a unique model $(\overline{X},\overline{L})$ for $(\overline{X}_T,\overline{L}_T)$ over S, compatible with all the morphisms above.

<u>Proof</u>: For any nonnegative integer n, we have a commutative diagram for $\overline{L}_T^{\otimes n}$, analogous to that in 12.15:

$$\begin{array}{ccc}
\overline{L}_T^{\otimes n} & \hookrightarrow & j_*L_T^{\otimes n} \oplus k_*\hat{L}_T^{\otimes n} \\
\downarrow & & \downarrow \\
\overline{\varphi}_*\overline{L}_T'^{\otimes n} & \hookrightarrow & \overline{\varphi}_*j'_*L_T'^{\otimes n} \oplus \overline{\varphi}_*k'_*\hat{L}_T'^{\otimes n} \cong j_*\varphi_*L_T'^{\otimes n} \oplus k_*\hat{\varphi}_*\hat{L}_T'^{\otimes n}
\end{array}$$

Since $\overline{L}_T$ is locally free, 12.15 implies that this diagram is also cartesian. Denote the structure morphisms by $\overline{f}: \overline{X}_T \to T$, $f: X_T \to T$, and $\hat{f}: \hat{X}_T \to T$ respectively, and analogously for $\overline{X}'$, etc. Applying $\overline{f}_*$ to the above diagram, we obtain the diagram

$$\begin{array}{ccc} \bar{f}_*(\bar{L}_T^{\otimes n}) & \hookrightarrow & f_{T,*}(L_T^{\otimes n}) \oplus \hat{f}_{T,*}(\hat{L}_T^{\otimes n}) \\ \downarrow & & \downarrow \\ \bar{f}'_*(\bar{L}_T'^{\otimes n}) & \hookrightarrow & f'_*L_T'^{\otimes n} \oplus \hat{f}'_*\hat{L}_T'^{\otimes n} \end{array}$$

which is, by the left-exactness of $\bar{f}_*$, again cartesian. In particular, $\bar{f}_*(\bar{L}_T^{\otimes n})$ is uniquely determined as a subsheaf of $f_{T,*}(L_T^{\otimes n}) \oplus \hat{f}_{T,*}(\hat{L}_T^{\otimes n})$ by the bottom part and the right hand part of this diagram. Now by our data, both these parts come from sheaves and morphisms over S. Invoking [SGA1] exp.VIII cor. 1.8, this shows that $\bar{f}_*(\bar{L}_T^{\otimes n})$ comes from a canonical quasi-coherent sheaf $\mathcal{Q}_n$ on S.

Put $\mathcal{Q} := \bigoplus_{n\geq 0} \mathcal{Q}_n$, then its pullback to T is canonically isomorphic to the graded algebra $\bigoplus_{n\geq 0} \bar{f}_*(\bar{L}_T^{\otimes n})$. By descent of homomorphisms of sheaves (see [SGA1] exp.VIII cor. 1.2), the algebra structure descends to one on $\mathcal{Q}$. Put $P := \mathbf{Proj}(\mathcal{Q})$, then by the relative ampleness of $\bar{L}_T$, there is a canonical open embedding $\bar{X}_T \hookrightarrow P_T$. The image of this embedding is the union of the images of X_T and of Y_T, each of which is stable by the descent data on P_T. Thus, by [SGA1] exp.VIII cor. 1.9, the complement of $\bar{X}_T$ in P_T descends to a closed subscheme of P, hence $\bar{X}_T$ descends to an open subscheme of P, as desired. The compatibility with the various morphisms holds by construction. q.e.d.

12.17. End of the proof of 12.4: It remains to prove 12.4 (a), and that in 12.4 (c) the isomorphism of formal neighborhoods of $\mathrm{Stab}_{\Delta_1}([\sigma])\backslash M^{\pi}[\sigma]^{(K_1^f)}(P_{1,[\sigma]}, \mathfrak{X}_{1,[\sigma]})$ descends to $E := E(P, \mathfrak{X})$. Let $(P, \mathfrak{X})$, K_f, $\mathcal{S}$ be as in 12.4. It suffices to prove the assertion for every $\mathcal{S}_i$, so we may assume $\mathcal{S} = \mathcal{S}_i$, and that $\mathcal{S}$ is finite. Then, by assumption and by 8.13, there is a line bundle $M_{\mathbb{C}}^{K'_f}(P', \mathfrak{X}', \mathcal{S}') \to M_{\mathbb{C}}^{K_f}(P, \mathfrak{X}, \mathcal{S})$ so that, if $\mathfrak{M}$ denotes the associated invertible sheaf, $\mathfrak{M}^{\otimes -1}$ is relatively ample with respect to $M_{\mathbb{C}}^{K_f}(P, \mathfrak{X}, \mathcal{S}) \to M_{\mathbb{C}}^{\pi(K_f)}((P, \mathfrak{X})/W)^*$. By 7.13 we may further assume that the condition 7.12 (*) is satisfied for $\mathcal{S}$.

We proceed by induction over the (finite) number of double classes in $P(\mathbb{Q})\backslash\mathcal{S}/K_f$. Fix $p_f \in P(\mathbb{A}_f)$, and a rational boundary component $(P_1,\mathfrak{X}_1)$ of $(P,\mathfrak{X})$. Let $K_f^1 := P_1(\mathbb{A}_f) \cap p_f \cdot K_f \cdot p_f^{-1}$ and $\mathcal{S}_1^o := (([\cdot p_f]^* \mathcal{S})|_{(P_1,\mathfrak{X}_1)})^o$, and fix a double coset $[\sigma] \in P_1(\mathbb{Q})\backslash\mathcal{S}_1^o/K_f^1$ such that $\sigma^o \subset C(\mathfrak{X}^o,P_1)$ for some $\mathfrak{X}^o$. By gluing, it suffices to prove the assertions in the case that $\mathcal{S}$ is the smallest K_f-admissible partial cone decomposition that contains σ. Let $\mathcal{S}_o \subset \mathcal{S}$ be the unique K_f-admissible partial cone decomposition that contains all proper faces of σ, but not σ itself. Let $\mathcal{S}'_o \subset \mathcal{S}'$ be analogous, then we may assume that both $M^{K_f}(P,\mathfrak{X},\mathcal{S}_o)$ and $M^{K'_f}(P',\mathfrak{X}',\mathcal{S}'_o)$ exist. We have to prove the same for $M^{K_f}(P,\mathfrak{X},\mathcal{S})$ and $M^{K'_f}(P',\mathfrak{X}',\mathcal{S}')$, and that the isomorphism 9.37 descends to E. This will follow by applying 12.16. Without loss of generality, we may assume $p_f=1$. By 12.8, we may assume that $(P_1,\mathfrak{X}_1)$ is a proper rational boundary component of $(P,\mathfrak{X})$.

In the setup of 12.16, put $T:=\mathrm{Spec}(\mathbb{C})$ and $S:=\mathrm{Spec}(E)$. As models over S we define $X := M^{K_f}(P,\mathfrak{X},\mathcal{S}_o)$, and let $\mathcal{X}$ be the formal completion of $\mathrm{Stab}_{\Delta_1}([\sigma])\backslash M^{K_f^1}(P_1,\mathfrak{X}_1,\mathcal{S}_1^o)$ along the stratum $Y := \mathrm{Stab}_{\Delta_1}([\sigma])\backslash M^{\pi_{[\sigma]}(K_f^1)}(P_{1,[\sigma]},\mathfrak{X}_{1,[\sigma]})$. We let $\overline{X}_T := M^{K_f}_{\mathbb{C}}(P,\mathfrak{X},\mathcal{S})$, with the isomorphism $\hat{X}_T \cong \mathcal{X} \times_S T$ given by 9.37. For the line bundle, fix any ample line bundle $\mathcal{N}$ on $M^{\pi(K_f)}((P,\mathfrak{X})/W)^*$. Then, for some positive integer n, $\overline{L}_T := \mathcal{M}^{\otimes -1} \otimes \mathcal{N}_T^{\otimes n}$ is ample. By assumption, its restriction to X_T has an obvious model over S, and the same holds for its pullback to $\mathcal{X}_T$; with $M^{K'_f}(P',\mathfrak{X}',\mathcal{S}'_o) \to M^{K_f}(P,\mathfrak{X},\mathcal{S}_o)$, respectively

$$\mathrm{Stab}_{\Delta_1}([\sigma])\backslash M^{K_f^{1\prime}}(P'_1,\mathfrak{X}'_1,\mathcal{S}'^o_1) \to \mathrm{Stab}_{\Delta_1}([\sigma])\backslash M^{K_f^1}(P_1,\mathfrak{X}_1,\mathcal{S}_1^o)$$

in place of $M^{K'_f}(P',\mathfrak{X}',\mathcal{S}') \to M^{K_f}(P,\mathfrak{X},\mathcal{S})$.

To define $\overline{X}'$, we apply 12.13 to $(P,\mathfrak{X})$ with all $U \subset U'_1 \subset U_1$ such that $\sigma^o \cap U'_1(\mathbb{R})(-1) = \emptyset$. Denote by $i_\alpha: (P_\alpha,\mathfrak{X}_\alpha) \hookrightarrow (P,\mathfrak{X})$ all embedded mixed Shimura varieties thus obtained. Denote by $w_{1,f}^\beta$ all elements of $W_1(\mathbb{A}_f)$. For all α and β, let $K_f^{\alpha\beta}$ be some open compact subgroup inside $P_\alpha(\mathbb{A}_f) \cap w^{\beta}_{1,f} \cdot K_f \cdot (w^{\beta}_{1,f})^{-1}$, and let $\mathcal{S}^*_{\alpha\beta}$ denote the smallest $K^{\alpha\beta}_f$-admissible partial cone decomposition for $(P_\alpha, \mathcal{X}_\alpha)$ that contains $U_\alpha(\mathbb{R})(-1) \cap \sigma \cdot (w^{\beta}_{1,f})^{-1}$. By 12.11, we can choose $K^{\alpha\beta}_f$ and a refinement $\mathcal{S}_{\alpha\beta}$ of $\mathcal{S}^*_{\alpha\beta}$ such that 12.4 holds for $(P_\alpha, \mathcal{X}_\alpha)$, $K^{\alpha\beta}_f$, and $\mathcal{S}_{\alpha\beta}$. Observe that, using 12.1, the characterization in 12.13 implies $E(P_\alpha, \mathcal{X}_\alpha) = E(P_1, \mathcal{X}_1) = E$. The inverse image of X_T under the map

$$[\cdot w^{\beta}_{1,f}] \circ [i_\alpha]: M^{K^{\alpha\beta}_f}_{\mathbb{C}}(P_\alpha, \mathcal{X}_\alpha, \mathcal{S}_{\alpha\beta}) \to M^{K_f}_{\mathbb{C}}(P, \mathcal{X}, \mathcal{S}) = \bar{X}_T$$

is just $M^{K^{\alpha\beta}_f}_{\mathbb{C}}(P_\alpha, \mathcal{X}_\alpha, \mathcal{S}_{\alpha\beta o})$ for some partial cone decomposition $\mathcal{S}_{\alpha\beta o} \subset \mathcal{S}_{\alpha\beta}$. We let $\bar{X}'$ be the disjoint union of all $M^{K^{\alpha\beta}_f}(P_\alpha, \mathcal{X}_\alpha, \mathcal{S}_{\alpha\beta})$, with $\bar{\varphi}: \bar{X}'_T \to \bar{X}_T$ the disjoint union of the above morphisms, and X' the disjoint union of all $M^{K^{\alpha\beta}_f}(P_\alpha, \mathcal{X}_\alpha, \mathcal{S}_{\alpha\beta o})$.

By 8.7, the pullback of the line bundle $M^{K'_f}_{\mathbb{C}}(P', \mathcal{X}', \mathcal{S}') \to M^{K_f}_{\mathbb{C}}(P, \mathcal{X}, \mathcal{S})$ is the line bundle $M^{K^{\alpha\beta\prime}_f}_{\mathbb{C}}(P'_\alpha, \mathcal{X}'_\alpha, \mathcal{S}'_\alpha) \to M^{K^{\alpha\beta}_f}_{\mathbb{C}}(P_\alpha, \mathcal{X}_\alpha, \mathcal{S}_{\alpha\beta})$ for some $(P'_\alpha, \mathcal{X}'_\alpha)$, $\mathcal{S}'_{\alpha\beta}$ and $K^{\alpha\beta\prime}_f$, for every α. Thus by 12.11 it has an obvious model over S. By the same argument as in 12.6, the composite morphism $[\pi]^* \circ [\cdot w^{\beta}_{1,f}] \circ [i_\alpha]$, descends to a morphism $M^{K^{\alpha\beta}_f}(P_\alpha, \mathcal{X}_\alpha, \mathcal{S}_{\alpha\beta}) \to M^{\pi(K_f)}((P, \mathcal{X})/W)^*$. Using the pullback of $\mathfrak{N}$ under this morphism, we get an obvious model of $\bar{\varphi}^*(\bar{L}_T)$ over S.

Now all objects needed in 12.16 have been defined. The conditions (i), (ii), and (iv) hold by construction. To get the desired assertion, it remains to show that the morphism $\hat{\varphi}: \hat{X}'_T \to \hat{X}_T$ is scheme-theoretically dominant. Recall how this morphism is defined: for all α and β, it comes by formal completion, and by taking the quotient by $\mathrm{Stab}_{\Delta_1}([\sigma])$, from a morphism

$$[\cdot w^{\beta}_{1,f}] \circ [i_\alpha]: M^{K^{\alpha\beta 1}_f}_{\mathbb{C}}(P_{\alpha 1}, \mathcal{X}_{\alpha 1}, \mathcal{S}^o_{\alpha\beta 1}) \to M^{K^1_f}_{\mathbb{C}}(P_1, \mathcal{X}_1, \mathcal{S}^o_1),$$

where $(P_{\alpha 1}, \mathcal{X}_{\alpha 1})$ is a rational boundary component of $(P_\alpha, \mathcal{X}_\alpha)$. It suffices to prove the same assertion without taking the quotient by $\mathrm{Stab}_{\Delta_1}([\sigma])$.

By 6.8-9, since $(P_1,\mathfrak{X}_1)$ is a proper rational boundary component, $M_{\mathbb{C}}^{K_f^1}(P_1,\mathfrak{X}_1,\mathcal{S}_1^o)$ is a relative torus embedding over $M_{\mathbb{C}}^{\pi'(K_f^1)}((P_1,\mathfrak{X}_1)/U_1)$ with respect to the split torus $\mathbb{G}_{m,\mathbb{C}}\otimes(U_1(\mathbb{Q})\cap K_f^1)$. A neighborhood of our stratum is given by the affine relative torus embedding with respect to the cone $\lambda(\sigma)\subset U_1(\mathbb{R})$. Let us introduce new notation. Let T be the torus over $\mathbb{C}$ with cocharacter group $\mathbb{R}\cdot\lambda(\sigma)\cap U_1(\mathbb{Q})\cap K_f^1$. Then

$$X := M_{\mathbb{C}}^{K_f^1}(P_1,\mathfrak{X}_1) \longrightarrow S := M_{\mathbb{C}}^{\pi_{[\sigma]}(K_f^1)}(P_{1,[\sigma]},\mathfrak{X}_{1,[\sigma]})$$

is a T-torsor, and letting $\hat{X}_{\lambda(\sigma)}$ be as in 5.7, our formal scheme $\mathfrak{X}_T$ becomes a finite quotient of $\hat{X}_{\lambda(\sigma)}$.

Next,

$$M_{\mathbb{C}}^{K_f^{\alpha\beta 1}}(P_{\alpha 1},\mathfrak{X}_{\alpha 1},\mathcal{S}_{\alpha\beta 1}^o) \to M_{\mathbb{C}}^{\pi(K_f^{\alpha\beta 1})}((P_{\alpha 1},\mathfrak{X}_{\alpha 1})/W_{\alpha 1})$$

is a relative torus embedding with respect to the torus with cocharacter group $U_{\alpha 1}(\mathbb{Q})\cap K_f^{\alpha\beta 1}$. We are only interested in a neighborhood of those strata that map to the closed stratum of $\hat{X}_{\lambda(\sigma)}$; this is a torus embedding with respect to the subtorus $T_{\alpha\beta}$ with cocharacter group $\mathbb{R}\cdot\lambda(\sigma)\cap U_{\alpha 1}(\mathbb{Q})\cap K_f^{\alpha\beta 1}$. The morphism

$$[\cdot w_{1,f}^{\beta}]\circ[i_\alpha]: M_{\mathbb{C}}^{K_f^{\alpha\beta 1}}(P_{\alpha 1},\mathfrak{X}_{\alpha 1}) \to M_{\mathbb{C}}^{K_f^1}(P_1,\mathfrak{X}_1,\mathcal{S}_1^o)$$

is still defined if we replace $K_f^{\alpha\beta 1}$ by $K_f^{\alpha\beta 1}\cdot(\mathbb{R}\cdot\lambda(\sigma)\cap U_{\alpha 1}(\mathbb{Q})\cap K_f^1)$, which corresponds to dividing by a finite subgroup of $T_{\alpha\beta}$. Then $T_{\alpha\beta}$ becomes canonically identified with a subtorus of T. Our neighborhood is the torus embedding with respect to some partial cone decomposition of $\mathbb{R}\cdot\lambda(\sigma)\cap U_{\alpha 1}(\mathbb{R})$ with support $\lambda(\sigma)\cap U_{\alpha 1}(\mathbb{R})$. Let $\tau_{\alpha\beta}^{\gamma}$ denote all those cones in this decomposition for which $(\tau_{\alpha\beta}^{\gamma})^o\subset\lambda(\sigma)^o$. Since W_1 centralizes U_1, we find that neither $T_{\alpha\beta}\subset T$, nor the set of these $\tau_{\alpha\beta}^{\gamma}$ depends on $w_{1,f}^{\beta}$. Therefore we may write T_α and τ_α^γ. Fixing α and γ, we now let $T_\alpha^\gamma\subset T_\alpha$ be the subtorus with cocharacter group $\mathbb{R}\cdot\tau_\alpha^\gamma\cap U_{\alpha 1}(\mathbb{Q})\cap K_f^1$. We let $X_\alpha^\gamma\to S_\alpha^\gamma$ be the disjoint union of the T_α^γ-torsors

$$M_{\mathbb{C}}^{K_f^{\alpha\beta 1}}(P_{\alpha 1},\mathfrak{X}_{\alpha 1}) \to M_{\mathbb{C}}^{\pi[\tau_\alpha^\gamma](K_f^{\alpha\beta 1})}(P_{\alpha 1,[\tau_\alpha^\gamma]},\mathfrak{X}_{\alpha 1,[\tau_\alpha^\gamma]})$$

for all β. The maps $[\cdot w_{1,f}^{\beta}]\circ[i_\alpha]$ induce a T_α^γ-equivariant morphism $(\Psi_\alpha^\gamma,\psi_\alpha^\gamma): (X_\alpha^\gamma,S_\alpha^\gamma)\to(X,S)$. Since the induced morphism of the formal completions $\hat{\Psi}_\alpha^\gamma: \hat{X}_{\alpha,\tau_\alpha^\gamma}^\gamma \to \hat{X}_{\lambda(\sigma)}$ factors through the morphism which we have to prove to be scheme-theoretically dominant, it suffices to prove that the disjoint union of all $\hat{\Psi}_\alpha^\gamma$ is scheme-theoretically dominant.

This will follow by applying 5.7. In fact, conditions (i) and (ii) of 5.7 hold by construction. To finish, it therefore remains to prove that every $\psi_\alpha^\gamma: S_\alpha^\gamma\to S$ is scheme-theoretically dominant. Since S is a normal scheme, it suffices to show that its image is Zariski-dense. Recall that by 12.13 (b), $P_{\alpha 1}$ has the same reductive part as P_1. Thus $P_1(\mathbb{A}_f) = P_{\alpha 1}(\mathbb{A}_f)\cdot W_1(\mathbb{A}_f)$, which, by the definition of ψ_α^γ, implies that its image is the same as the union of the images of maps

$$[\cdot p_{1,[\sigma],f}]\circ[i_\alpha]: M_{\mathbb{C}}^{K_f^{\alpha 1*}}(P_{\alpha 1,[\tau_\alpha^\gamma]},\mathfrak{X}_{\alpha 1,[\tau_\alpha^\gamma]}) \longrightarrow M_{\mathbb{C}}^{\pi[\sigma](K_f^1)}(P_{1,[\sigma]},\mathfrak{X}_{1,[\sigma]})$$

for all $p_{1,[\sigma],f}\in P_{1,[\sigma]}(\mathbb{A}_f)$, and some $K_f^{\alpha 1*}$ for which these morphisms are defined. By 11.7 this is Zariski-dense, as desired. This finishes the proof of 12.3 and of 12.4. q.e.d.

As an application of 12.4, we study q-expansions in our setting.

12.18. Completions and coherent sheaves: Consider irreducible mixed Shimura data $(P,\mathfrak{X})$, a neat open compact subgroup $K_f\subset P(\mathbb{A}_f)$, a K_f-admissible cone decomposition $\mathfrak{S}$ for $(P,\mathfrak{X})$. Assume that $\mathfrak{S}$ is concentrated in the unipotent fibre, and that there exists a morphism $(P,\mathfrak{X})\to(\mathbb{G}_{m,\mathbb{Q}},\mathcal{H}_0)$. Then by 6.8–9 (and, of course, 12.4) $M^{K_f}(P,\mathfrak{X},\mathfrak{S})$ is a relative torus embedding with respect to a split torus $\mathbb{G}_m\otimes\Gamma_U$ and a rational partial polyhedral cone decomposition $\mathfrak{S}_U$ of $U(\mathbb{R})$. Fix a coset $[\sigma]\in\mathfrak{S}/P(\mathbb{A}_f)$, and assume that $\mathfrak{S}$ is the smallest K_f-admissible cone decomposition containing σ. Then $\mathcal{S}_U$ consists of all faces of the single cone $\lambda(\sigma)$, for λ associated to the connected component $\mathfrak{X}^o$ with $\sigma\in\mathcal{S}(P,\mathfrak{X}^o,p_f)$. Letting $T:=\mathbb{G}_m\otimes(\Gamma_U\cap\mathbb{R}\cdot\sigma)$, we have $\mathbb{R}\cdot\sigma=Y_*(T)_{\mathbb{R}}$, and $M^{K_f}(P,\mathfrak{X},\mathcal{S})$ is also a torus embedding with respect to T. The closed stratum is just $M^{\pi[\sigma](K_f)}(P_{[\sigma]},\mathfrak{X}_{[\sigma]})$ in the notation of 7.1. Let us denote the canonical projection $M^{K_f}(P,\mathfrak{X},\mathcal{S})\to M^{\pi[\sigma](K_f)}(P_{[\sigma]},\mathfrak{X}_{[\sigma]})$ by π_σ.

Every character $\chi\in X^*(T)=\mathrm{Hom}(\Gamma_U\cap\mathbb{R}\cdot\lambda(\sigma),\mathbb{Z})$ induces an isomorphism $U(\mathbb{Q})\cap\mathbb{R}\cdot\lambda(\sigma)/\mathbb{Q}\cdot\ker(\chi)\cong\mathbb{Q}=U_0(\mathbb{Q})$, and so defines the structure of a $((P_0,\mathfrak{X}_0)\to(\mathbb{G}_{m,\mathbb{Q}},\mathcal{H}_0))$-torsor on $(P,\mathfrak{X})/\langle\ker(\chi)\rangle\to(P_{[\sigma]},\mathfrak{X}_{[\sigma]})$. This defines a $\mathbb{G}_m$-torsor structure on

$$M^{(K_f \bmod \langle\ker(\chi)\rangle)}((P,\mathfrak{X})/\langle\ker(\chi)\rangle)\longrightarrow M^{\pi[\sigma](K_f)}(P_{[\sigma]},\mathfrak{X}_{[\sigma]}).$$

Let us denote the associated invertible sheaf by $\mathcal{L}_\chi$. By 5.13, the direct image under π_σ of the structure sheaf of $M^{K_f}(P,\mathfrak{X},\mathcal{S})$ is the direct sum of all those $\mathcal{L}_\chi$, for which χ is nonpositive on $\lambda(\sigma)$. Denote by $\hat{M}$ the formal completion of $M^{K_f}(P,\mathfrak{X},\mathcal{S})$ along the closed stratum, and by $\hat{\pi}_\sigma$: $\hat{M}\to M^{\pi[\sigma](K_f)}(P_{[\sigma]},\mathfrak{X}_{[\sigma]})$ the retraction induced by π_σ. By 5.13, we have a canonical isomorphism

$$\hat{\pi}_{\sigma,*}\mathcal{O}_{\hat{M}}\cong\prod_\chi\mathcal{L}_\chi,$$

the sum extended over the same $\chi\in X^*(T)$.

More generally, let $\mathcal{F}$ be a locally free coherent sheaf on $M^{K_f}(P,\mathfrak{X},\mathcal{S})$, and $\hat{\mathcal{F}}$ the associated coherent sheaf on $\hat{M}$. Assume that we are given a coherent sheaf $\mathcal{G}$ on $M^{\pi[\sigma](K_f)}(P_{[\sigma]},\mathfrak{X}_{[\sigma]})$ and an isomorphism $\mathcal{F}\cong\pi_\sigma^*\mathcal{G}$, whence an isomorphism $\hat{\mathcal{F}}\cong\hat{\pi}_\sigma^*\mathcal{G}$. Then we have canonical isomorphisms

$$\pi_{\sigma,*}\mathcal{F}\cong\bigoplus_\chi\mathcal{L}_\chi\otimes\mathcal{G}\quad\text{and}\quad\hat{\pi}_{\sigma,*}\hat{\mathcal{F}}\cong\prod_\chi\mathcal{L}_\chi\otimes\mathcal{G}.$$

In particular, for any $E=E(P,\mathfrak{X})$-algebra R the given data determines an isomorphism

$$\Gamma(\hat{M}\times_E \mathrm{Spec}(R),\hat{\mathfrak{F}}) \cong \prod_\chi \Gamma(M^{\pi[\sigma](K_f)}(P_{[\sigma]},\mathfrak{X}_{[\sigma]})\times_E \mathrm{Spec}(R),\mathfrak{L}_\chi\otimes\mathfrak{G}).$$

<u>12.19. Definition of q-expansions</u>: Let $M^{K_f}(P,\mathfrak{X},\mathfrak{S})$ be a toroidal compactification; for simplicity we assume that K_f is neat. Fix a stratum $\mathrm{Stab}_{\Delta_1}([\sigma])\backslash M^{\pi[\sigma](K_f^1)}(P_{1,[\sigma]},\mathfrak{X}_{1,[\sigma]})$ of $M^{K_f}(P,\mathfrak{X},\mathfrak{S})\smallsetminus M^{K_f}(P,\mathfrak{X})$. Let $\hat{M}_1$ denote the formal completion of $M^{K_f^1}(P_1,\mathfrak{X}_1,\mathfrak{S}_1^0)$ along $M^{\pi[\sigma](K_f^1)}(P_{1,[\sigma]},\mathfrak{X}_{1,[\sigma]})$, and $\hat{\pi}_1\colon \hat{M}_1 \to M^{\pi[\sigma](K_f^1)}(P_{1,[\sigma]},\mathfrak{X}_{1,[\sigma]})$ the retraction, as in 12.18. Let $\mathfrak{F}$ be a locally free coherent sheaf on $M^{K_f}(P,\mathfrak{X},\mathfrak{S})$, and $\hat{\mathfrak{F}}_1$ its pullback to $\hat{M}_1$ under the morphism 12.4. Assume that we are given an isomorphism $\hat{\mathfrak{F}}_1\cong\hat{\pi}_1^*\mathfrak{G}_1$, for some locally free sheaf $\mathfrak{G}_1$ on $M^{\pi[\sigma](K_f^1)}(P_{1,[\sigma]},\mathfrak{X}_{1,[\sigma]})$. Let $E:=E(P,\mathfrak{X})$, and R an E-algebra. By pullback of sections and by 12.18, this data determines a map

$$\Gamma(M^{K_f}(P,\mathfrak{X},\mathfrak{S})\times_E \mathrm{Spec}(R),\mathfrak{F}) \longrightarrow \prod_\chi \Gamma(M^{\pi[\sigma](K_f^1)}(P_{1,[\sigma]},\mathfrak{X}_{1,[\sigma]})\times_E \mathrm{Spec}(R),\mathfrak{L}_\chi\otimes\mathfrak{G}_1),$$

the product being extended over all χ as in 12.18. This map can be interpreted as associating to each section of $\mathfrak{F}$ its "<u>q-expansion coefficients</u>".

For certain $\mathfrak{F}$ this can be made more explicit. Assume first that $\mathfrak{F}$ is defined in terms of the internal geometry of our toroidal compactifications: for instance in terms of sheaves of differentials possibly of logarithmic poles (compare 8.1), for the given $(P,\mathfrak{X})$ or for other mixed Shimura varieties, and/or in terms of the line bundles in 8.6. Then the same definition gives rise to a coherent sheaf $\mathfrak{F}_1$ on $M^{K_f^1}(P_1,\mathfrak{X}_1,\mathfrak{S}_1^0)$, and 12.4 (c) yields a canonical isomorphism between $\hat{\mathfrak{F}}_1$ and the coherent sheaf on $\hat{M}_1$ associated to $\mathfrak{F}_1$. Next, by 8.1 a canonical $\mathfrak{G}_1$ exists in the case $\mathfrak{F}=\omega[\mathrm{dlog}]$. By functoriality, the same follows for any other sheaf defined in terms of $\omega[\mathrm{dlog}]$ for the given $(P,\mathfrak{X})$ or for other mixed Shimura varieties. In particular, for $\mathfrak{F}=\omega[\mathrm{dlog}]^{\otimes n}$ and $\mathcal{S}$ complete, the above definition of q-expansion applies to the space

$$\Gamma(M^{K_f}(P,\mathfrak{X},\mathcal{S})\times_E \mathrm{Spec}(R),\omega[\mathrm{dlog}]^{\otimes n})$$

of all R-valued automorphic forms f of a certain weight. The "q-expansion" of f is a (in general infinite) collection of R-valued automorphic forms $f_{1,\alpha}$ on $M^{\pi[\sigma](K_f^1)}(P_{1,[\sigma]},\mathfrak{X}_{1,[\sigma]})$ of certain other weights. These automorphic forms are generalizations of Jacobi modular forms; the condition at infinity being played by extendability to some toroidal compactification.

12.20. A q-expansion principle for E(P,X)-rationality: In the situation of 12.19, let us now consider a finite number of strata $\mathrm{Stab}_{\Delta_i}([\sigma_i])\backslash M^{\pi[\sigma_i](K_f^1)}(P_{i,[\sigma_i]},\mathfrak{X}_{i,[\sigma_i]})$, and write $\hat{M}_i$, $\hat{\mathfrak{F}}_i$, $\hat{\pi}_i$ accordingly. We assume that the map

$$\coprod_i \mathrm{Stab}_{\Delta_i}([\sigma_i])\backslash M^{\pi[\sigma_i](K_f^1)}(P_{1,[\sigma_i]},\mathfrak{X}_{i,[\sigma_i]})(\mathbb{C}) \longrightarrow M^{K_f}(P,\mathfrak{X},\mathcal{S})(\mathbb{C})$$

induces a surjection on the sets of connected components. Then, since $M^{K_f}(P,\mathfrak{X},\mathcal{S})$ is a normal scheme, our assumption implies that pullback of sections induces injections

$$\begin{array}{ccc}\Gamma(M^{K_f}(P,\mathfrak{X},\mathcal{S}),\mathfrak{F}) & \hookrightarrow & \bigoplus_i \Gamma(\hat{M}_i,\hat{\mathfrak{F}}_i)\\ \downarrow & & \downarrow\\ \Gamma(M^{K_f}_{\mathbb{C}}(P,\mathfrak{X},\mathcal{S}),\mathfrak{F}) & \hookrightarrow & \bigoplus_i \Gamma(\hat{M}_{i,\mathbb{C}},\hat{\mathfrak{F}}_i).\end{array}$$

Clearly this diagram is cartesian, i.e. $\Gamma(M^{K_f}(P,\mathfrak{X},\mathcal{S}),\mathfrak{F})$ consists of those sections in $\Gamma(M^{K_f}_{\mathbb{C}}(P,\mathfrak{X},\mathcal{S}),\mathfrak{F})$ whose pullback to every $\hat{M}_{i,\mathbb{C}}$ is already defined over E. If for every i we are given a product decomposition $\hat{\pi}_{i,*}(\hat{\mathfrak{F}}_i) \cong \prod_\alpha(\mathcal{G}_i\otimes\mathfrak{L}_{i,\alpha})$, then this means that a section in

$\Gamma(M_{\mathbb{C}}^{K}f(P,\mathfrak{X},\mathcal{S}),\mathcal{F})$ is defined over E if and only if every "coefficient in the q-expansion" is defined over E, i.e. lies in

$$\Gamma(M^{\pi[\sigma_1](K_1^f)}(P_{1,[\sigma_1]},\mathfrak{X}_{1,[\sigma_1]}),\mathcal{G}_1\otimes\mathcal{L}_{1,\alpha}).$$

Sometimes the above surjectivity condition already holds for just one stratum. This is so for instance if $(P,\mathfrak{X})$ is irreducible, the semi-simple part, $(P/W)^{der}$, of P is simply connected, and there exists an algebraic "local cross-section" $(P/W)/(P/W)^{der} \hookrightarrow P_1$. Indeed, under these circumstances we have $P(\mathbb{A}) = P_1(\mathbb{A})\cdot P^{der}(\mathbb{A})$, and the surjectivity follows from 3.9.

Remark: The same result has been obtained by M. Harris for arbitrary "arithmetic vector bundles" on pure Shimura varieties, in terms of the Baily-Borel compactification (see [Ha1], [Ha2]).

12.21. Example: Hilbert modular varieties: Fix a totally real number field F. Let $P:=\mathcal{R}_{F/\mathbb{Q}}GL_{2,F}$, and $\mathfrak{X}\cong h(\mathfrak{X})$ be the $P(\mathbb{R})$-orbit generated by

$$h: \mathbb{S}(\mathbb{R})\cong\mathbb{C}^\times \longrightarrow P(\mathbb{R})\cong(GL_2(\mathbb{R}))^{[F/\mathbb{Q}]},$$
$$x+\sqrt{-1}y \mapsto \left(\begin{pmatrix} x & y \\ -y & x\end{pmatrix},\dots,\begin{pmatrix} x & y \\ -y & x\end{pmatrix}\right).$$

This orbit does not depend on the choice of $\sqrt{-1}$ (of course, using such a choice it can be identified with $(\mathbb{C}\smallsetminus\mathbb{R})^{[F/\mathbb{Q}]}$). The reflex field of $(P,\mathfrak{X})$ is $\mathbb{Q}$, not F! There is precisely one conjugacy class of proper rational boundary components of $(P,\mathfrak{X})$, e.g. with representative

$$P_1 := \begin{pmatrix} \mathbb{G}_{m,\mathbb{Q}} & \mathcal{R}_{F/\mathbb{Q}}\mathbb{G}_{a,F} \\ 0 & 1 \end{pmatrix}.$$

Here every h_{x_1} for $x_1\in\mathfrak{X}_1$ is of the form $\mathbb{C}^\times\ni z\mapsto\begin{pmatrix} z\bar{z} & * \\ 0 & 1\end{pmatrix}$, and $(P_1,\mathfrak{X}_1)$ is isomorphic to the $[F/\mathbb{Q}]$-fold fibre product of $(P_0,\mathfrak{X}_0)$ of 2.24 with itself over $(\mathbb{G}_{m,\mathbb{Q}},\mathcal{H}_0)$. In particular, $(P_1,\mathfrak{X}_1)/W_1\cong(\mathbb{G}_{m,\mathbb{Q}},\mathcal{H}_0)$.

It follows from 12.3 that in the Baily-Borel compactification $M^{K_f}(P,\mathfrak{X})^*$, the boundary is a finite disjoint union of certain $M^{K_f^0}(\mathbb{G}_{m,\mathbb{Q}},\mathcal{H}_0)$. In particular, the field of definition of every geometric point of the boundary, i.e. of $M^{K_f}(P,\mathfrak{X})^*(\mathbb{C}) \smallsetminus M^{K_f}(P,\mathfrak{X})(\mathbb{C})$, is an abelian extension of $\mathbb{Q}$. Of course, this field of definition may be larger than the field of definition of the associated connected component of $M^{K_f}(P,\mathfrak{X})^*(\mathbb{C})$. In the toroidal compactification, it is well-known how the arithmetic of F comes into play. This is described in M. Rapoport's thesis [Rap], in a strictly more general setting (i.e. not restricted to characteristic zero).

<u>12.22. Example: The group $CU(n,1)$</u>: Let $E \subset \mathbb{C}$ be an imaginary quadratic field (with a fixed embedding into $\mathbb{C}$), V an E-vector space of dimension $n+1 \geq 3$, and $H: V \times V \to E$ a nondegenerate hermitian form that becomes of type $(n,1)$ over $\mathbb{C}$. We let P be the group of unitary similitudes of V with respect to H, i.e.

$$P := \{g \in \mathcal{R}_{E/\mathbb{Q}}GL_E(V) \mid H(gv,gv') = t \cdot H(v,v') \text{ for some } t \in \mathcal{R}_{E/\mathbb{Q}}\mathbb{G}_{m,E}\}.$$

Every homomorphism $h: \mathbb{S} \to \mathcal{R}_{\mathbb{C}/\mathbb{R}}GL_{\mathbb{C}}(V \otimes_E \mathbb{C})$ such that $V \otimes_E \mathbb{C}$ becomes of Hodge type $\{(-1,0),(0,-1)\}$ induces another structure of $\mathbb{C}$-vector space on $V \otimes_E \mathbb{C}$, which commutes with the old $\mathbb{C}$-structure. Thus it defines a decomposition $V \otimes_E \mathbb{C} = V^+ \oplus V^-$, where V^+ (resp. V^-) is the subspace where the two $\mathbb{C}$-structures coincide (resp. differ by complex conjugation). If we further require that V^+ and V^- are orthogonal with respect to H and that $H|_{V^-}$ is negative definite (whence $\dim(V^-)=1$, $\dim(V^+)=n$, and $H|_{V^+}$ is positive definite), then h must factor through $P_{\mathbb{R}}$. The set $\mathfrak{X}$ of all such h constitutes a $P(\mathbb{R})$-orbit, and $(P,\mathfrak{X})$ is pure Shimura data. Since $n \geq 2$, $P(\mathbb{R})$ and hence $\mathfrak{X}$ are connected. The reflex field of $(P,\mathfrak{X})$ is E.

Every maximal proper $\mathbb{Q}$-parabolic subgroup $Q \subset P$ is the normalizer of a nontrivial isotropic subspace $V' \subset V$. In the given case, we must have $\dim(V')=1$, and Q also stabilizes $(V')^{\perp}$ which is of codimension 1. If $(P_1,\mathfrak{X}_1)$ is the associated rational boundary component of $(P,\mathfrak{X})$, then for every $x_1 \in \mathfrak{X}_1$ and $z \in \mathbb{S}(\mathbb{R}) = \mathbb{C}^\times$, $h_{x_1}(z)$ acts on $V' \otimes_E \mathbb{C}$ through multiplication by $z\bar{z}$, on $((V')^{\perp}/V') \otimes_E \mathbb{C}$ through multiplication by z, and trivially on $(V/(V')^{\perp}) \otimes_E \mathbb{C}$. Since the semisimple part of G is totally anisotropic over $\mathbb{R}$, P_1 is solvable, and, in fact, $P_1/W_1 \cong \mathcal{R}_{E/\mathbb{Q}}\mathbb{G}_{m,E}$. Since $\mathfrak{X}$ is connected, we have $|\mathfrak{X}_1|=1$, and the associated homomorphism is the isomorphism

$$\mathbb{S} = \mathcal{R}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_{m,\mathbb{C}} \cong (\mathcal{R}_{E/\mathbb{Q}}\mathbb{G}_{m,E}) \times_E \mathbb{C}.$$

Moreover $\dim(U_1)=1$, and $\dim(V_1)=2(n-1)$.

By 12.3, the boundary $M^{K_f}(P,\mathfrak{X})^* \smallsetminus M^{K_f}(P,\mathfrak{X})$ of the Baily-Borel compactification is a finite disjoint union of certain $M^{\pi(K_f^1)}((P_1,\mathfrak{X}_1)/W_1)$. In particular, the field of definition of every geometric point of the boundary is an abelian extension of E. Since $\dim(U_1)=1$, there is a unique complete cone decomposition $\mathcal{S}$ for $(P,\mathfrak{X})$. By 12.4, the boundary of the associated toroidal compactification is a finite disjoint union of certain $M^{\pi'(K_f^1)}((P_1,\mathfrak{X}_1)/U_1)$. The family $M^{\pi'(K_f^1)}((P_1,\mathfrak{X}_1)/U_1) \to M^{\pi(K_f^1)}((P_1,\mathfrak{X}_1)/W_1)$ is a torsor under a family of abelian varieties of dimension $n-1$, isogenous to the $(n-1)$-fold product with itself of an elliptic curve with complex multiplication by E. By 12.4 (c) the normal bundle of $M^{\pi'(K_f^1)}((P_1,\mathfrak{X}_1)/U_1)$ in $M^{K_f}(P,\mathfrak{X},\mathcal{S})$ is isomorphic to the line bundle $M^{K_f^1}(P_1,\mathfrak{X}_1,\mathcal{S}_1) \to M^{\pi'(K_f^1)}((P_1,\mathfrak{X}_1)/U_1)$. The coefficients of the q-expansion of a modular form

$$f \in \Gamma(M^{K_f}(P,\mathfrak{X},\mathcal{S}),\omega[\mathrm{dlog}]^{\otimes n})$$

are sections of all nonpositive powers of the invertible sheaf associated to this line bundle. (Since the stratum is blown down in the Baily-Borel