faithful representation of G, and ρ' another representation of G, then the group Γ_p above for ρ is the same as that for $\rho\oplus\rho'$, and contains that for ρ'. Thus if g_f is neat, then its image in any representation of G is neat. A subgroup of $G(\mathbb{A}_f)$ is called <u>neat</u> if all its elements are neat. Clearly a subgroup of a neat subgroup is again neat. If H is a linear algebraic subgroup of G, and $K_f\subset G(\mathbb{A}_f)$ is neat, then $H(\mathbb{A}_f)\cap K_f$ is neat. If $\varphi: G\to H$ is a homomorphism and $K_f\subset G(\mathbb{A}_f)$ is neat, then $\varphi(K_f)$ is neat, since we can consider any faithful representation of H as a representation of G. The neatness property is invariant under all automorphisms of G, and all inner automorphisms of $G(\mathbb{A}_f)$. From the definition it is clear that if $K_f\subset G(\mathbb{A}_f)$ is neat, then $G(\mathbb{Q})\cap K_f$ is neat in the usual sense.

Finally fix an integer $d\geq 3$, and let $K_f(d)\subset GL_n(\hat{\mathbb{Z}})$ be the subgroup of all elements that act trivially on $\hat{\mathbb{Z}}^n/d\cdot\hat{\mathbb{Z}}^n$. We shall show that $K_f(d)$ is neat. It follows that every sufficiently small open compact subgroup of a linear algebraic group is neat. The proof of the neatness of $K_f(d)$ is standard. Note that d is divisible by some prime power $p^i\geq 3$. Consider an element $g_p\in GL_n(\mathbb{Z}_p)$ that acts trivially on $\mathbb{Z}_p^n/p^i\cdot\mathbb{Z}_p^n$, then it suffices to show that the group Γ_p above does not contain a nontrivial root of unity. By assumption every eigenvalue of g_p in $\bar{\mathbb{Z}}_p$ (the algebraic closure) is congruent to 1 modulo $p^i\cdot\bar{\mathbb{Z}}_p$. Thus the same follows for every element of Γ_p, and we have to show that every root of unity $\zeta\in\bar{\mathbb{Z}}_p$, that is congruent to 1 modulo $p^i\cdot\bar{\mathbb{Z}}_p$, is equal to 1. Since the residue class of ζ in $\bar{\mathbb{F}}_p^\times$ is 1, the order of ζ must be a power of p. Suppose this order is p^j with $j\geq 1$, then the following inequalities for the p-adic valuation

$$v_p(1-\zeta) \geq v_p(p^i) = i\cdot v_p(p) = i\cdot p^{j-1}\cdot(p-1)\cdot v_p(1-\zeta) > 0$$

imply $i\cdot(p-1) \leq i\cdot p^{j-1}\cdot(p-1) \leq 1$, which contradicts the assumption that $p^i\geq 3$. Thus $\zeta=1$, as desired.

1. Equivariant families of mixed Hodge structures

Usual Shimura varieties can be viewed analytically as moduli spaces for variations of certain polarized pure Hodge structures, or combinations thereof. Thus, in analogy to [D2] §1.1, a natural starting point for us is to study how certain variations of mixed Hodge structures can be expressed in group theoretic terms. For most of the chapter we consider the case of rational weights, that is where the weight filtration is defined over $\mathbb{Q}$. In 1.3-1.8 we describe equivariant families of rational mixed Hodge structures: these form naturally a complex manifold (1.7). Then we analyze which of these satisfy the two requirements: (a) Griffiths' transversality (1.9-10), and (b) polarizability of all pure constituents (1.11-12). To avoid some pathologies we replace the group in question by a smaller one (1.13-17). Thus we are lead naturally to the group theoretic data described in 1.18. Since the rationality of the weight is a true restriction for usual Shimura varieties, it is desirable to drop this requirement. It turns out that to define mixed Shimura varieties it is enough to assume that the weight acts rationally on the "mixed" part of the data (see 3.3 (a)). The modification 1.19 of the data of 1.18 achieves this.

1.1. Hodge structures: (compare [D3] §§2.1 and 2.3) Let M be a finite dimensional $\mathbb{Q}$-vector space. A pure Hodge structure of weight $n \in \mathbb{Z}$ on M is a decomposition $M_{\mathbb{C}} = \bigoplus_{p+q=n} M^{p,q}$ into $\mathbb{C}$-vector spaces, such that for all $p,q \in \mathbb{Z}$ with $p+q=n$ one has $\overline{(M^{q,p})} = M^{p,q}$. The associated (descending) Hodge filtration on $M_{\mathbb{C}}$ is defined by $F^pM_{\mathbb{C}} := \bigoplus_{p' \geq p} M^{p',q}$. It determines the Hodge structure uniquely, because $M^{p,q} = F^pM_{\mathbb{C}} \cap \overline{(F^qM_{\mathbb{C}})}$.

A rational mixed Hodge structure on M consists of an ascending, exhausting, separated filtration $\{W_nM\}_{n \in \mathbb{Z}}$ of M by $\mathbb{Q}$-vector spaces, called weight filtration, together with a descending, exhausting, separated filtration $\{F^pM_{\mathbb{C}}\}_{p\in\mathbb{Z}}$ of $M_{\mathbb{C}}$, called Hodge filtration, such that for all $n\in\mathbb{Z}$ the Hodge filtration induces a pure Hodge structure of weight n on $Gr^W_nM := W_nM/W_{n-1}M$. A pure Hodge structure of weight n is considered a special case of a mixed Hodge structure by defining the weight filtration as $W_{n'}M = M$ for $n'\geq n$, and $W_{n'}M = 0$ for $n'<n$.

convention de signe opposée à [D-3]

The Hodge numbers are defined as $h^{p,q} := \dim_{\mathbb{C}}(Gr^W_{p+q}M)^{p,q}$. They satisfy $h^{q,p} = h^{p,q}$, almost all $h^{p,q}$ are zero, and their sum is equal to the dimension of M. If $A\subset\mathbb{Z}\oplus\mathbb{Z}$ is an arbitrary subset, then we call a Hodge structure of type A, if $h^{p,q} = 0$ for all $(p,q)\notin A$. The weights that occur in a Hodge structure are the numbers $p+q$ for all pairs (p,q), for which $h^{p,q}\neq 0$. The notions "of weight $\leq n$" and "of weight $\geq n$" are defined in the obvious way.

A morphism of rational mixed Hodge structures is a homomorphism $f: M\to M'$, such that $f(W_nM)\subset W_nM'$ and $f(F^pM_{\mathbb{C}})\subset F^pM'_{\mathbb{C}}$ for all $n,p\in\mathbb{Z}$. By [D3] 2.3.5 (i) the rational mixed Hodge structures form an abelian category with these morphisms. Given rational mixed Hodge structures on M_1 and M_2, there are canonical rational mixed Hodge structures on $M_1\oplus M_2$, the dual $(M_1)^\vee$, on $\mathrm{Hom}(M_1,M_2)$, and on $M_1\otimes M_2$ (see [D3] 1.1.6, 1.1.12 and 2.1.11).

A mixed Hodge structure on M splits over $\mathbb{R}$, if there exists a decomposition $M_{\mathbb{C}} = \bigoplus_{p,q} M^{p,q}$, such that $W_nM_{\mathbb{C}} = \bigoplus_{p+q\leq n} M^{p,q}$, $F^pM_{\mathbb{C}} = \bigoplus_{p'\geq p} M^{p',q}$, and $\overline{(M^{q,p})} = M^{p,q}$. This decomposition is then uniquely determined by these properties. Every pure Hodge structure splits over $\mathbb{R}$, but not every mixed Hodge structure does. If one weakens the requirements, however, one can still associate to every mixed Hodge structure a canonical decomposition $M_{\mathbb{C}} = \bigoplus_{p,q} M^{p,q}$, as in the following proposition.

1.2. Proposition: Fix a rational mixed Hodge structure on M.

(a) There exists a decomposition $M_{\mathbb{C}} = \bigoplus_{p,q} M^{p,q}$, such that $W_n M_{\mathbb{C}} = \bigoplus_{p+q \leq n} M^{p,q}$ and $F^p M_{\mathbb{C}} = \bigoplus_{p' \geq p} M^{p',q}$.

ℚ-structure ? (b) The Hodge structure is uniquely determined by any such decomposition.

(c) There exists a unique decomposition as in (a), which also satisfies

$$\overline{(M^{q,p})} \equiv M^{p,q} \bmod \bigoplus_{p'<p,q'<q} M^{p',q'}.$$

Proof: (a) follows from (c), and (b) is obvious. For the existence in (c) see [D3] 1.2.11, for the uniqueness [CK] 2.2. q.e.d.

Remark: In general there exist different decompositions satisfying (a).

1.3. The Deligne-torus: (compare [D2] 1.1.1. Attention: Our convention is opposite to that in [D1] 1.3 and [D3] 2.1.5.1.) Consider the torus $\mathbb{S} := \mathcal{R}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_{m,\mathbb{C}}$. Over $\mathbb{C}$ it is canonically isomorphic to $\mathbb{G}_{m,\mathbb{C}} \times \mathbb{G}_{m,\mathbb{C}}$, but the action of complex conjugation is twisted by the automorphism c that interchanges the two factors. In particular $\mathbb{S}(\mathbb{R}) = \mathbb{C}^\times$ corresponds to the points of the form $(z,\bar{z})$ with $z \in \mathbb{C}^\times$. While the character group of $\mathbb{G}_{m,\mathbb{C}}$ is $\mathbb{Z}$ in the standard way, we identify the character group of $\mathbb{S}$ with $\mathbb{Z} \oplus \mathbb{Z}$ such that the character (p,q) maps $z \in \mathbb{S}(\mathbb{R}) = \mathbb{C}^\times$ to $z^{-p} \cdot (\bar{z})^{-q} \in \mathbb{C}^\times$. Under this identification the complex conjugation still operates on $\mathbb{Z} \oplus \mathbb{Z}$ by interchanging the two factors. Consider the homomorphisms $w: \mathbb{G}_{m,\mathbb{R}} \hookrightarrow \mathbb{S}$, $\mathbb{R}^\times \ni t \mapsto t \in \mathbb{C}^\times$; $\mu: \mathbb{G}_{m,\mathbb{C}} \to \mathbb{S}_{\mathbb{C}}$, $\mathbb{C}^\times \ni z \mapsto (z,1) \in \mathbb{C}^\times \times \mathbb{C}^\times = \mathbb{S}(\mathbb{C})$; and the norm $N: \mathbb{S} \twoheadrightarrow \mathbb{G}_{m,\mathbb{R}}$, $\mathbb{S}(\mathbb{R}) = \mathbb{C}^\times \ni z \mapsto z \cdot \bar{z} \in \mathbb{R}^\times$. The kernel $\mathbb{S}^1$ of N is anisotropic over $\mathbb{R}$, and we have a short exact sequence $1 \to \mathbb{S}^1 \to \mathbb{S} \to \mathbb{G}_{m,\mathbb{R}} \to 1$.

Let M be a finite dimensional $\mathbb{Q}$-vector space. The choice of a representation $k: \mathbb{S}_{\mathbb{C}} \to GL(M_{\mathbb{C}})$ is equivalent to the choice of a decomposition $M_{\mathbb{C}} = \bigoplus_{p,q} M^{p,q}$, where $M^{p,q}$ is the eigenspace in $M_{\mathbb{C}}$ to the character (p,q). Like in 1.1 we call $W_n M_{\mathbb{C}} = \bigoplus_{p+q \leq n} M^{p,q}$ and $F^p M_{\mathbb{C}} = \bigoplus_{p' \geq p} M^{p',q}$ the associated weight filtration, respectively Hodge filtration, and define the notions "of type A", pure, etc. in the same way. These notions coincide with those of 1.1, if the filtrations are those of a rational mixed Hodge structure on M. We next study the question, under which condition on k this is the case.

1.4. Proposition: Let P be a connected linear algebraic group over $\mathbb{Q}$. Let W be the unipotent radical of P, let $G := P/W$ and $\pi: P \to G$ the canonical projection. Let $h: \mathbb{S}_{\mathbb{C}} \to P_{\mathbb{C}}$ be a homomorphism, such that the following conditions are satisfied:

(i) $\pi \circ h: \mathbb{S}_{\mathbb{C}} \to G_{\mathbb{C}}$ is already defined over $\mathbb{R}$.

(ii) $\pi \circ h \circ w: \mathbb{G}_{m,\mathbb{R}} \to G_{\mathbb{R}}$ is a cocharacter of the center of P/W, that is already defined over $\mathbb{Q}$.

(iii) Under the weight filtration on $\mathrm{Lie}\, P$ defined by $\mathrm{Ad}_P \circ h$ we have $W_{-1}(\mathrm{Lie}\, P) = \mathrm{Lie}\, W$.

Then:

(a) For every representation $\rho: P \to GL(M)$, defined over $\mathbb{Q}$, the homomorphism $\rho \circ h: \mathbb{S}_{\mathbb{C}} \to GL(M_{\mathbb{C}})$ induces a rational mixed Hodge structure on M.

(b) The weight filtration on M is stable under P.

(c) For any $p \in P(\mathbb{R}) \cdot W(\mathbb{C})$ the assertions (a) and (b) also hold for $\mathrm{int}(p) \circ h$ in place of h. The weight filtration and the Hodge numbers in any representation are the same for $\mathrm{int}(p) \circ h$ and for h.

Proof: Assertion (c) follows directly from (a) and (b), because the conditions (i) to (iii) are invariant under conjugation by an element of $P(\mathbb{R}) \cdot W(\mathbb{C})$. For (a) and (b) we first use induction to prove the statement: The weight filtration is defined over $\mathbb{Q}$ and stable under P.

In the case $\dim M = 0$ there is nothing to prove. Otherwise there exists a short exact sequence $0 \to M' \to M \to M'' \to 0$ of representations of P, with M'' nontrivial and irreducible. Let M_W be the space of W-coinvariants of M, then M'' is a quotient of M_W. By definition M_W is completely reducible under P, so there exists a P-invariant subspace $N \subset M$ with $N+M'=M$ and $M'' \cong \mathrm{im}(N) \subset M_W$. If N is properly contained in M, then the assertion follows from the formula $W_nM_{\mathbb{C}} = W_nM'_{\mathbb{C}} + W_nN_{\mathbb{C}}$ for all $n \in \mathbb{Z}$ and the inductive assumption. Otherwise we have $M''=M_W$. The differential of ρ induces a P-equivariant linear map $(\mathrm{Lie}\ W) \otimes M \to M$, whose cokernel is precisely the space of $(\mathrm{Lie}\ W)$-coinvariants, so again equal to M''. Thus we obtain a surjective P-equivariant map $(\mathrm{Lie}\ W) \otimes M \to M'$. Let n be the highest weight that occurs in M. Since by assumption $\mathrm{Lie}\ W$ is of weight ≤ -1, M' must be of weight $\leq n-1$. Thus the weight n does not occur in M', hence it must occur in M''. By condition (ii) every irreducible representation is pure of some weight, so M'' must be pure of weight n. For the weight filtration we thus get: $W_mM_{\mathbb{C}} = W_mM'_{\mathbb{C}}$ for $m<n$, and $W_mM_{\mathbb{C}} = M_{\mathbb{C}}$ for $m \geq n$. Therefore the assertion follows from the inductive assumption for M'.

With this (b) is proved. For (a) it remains to show that for every $n \in \mathbb{Z}$ the Hodge filtration induces a pure Hodge structure of weight n on $\mathrm{Gr}^W_n M := W_nM/W_{n-1}M$. Since $\mathrm{Gr}^W_n M$, as we have just proved, is again a representation of P over $\mathbb{Q}$, it suffices to show: If M is already pure of weight n, then the decomposition $M_{\mathbb{C}} = \bigoplus_{p+q=n} M^{p,q}$ is a pure Hodge structure of weight n. Let us again consider the map $(\mathrm{Lie}\ W) \otimes M \to M$ of above. The left hand side is mixed of weight $\leq n-1$, the right hand side pure of weight n. Thus this map must be zero, which means that W operates trivially on M. By (i) the operation of $S_{\mathbb{C}}$ on $M_{\mathbb{C}}$ is therefore defined over $\mathbb{R}$, so $\overline{(M^{q,p})} = M^{p,q}$ for all p, q with $p+q=n$. q.e.d.

<u>1.5. Proposition</u>: Let M be a finite dimensional $\mathbb{Q}$-vector space. A representation $k: \mathbb{S}_{\mathbb{C}} \to GL(M_{\mathbb{C}})$ defines a rational mixed Hodge structure on M, if and only if there exists a connected linear algebraic group P over $\mathbb{Q}$, a representation $\rho: P \to GL(M)$ defined over $\mathbb{Q}$, and a homomorphism $h: \mathbb{S}_{\mathbb{C}} \to P_{\mathbb{C}}$, such that $k = \rho \circ h$ and the conditions in 1.4 are satisfied.

<u>Proof</u>: One implication is already contained in 1.4. So let us assume that k induces a rational mixed Hodge structure on M. Let $P \subset GL(M)$ be the stabilizer of the weight filtration, this is a $\mathbb{Q}$-parabolic, hence connected subgroup of $GL(M)$. Let $\rho: P \hookrightarrow GL(M)$ be the inclusion, then k factors in exactly one way through a homomorphism $h: \mathbb{S}_{\mathbb{C}} \to P_{\mathbb{C}}$. By assumption $Gr^W_n M$ is a pure Hodge structure of weight n for every $n \in \mathbb{Z}$. Thus for all $p,q \in \mathbb{Z}$ with $p+q=n$ the space $\overline{(Gr^W_n M)^{q,p}} = (Gr^W_n M)^{p,q}$ is the eigenspace in $Gr^W_n M$ associated to the character $c((q,p)) = (p,q)$, so the operation of $\mathbb{S}_{\mathbb{C}}$ on $Gr^W_n M_{\mathbb{C}}$ is already defined over $\mathbb{R}$. Since furthermore $\mathbb{G}_{m,\mathbb{R}} \subset \mathbb{S}$ acts by scalars on $Gr^W_n M_{\mathbb{R}}$, its operation is given by a cocharacter $\mathbb{G}_{m,\mathbb{Q}} \to Z(GL(Gr^W_n M))$ that is defined over $\mathbb{Q}$. The properties 1.4 (i) and (ii) now follow from the isomorphism $P/W \cong \prod_n GL(Gr^W_n M)$.

Finally $Lie\, P$ is the successive extension of $Hom(Gr^W_n M, Gr^W_m M)$ for all $n \geq m$, and $Lie\, W$ the successive extension of $Hom(Gr^W_n M, Gr^W_m M)$ for all $n > m$. In $Hom(Gr^W_n M, Gr^W_m M)_{\mathbb{C}}$ as representation of $\mathbb{S}_{\mathbb{C}}$ only characters (p,q) occur with $p+q = m-n$, so $Lie\, W$ contains precisely all eigenspaces in $Lie\, P$ associated to the characters (p,q) with $p+q \leq -1$. This implies 1.4 (iii). <u>q.e.d.</u>

<u>1.6. Equivariant families of Hodge structures</u>: Generalizing [D2] 1.1.11 and 1.1.12 we want to parametrize certain rational mixed Hodge structures by a homogeneous space under a real Lie group. Furthermore we

want this group to come from a linear algebraic group over $\mathbb{Q}$. Finally the homogeneous space should be described in terms of homomorphisms $\mathbb{S}_{\mathbb{C}} \to P_{\mathbb{C}}$ and should carry a canonical complex structure.

More concretely let P be a connected linear algebraic group over $\mathbb{Q}$. In a crucial case (see 4.7) P will be strongly related to a parabolic subgroup of another group, whence the unusual notation. Let W be the unipotent radical of P, and $\mathcal{H}_W$ a $P(\mathbb{R})\cdot W(\mathbb{C})$-conjugacy class in $\mathrm{Hom}(\mathbb{S}_{\mathbb{C}}, P_{\mathbb{C}})$. We assume that for one, hence by 1.4 (c) for all $h \in \mathcal{H}_W$ the conditions in 1.4 hold. We call a representation M of P pure of weight n, mixed of weight $\leq n$, of type $A \subset \mathbb{Z} \oplus \mathbb{Z}$, etc. if and only if for some ($\Leftrightarrow$ by 1.4 (c) for all) $h \in \mathcal{H}_W$ the induced Hodge structure on M has the respective property. Note that by 1.4 (ii) $\mathrm{Lie}\, G = \mathrm{Lie}\, P/W$ is pure of weight 0, so by 1.4 (iii) $\mathrm{Lie}\, P$ is mixed of weight ≤ 0. The following propositions 1.7, 1.10 and 1.12 are generalizations of [D2] 1.1.14.

<u>1.7. Proposition</u>: Let P and $\mathcal{H}_W$ be as in 1.6, and let M be a faithful representation of P. Let φ be the obvious map

$$\mathcal{H}_W \to \{\text{rational mixed Holdge structures on } M\}.$$

(a) There exists a unique structure on $\varphi(\mathcal{H}_W)$ as a complex manifold, such that the Hodge filtration on $M_{\mathbb{C}}$ depends analytically on $\varphi(h) \in \varphi(\mathcal{H}_W)$. This structure is $P(\mathbb{R})\cdot W(\mathbb{C})$-invariant, and $W(\mathbb{C})$ acts analytically on $\varphi(\mathcal{H}_W)$.

(b) For any other representation M' of P the analogous map

$$\varphi' : \mathcal{H}_W \to \{\text{rational mixed Hodge structures on } M'\}$$

factors through $\varphi(\mathcal{H}_W)$. The Hodge filtration on M' varies analytically with $\varphi(h) \in \varphi(\mathcal{H}_W)$.

(c) If in addition M' is faithful, then $\varphi(\mathcal{H}_W)$ and $\varphi'(\mathcal{H}_W)$ are canonically isomorphic and the isomorphism is compatible with the complex structure.

Proof: (a) Since the weight filtration on M is constant, the Hodge filtration gives an injective map $\varphi(\mathcal{H}_W) \hookrightarrow \mathrm{Grass}(M)(\mathbb{C})$ to a certain Grassmann variety. To prove the first assertion we have to show that its image is a locally closed analytic subvariety. Fix $h \in \mathcal{H}_W$ and let $C(h)$ be the centralizer of h in $P(\mathbb{R})\cdot W(\mathbb{C})$, then φ factors through

$$\mathcal{H}_W \cong P(\mathbb{R})\cdot W(\mathbb{C})/C(h) \to P(\mathbb{C})/\exp F^0(\mathrm{Lie}\ P)_{\mathbb{C}} \not\hookrightarrow \mathrm{Grass}(M)(\mathbb{C}).$$

The map on the right hand side is a closed embedding of complex manifolds. The complex structure in the middle is $P(\mathbb{C})$-invariant, and $P(\mathbb{C})$ operates analytically. Thus it is enough to show that the image of $\mathcal{H}_W$ is an open subset of $P(\mathbb{C})/\exp F^0(\mathrm{Lie}\ P)_{\mathbb{C}}$. Let $L := (\mathrm{Lie}\ P)_{\mathbb{R}} + (\mathrm{Lie}\ W)_{\mathbb{C}} \subset (\mathrm{Lie}\ P)_{\mathbb{C}}$, and let $L^{0,0} \subset L$ be the eigenspace corresponding to the trivial operation of $\mathrm{Ad}\circ h$. Then on the tangent space at h the map is given by

$$T_{\mathcal{H}_W,h} \cong L/L^{0,0} \to (\mathrm{Lie}\ P)_{\mathbb{C}}/F^0(\mathrm{Lie}\ P)_{\mathbb{C}},$$

and we have to show that this map is surjective. This is equivalent to the equality $(\mathrm{Lie}\ P)_{\mathbb{C}} = L + F^0(\mathrm{Lie}\ P)_{\mathbb{C}}$. Since both sides of this contain $(\mathrm{Lie}\ W)_{\mathbb{C}}$, it suffices to show the equality $(\mathrm{Lie}\ G)_{\mathbb{C}} = (\mathrm{Lie}\ G)_{\mathbb{R}} + F^0(\mathrm{Lie}\ G)_{\mathbb{C}}$. But $\mathrm{Lie}\ G$ carries a pure Hodge structure of weight 0, so this equality follows from $(\mathrm{Lie}\ G)_{\mathbb{R}} \cap F^0(\mathrm{Lie}\ G)_{\mathbb{C}} \subset (\mathrm{Lie}\ G)^{0,0}$ and dimension count.

(b) follows directly from (c), applied to $M \oplus M'$ in the place of M'. By part (b) of the following lemma the fibres of φ do not depend on the choice of M, so the two maps φ and φ' have identical fibres. This implies the first assertion of (c). Finally the proof of (a) shows that the complex structure on $\varphi(\mathcal{H}_W)$ can be defined independently of M. q.e.d.

<u>1.8. Lemma</u>: Let P, $\mathcal{H}_W$, M and φ be as in 1.7, and let $h \in \mathcal{H}_W$.

(a) The projection $\pi: P \to G$ induces an isomorphism

$$\mathrm{Cent}_{P(\mathbb{R})\cdot W(\mathbb{C})}(h) \to \mathrm{Cent}_{G(\mathbb{R})}(\pi \circ h).$$

(b) We have

$$\mathrm{Stab}_{P(\mathbb{R})\cdot W(\mathbb{C})}(\varphi(h)) = \exp(F^0(\mathrm{Lie}\ W)_{\mathbb{C}}) \rtimes \mathrm{Cent}_{P(\mathbb{R})\cdot W(\mathbb{C})}(h).$$

<u>Proof</u>: (a) By 1.4 (ii) and (iii) the space of invariants in $\mathrm{Lie}\ W_{\mathbb{C}}$ under $(\mathrm{Ad}|_W) \circ h \circ w$ is the zero subspace. Hence h defines a canonical Levi decomposition $P_{\mathbb{C}} = W_{\mathbb{C}} \rtimes \mathrm{Cent}_{P_{\mathbb{C}}}(h \circ w)$. Since moreover $\pi \circ h$ is by 1.4 (ii) already defined over $\mathbb{R}$, we have a corresponding decomposition

$$P(\mathbb{R})\cdot W(\mathbb{C}) = W(\mathbb{C}) \rtimes \mathrm{Cent}_{P(\mathbb{R})\cdot W(\mathbb{C})}(h \circ w).$$

Thus $\mathrm{Cent}_{P(\mathbb{R})\cdot W(\mathbb{C})}(h \circ w) \cong G(\mathbb{R})$, so the centralizers of h in both these groups are isomorphic.

(b) First suppose that $W = 1$. Then there exists a unique decomposition $M = \bigoplus_{n \in \mathbb{Z}} M_n$ into representations M_n of $P = G$ that are pure of weight n. For all p, q we therefore have

$$M^{p,q} = F^p M_{\mathbb{C}} \cap \overline{F^q M_{\mathbb{C}}} \cap W_{p+q} M_{\mathbb{C}},$$

so the decomposition into eigenspaces under $\rho \circ h$ is already uniquely determined by $\varphi(h)$. Hence we even have $\mathrm{Stab}_{P(\mathbb{R})}(\varphi(h)) = \mathrm{Cent}_{P(\mathbb{R})}(h)$. In the general case this together with (a) implies

$$\mathrm{Cent}_{P(\mathbb{R})\cdot W(\mathbb{C})}(h) \subset \mathrm{Stab}_{P(\mathbb{R})\cdot W(\mathbb{C})}(\varphi(h)) \subset W(\mathbb{C}) \rtimes \mathrm{Cent}_{P(\mathbb{R})\cdot W(\mathbb{C})}(h),$$

so it only remains to show that $\mathrm{Stab}_{W(\mathbb{C})}(\varphi(h)) = \exp(F^0(\mathrm{Lie}\ W)_{\mathbb{C}})$. This follows from

$$\mathrm{Stab}_{(\mathrm{Lie}\ W)_{\mathbb{C}}}(\varphi(h)) = (\mathrm{Lie}\ W)_{\mathbb{C}} \cap F^0(\mathrm{End}(M))_{\mathbb{C}} = F^0(\mathrm{Lie}\ W)_{\mathbb{C}}.$$

<u>q.e.d.</u>

1.9. Variation of Hodge structures: Let X be a complex manifold. A variation of rational mixed Hodge structures over X consists of a local system of finite dimensional $\mathbb{Q}$-vector spaces, together with a rational mixed Hodge structure on every fibre, such that:

(a) The weight filtration is locally constant and the Hodge filtration varies holomorphically. Let $\mathfrak{M}$ be the locally free coherent $\mathcal{O}_X$-sheaf associated to the vector bundle, and let $\mathcal{F}^p\mathfrak{M}$ be the coherent subsheaves for all $p\in\mathbb{Z}$ that induce the Hodge filtration in each ~~stalk~~. fibre

(b) Transversality: For every $p\in\mathbb{Z}$ the canonical connection $\nabla: \mathfrak{M} \to \mathfrak{M} \otimes_{\mathcal{O}_X} \Omega^1_X$ maps the subsheaf $F^p\mathfrak{M}$ to $F^{p-1}\mathfrak{M} \otimes_{\mathcal{O}_X} \Omega^1_X$.

1.10. Proposition: Let P, $\mathcal{H}_W$, M and φ be as in 1.7. Then we have a variation of rational mixed Hodge structures on M over $\varphi(\mathcal{H}_W)$, if and only if for one ($\Leftrightarrow$ for all) $h\in\mathcal{H}_W$ the Hodge structure on $\operatorname{Lie} P$ is of type

$$\{(-1,1),(0,0),(1,-1)\} \cup \{(-1,0),(0,-1)\} \cup \{(-1,-1)\}.$$

Proof: With the notation of the proof of 1.7 the transversality condition in the point $\varphi(h)$ is equivalent to

$$\operatorname{im}(d\Phi) \subset F^{-1}\operatorname{End}(M)_{\mathbb{C}}/F^0\operatorname{End}(M)_{\mathbb{C}}.$$

Since this image equals $(\operatorname{Lie} P)_{\mathbb{C}}/F^0(\operatorname{Lie} P)_{\mathbb{C}}$, this means $F^{-1}(\operatorname{Lie} P)_{\mathbb{C}} = (\operatorname{Lie} P)_{\mathbb{C}}$. In other words the Hodge numbers $h^{p,q}$ must vanish for all $p,q\in\mathbb{Z}$ with $p<-1$. If they vanish, then so do all $h^{p,q}$ with $q<-1$. Since $\operatorname{Lie} P$ is mixed of weight ≤ 0, one easily finds the condition equivalent to the stated type restriction. q.e.d.

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1.11. Polarization of pure Hodge structures: Let $\mathcal{A}$ be a primitive fourth root of unity in $\mathbb{C}$, then $\mathbb{Z}(1) := 2\pi\mathcal{A}\cdot\mathbb{Z}$ is a submodule of $\mathbb{C}$ that is independent of the choice of $\mathcal{A}$. For all $n \in \mathbb{Z}$ and any $\mathbb{Z}$-module M define $M(n) := M \otimes \mathbb{Z}(1)^{\otimes n}$. By the given embedding $\mathbb{Z}(1) \hookrightarrow \mathbb{C}$ we get a canonical isomorphism $M(n)_{\mathbb{C}} \cong M_{\mathbb{C}}$. The Tate Hodge structure is the $\mathbb{Q}$-vector space $\mathbb{Q}(1)$ with the pure Hodge structure of type $(-1,-1)$. For every $n \in \mathbb{Z}$ we get a pure Hodge structure of type $(-n,-n)$ on $\mathbb{Q}(n) = \mathbb{Q}(1)^{\otimes n}$, and again a canonical isomorphism $\mathbb{Q}(n)_{\mathbb{C}} \cong \mathbb{C}$.

Let M be a $\mathbb{Q}$-vector space and $k: \mathbb{S} \to GL(M_{\mathbb{R}})$ a representation that induces a pure Hodge structure of weight n on M. Let $\Psi: M \otimes M \to \mathbb{Q}(-n)$ be a morphism of Hodge structures. Choose a primitive fourth root of unity $\mathcal{A} \in \mathbb{C}^{\times} = \mathbb{S}(\mathbb{R})$, and consider the bilinear form

$$M_{\mathbb{R}} \times M_{\mathbb{R}} \to \mathbb{Q}(-n)_{\mathbb{C}} = \mathbb{C}, \ (m,m') \mapsto (2\pi\mathcal{A})^n \cdot \Psi(m, k(\mathcal{A})m').$$

Claim: It does not depend on the choice of $\mathcal{A}$, and takes values in $\mathbb{R}$.

Proof: Since M is pure of weight n, we have $k(-\mathcal{A}) = k(-1)\cdot k(\mathcal{A}) = (-1)^n \cdot k(\mathcal{A})$. Thus on replacing $\mathcal{A}$ by $-\mathcal{A}$ the bilinear form is multiplied by $(-1)^n \cdot (-1)^n = 1$. So it does not depend on the choice of $\mathcal{A}$. Also note that $k(\mathcal{A})$ lies in $GL(M)(\mathbb{R})$, so $\Psi(m, k(\mathcal{A})m')$ lies in $\mathbb{Q}(-n)_{\mathbb{R}} = (2\pi\mathcal{A})^{-n} \cdot \mathbb{R} \subset \mathbb{C}$. This implies the second assertion. q.e.d.

Having shown this the following definition is meaningful. A polarization of a rational Hodge structure pure of weight n is a morphism of Hodge structures $\Psi: M \otimes M \to \mathbb{Q}(-n)$, such that the bilinear form

$$M_{\mathbb{R}} \times M_{\mathbb{R}} \to \mathbb{R}, \ (m,m') \mapsto (2\pi\mathcal{A})^n \cdot \Psi(m, k(\mathcal{A})m')$$

is symmetric and positive definite. Note that this pairing is symmetric if and only if n is even and Ψ symmetric, or n odd and Ψ alternating.

<u>1.12. Proposition</u>: Let P and $\mathcal{H}_W$ be as in 1.6. Let G_1 be a normal subgroup of $G=P/W$, defined over $\mathbb{Q}$, which for one ($\Leftrightarrow$ for all) $h\in\mathcal{H}_W$ contains the image of $h(S^1)$. The following assertions are equivalent:

(a) $\mathrm{int}(\pi(h(\mathcal{A})))$ is a Cartan involution on G_1 for one ($\Leftrightarrow$ for all) $h\in\mathcal{H}_W$. In particular the center of G_1 is compact, hence splits over a CM-field.

(b) For every representation M of P which is pure of some weight n there exist
- a representation N of P, defined over $\mathbb{Q}$, which factors through G/G_1 and is pure of type (n,n),
- a P-equivariant nondegenerate pairing $\Psi: M\otimes M \to N$, and
- for every $h\in\mathcal{H}_W$ a morphism of rational Hodge structures $\lambda_h: N \to \mathbb{Q}(-n)$,

such that for every $h\in\mathcal{H}_W$ the pairing $\lambda_h\circ\Psi: M\otimes M\to\mathbb{Q}(-n)$ is a polarization for the Hodge structure on M defined by h.

<u>Proof</u>: Both conditions depend only on P/W, so we may assume that $P=G$. Fix a pure representation $\rho: G\to GL(M)$. Fix $h_o\in\mathcal{H}_W$ and a choice of $\mathcal{A}$, then we shall show that condition (b) for the given representation M is equivalent to the following one:

(c) There exists a $G_{1,\mathbb{R}}$-invariant pairing $\Psi': M_\mathbb{R}\times M_\mathbb{R}\to\mathbb{R}$, defined over $\mathbb{R}$, such that $(m,m')\mapsto\Psi'(m,\rho\circ h_o(\mathcal{A})m')$ is symmetric and positive definite.

Clearly (b) implies (c) with $\Psi':=(2\pi\mathcal{A})^n\cdot\lambda_{h_o}\circ\Psi$. Conversely assume (c). Note first that since $(\rho\circ h_o(\mathcal{A}))^2 = \rho\circ h_o(-1) = (-1)^n$, the pairing Ψ' is symmetric if n is even, alternating if n is odd. So let $N := (S^2M)_{G_1}$ if n is even, respectively $(\Lambda^2M)_{G_1}$ if n is odd, where $(.)_{G_1}$ denotes the G_1-coinvariants. Let $\Psi: M\otimes M\to N$ be the canonical pairing, then by construction Ψ' factors through N, hence $\Psi'=\mu_o\circ\Psi$ for some homomorphism $\mu_o: N_{\mathbb{R}} \to \mathbb{R}$. Since G_1 is a normal subgroup of G, N is a G-invariant quotient of $M \otimes M$, hence necessarily pure of weight $2n$. Since moreover G_1 operates trivially on N, it is already pure of type (n,n). Since by assumption Ψ' is nontrivial, so is Ψ.

By the construction of N for every $\mu \in \mathrm{Hom}(N_{\mathbb{R}}, \mathbb{R})$ the pairing

$$M_{\mathbb{R}} \times M_{\mathbb{R}} \to \mathbb{R}, \ (m,m') \mapsto \mu \circ \Psi(m, \rho \circ h_o(\sqrt{-1})m')$$

is automatically symmetric. Thus the set of all μ, such that this pairing is symmetric and positive definite, is open in the usual topology. Furthermore the existence of μ_o shows that this set is nonempty. Since the homomorphisms that are defined over $\mathbb{Q}$ are dense, we can replace μ_o by a homomorphism $N \to \mathbb{Q}$ that is defined over $\mathbb{Q}$. Let $\lambda_{h_o} := (2\pi\sqrt{-1})^{-n} \cdot \mu_o: N \to \mathbb{Q}(-n)$, this is automatically a morphism of rational Hodge structures, and by construction $\lambda_{h_o} \circ \Psi$ is a polarization for the Hodge structure on M defined by h_o.

Now let $g \in G(\mathbb{R})$ and $h = \mathrm{int}(g) \circ h_o$. Denote by $\tau: G \to GL(N)$ the representation of G on N. Then

$$\mu_o \circ \tau(g^{-1}) \circ \Psi(m, \rho \circ h(\sqrt{-1})m') = \mu_o \circ \Psi(\rho(g^{-1})m, \rho \circ h_o(\sqrt{-1}) \cdot \rho(g^{-1})m'),$$

so after replacing μ_o by $\mu_o \circ \tau(g)^{-1}$, condition (c) holds for h in place of h_o. Since the above definitions of N and Ψ do not depend on h_o, the desired λ_h exists for every $h \in \mathcal{H}_W$. Thus (c) implies (b).

By [D2] 1.1.15 condition (c) holds for all M if and only if $\mathrm{int}(h_o(\sqrt{-1}))$ is a Cartan involution on $G_{1,\mathbb{R}}$. Clearly this second property is invariant under conjugation by $G(\mathbb{R})$. It also implies that $Z(G_1)(\mathbb{R})$ is compact, so the torus $Z(G_1)^o$ splits over a CM-field. Thus (c) is equivalent to (a). q.e.d.

1.13. Restriction of the group: In the situation of 1.12, we encounter the following phenomenon. The quotient G/G_1 is entirely arbitrary, but it contributes almost nothing to the family of Hodge structures. In particular the semisimple part of G/G_1 is ballast. On the other hand, N is an arbitrary representation of G/G_1, so the G-orbit generated by λ_h may be disturbingly large. For our purposes we want to have that this orbit consists only of multiples of λ_h, at least for irreducible M. By 1.12, this clearly holds if G/G_1 is a $\mathbb{Q}$-split torus. Thus, to solve both problems at the same time, we impose the following condition on P. Let G_2 be the smallest normal subgroup of G, defined over $\mathbb{Q}$, which contains the image $\pi \circ h(\mathbb{S})$ for one ($\Leftrightarrow$ for all) $h \in \mathcal{H}_W$. This group essentially determines the family of Hodge structures. We require:

(*) G/G_2 is an almost direct product of a $\mathbb{Q}$-split torus with a torus of compact type defined over $\mathbb{Q}$.

Consider the situation of 1.12, with G_1 the smallest normal $\mathbb{Q}$-subgroup of G such that G/G_1 is a $\mathbb{Q}$-split torus. Clearly the condition (*) above together with 1.12 (a) is equivalent to:

(a') $\mathrm{int}(\pi(h(\sqrt{-1})))$ induces a Cartan involution on $G^{ad}_{\mathbb{R}}$, and G^{ad} possesses no nontrivial factors of compact type that are defined over $\mathbb{Q}$. Furthermore the center of G is an almost direct product of a $\mathbb{Q}$-split torus with a torus of compact type defined over $\mathbb{Q}$.

Assume that this condition holds. Then in 1.12 (b), N is a direct sum of onedimensional representations of G. If M is irreducible, then the space of G_1-coinvariants of $M \otimes M$ has dimension ≤ 1, so we can choose N to have dimension 1. If M is arbitrary, but $\dim(G/G_1) \leq 1$, then G operates by scalars on N through a character that depends only on n. From the proof of 1.12 we see that again we can choose N to have dimension 1. Observe that if $n=0$, the G-orbit generated by λ_h

always has dimension ≥ 1. Moreover, as in the examples 2.7-8 below, the different λ_h need not have the same sign.

__1.14. Proposition:__ Suppose that $W=1$ and the conditions of 1.10 and 1.12 are satisfied. Then every connected component of $\mathcal{H}_W \cong \varphi(\mathcal{H}_W)$ is a hermitian symmetric domain.

__Proof:__ [D2] 1.1.14 (iii). __q.e.d.__

__1.15. Replacing by a smaller orbit:__ In general the map φ from proposition 1.7 is not injective. We can, however, replace $\mathcal{H}_W$ by an orbit under a subgroup of $P(\mathbb{R})\cdot W(\mathbb{C})$, and still get the same image under φ. Which subgroup we can choose is seen from 1.2 (c). In particular, if we want a subgroup of the form $P(\mathbb{R})\cdot U(\mathbb{C})$, where $U \subset W$ is a subgroup defined over $\mathbb{Q}$, in general there is only the following possibility. Although in general the resulting map will still not be injective, it will be injective in the case that is of interest to us, namely when the condition of 1.10 is satisfied.

Explicitly let P and $\mathcal{H}_W$ be as in 1.6. Let $U \subset W$ be the unique connected subgroup, defined over $\mathbb{Q}$, such that $\mathrm{Lie}\, U = W_{-2}(\mathrm{Lie}\, W)$. By 1.4 (c) it does not depend on $h \in \mathcal{H}_W$. Let π' be the canonical projection $P \to P/U$. Let $V := W/U$, then $\mathrm{Lie}\, V$ is pure of weight -1. Since $[\mathrm{Lie}\, V, \mathrm{Lie}\, V] \subset \mathrm{Lie}\, V$ is at the same time pure of weight -1 and -2, V must be abelian. Consider instead of 1.4 (i) the following stronger condition for $h \in \mathrm{Hom}(\mathbb{S}_{\mathbb{C}}, P_{\mathbb{C}})$:

(i') $\pi' \circ h: \mathbb{S}_{\mathbb{C}} \to (P/U)_{\mathbb{C}}$ is already defined over $\mathbb{R}$.

Finally let

$$\mathcal{H} := \{h \in \mathcal{H}_W \mid h \text{ satisfies (ii)}\}.$$

$$\mathcal{H} := \{ h \in \mathcal{H}_W \mid h \text{ satisfies } (i') \}$$

__1.16. Proposition:__ With the notations of 1.15 we have:

(a) $\mathcal{H}$ is a nonempty $P(\mathbb{R})\cdot U(\mathbb{C})$-orbit in $\mathrm{Hom}(S_{\mathbb{C}}, P_{\mathbb{C}})$.

(b) For M and φ as in 1.7 we have $\varphi(\mathcal{H}) = \varphi(\mathcal{H}_W)$.

(c) If moreover $F^0(\mathrm{Lie}\, U)_{\mathbb{C}} = 0$, then $\varphi(\mathcal{H}) \cong \mathcal{H}$.

__Proof:__ Since Lie V is pure of weight -1, we have

$$F^0(\mathrm{Lie}\, V)_{\mathbb{C}} \cap \overline{F^0(\mathrm{Lie}\, V)_{\mathbb{C}}} = (\mathrm{Lie}\, V)_{\mathbb{R}} \cap F^0(\mathrm{Lie}\, V)_{\mathbb{C}} = 0,$$

and by dimension count we get $(\mathrm{Lie}\, V)_{\mathbb{C}} = (\mathrm{Lie}\, V)_{\mathbb{R}} \oplus F^0(\mathrm{Lie}\, V)_{\mathbb{C}}$. This implies on the one hand $(\mathrm{Lie}\, W)_{\mathbb{C}} = ((\mathrm{Lie}\, W)_{\mathbb{R}} + (\mathrm{Lie}\, U)_{\mathbb{C}}) + F^0(\mathrm{Lie}\, W)_{\mathbb{C}}$, that is

$$W(\mathbb{C}) = (W(\mathbb{R})\cdot U(\mathbb{C}))\cdot \exp(F^0(\mathrm{Lie}\, W)_{\mathbb{C}}),$$

on the other hand

$$((\mathrm{Lie}\, W)_{\mathbb{R}} + (\mathrm{Lie}\, U)_{\mathbb{C}}) \cap F^0(\mathrm{Lie}\, W)_{\mathbb{C}} = (\mathrm{Lie}\, U)_{\mathbb{C}} \cap F^0(\mathrm{Lie}\, W)_{\mathbb{C}} = F^0(\mathrm{Lie}\, U)_{\mathbb{C}},$$

that is

$$(W(\mathbb{R})\cdot U(\mathbb{C})) \cap \exp(F^0(\mathrm{Lie}\, W)_{\mathbb{C}}) = \exp(F^0(\mathrm{Lie}\, U)_{\mathbb{C}}).$$

As Lie W is of weight ≤ -1, there exists no nontrivial subgroup of W on which P operates trivially. Thus the variety of all Levi decompositions of P is a principal homogeneous space under W.

Let us first prove (b). We have to show that for every $h \in \mathcal{H}_W$ there exists $p \in \mathrm{Stab}_{P(\mathbb{R})\cdot W(\mathbb{C})}(\varphi(h))$, such that $\mathrm{int}(p)\circ h \in \mathcal{H}$, that is $\pi' \circ \mathrm{int}(p) \circ h \colon S_{\mathbb{C}} \to (P/U)_{\mathbb{C}}$ is defined over $\mathbb{R}$. As in the proof of 1.8 (a) consider the Levi decomposition of $P_{\mathbb{C}}$ defined by h. Since there also exists a Levi decomposition that is defined over $\mathbb{R}$, there exists $w_0 \in W(\mathbb{C})$

such that $\mathrm{int}(w_o)\circ h\circ w$, hence also $\mathrm{int}(w_o)\circ h$ is defined over $\mathbb{R}$. Write $w_o = w'\cdot w''$ with $w' \in W(\mathbb{R})\cdot U(\mathbb{C})$ and $w'' \in \exp(F^0(\mathrm{Lie}\ W)_{\mathbb{C}})$. Then $\pi'\circ\mathrm{int}(w'')\circ h = \pi'\circ\mathrm{int}(w')^{-1}\circ\mathrm{int}(w_o)\circ h$ is defined over $\mathbb{R}$, and by 1.8 (b) we have $\varphi(\mathrm{int}(w'')\circ h) = \varphi(h)$. Thus the assertion follows with $p = w''$.

This also proves that $\mathcal{H}$ is nonempty. (a) now follows from the stronger statement:

(*) For any $h \in \mathcal{H}$ and $p \in P(\mathbb{R})\cdot W(\mathbb{C})$, we have $\mathrm{int}(p)\circ h \in \mathcal{H}$ if and only if $p \in P(\mathbb{R})\cdot U(\mathbb{C})$.

To prove this we may assume without loss of generality that U is trivial. Writing $p = p'\cdot w'$ with $p' \in P(\mathbb{R})$ and $w'' \in W(\mathbb{C})$ we find that $\mathrm{int}(p)\circ h$ is defined over $\mathbb{R}$ if and only if $\mathrm{int}(w')\circ h$ is defined over $\mathbb{R}$. That again is true if and only if the Levi decomposition of $P_{\mathbb{C}}$ that is defined by $\mathrm{int}(w')\circ h\circ w$ is defined over $\mathbb{R}$. By the above remark the map

$$w' \mapsto (\text{the Levi decomposition of } P_{\mathbb{C}} \text{ defined by } \mathrm{int}(w')\circ h\circ w)$$

is an algebraic isomorphism of W to the variety of Levi decompositions of P, and is defined over $\mathbb{R}$ since $h\circ w$ is defined over $\mathbb{R}$. Thus the Levi decomposition associated to $\mathrm{int}(w')\circ h\circ w$ is defined over $\mathbb{R}$ if and only if $w' \in W(\mathbb{R})$, as desired.

For (c) consider an element $h \in \mathcal{H}$. The fibre $(\varphi|_{\mathcal{H}})^{-1}(\varphi(h))$ is the $\mathrm{Stab}_{P(\mathbb{R})\cdot U(\mathbb{C})}(\varphi(h))$-orbit in $\mathcal{H}$ that is generated by h. If $F^0(\mathrm{Lie}\ U)_{\mathbb{C}} = 0$, then by part (b) of the following lemma this fibre consists only of h. Thus $\varphi|_{\mathcal{H}}$ is a bijection. <u>q.e.d.</u>

<u>1.17. Lemma</u>: Let P, $\mathcal{H}$, etc. be as in 1.15, and let $h \in \mathcal{H}$. Then

(a) $\mathrm{Cent}_{P(\mathbb{R})\cdot U(\mathbb{C})}(h) = \mathrm{Cent}_{P(\mathbb{R})\cdot W(\mathbb{C})}(h) \cong \mathrm{Cent}_{G(\mathbb{R})}(\pi\circ h)$, and

(b) $\mathrm{Stab}_{P(\mathbb{R})\cdot U(\mathbb{C})}(\varphi(h)) = \exp(F^0(\mathrm{Lie}\ U)_{\mathbb{C}}) \rtimes \mathrm{Cent}_{P(\mathbb{R})\cdot U(\mathbb{C})}(h)$.

Proof: (a) By 1.8 (a) it suffices to show that $\mathrm{Cent}_{P(\mathbb{R})\cdot W(\mathbb{C})}(h)$ is contained in $P(\mathbb{R})\cdot U(\mathbb{C})$. This follows from the assertion (*) in the proof of 1.16.

(b) The assertions (a) and 1.8 (b) imply

$$\begin{aligned}\mathrm{Stab}_{P(\mathbb{R})\cdot U(\mathbb{C})}(\varphi(h)) &= (P(\mathbb{R})\cdot U(\mathbb{C})) \cap \mathrm{Stab}_{P(\mathbb{R})\cdot W(\mathbb{C})}(\varphi(h)) \\ &= (P(\mathbb{R})\cdot U(\mathbb{C})) \cap (\exp(F^0(\mathrm{Lie}\ W)_{\mathbb{C}}) \rtimes \mathrm{Cent}_{P(\mathbb{R})\cdot W(\mathbb{C})}(h)) \\ &= ((P(\mathbb{R})\cdot U(\mathbb{C})) \cap \exp(F^0(\mathrm{Lie}\ W)_{\mathbb{C}})) \rtimes \mathrm{Cent}_{P(\mathbb{R})\cdot U(\mathbb{C})}(h)),\end{aligned}$$

and in the proof of 1.16 we showed

$$\begin{aligned}(P(\mathbb{R})\cdot U(\mathbb{C})) \cap \exp(F^0(\mathrm{Lie}\ W)_{\mathbb{C}}) &= (P(\mathbb{R})\cdot U(\mathbb{C})) \cap \exp(F^0(\mathrm{Lie}\ W)_{\mathbb{C}}) \\ &= \exp(F^0(\mathrm{Lie}\ U)_{\mathbb{C}}) \rtimes \mathrm{Cent}_{P(\mathbb{R})\cdot U(\mathbb{C})}(h).\end{aligned}$$

q.e.d.

1.18. Summary: Putting together the results of 1.4, 1.7, 1.10, 1.12, 1.13 and 1.16 we get the following, albeit somewhat complicated picture. Let P be a connected linear algebraic group over $\mathbb{Q}$. Let W be the unipotent radical of P, and $U \subset W$ a subgroup that is normal in P. Let $V := W/U$, $G := P/W$, and $\pi: P \to G$ and $\pi': P \to P/U$ be the canonical projections. Let $\mathcal{H}$ be a $P(\mathbb{R})\cdot U(\mathbb{C})$-orbit in $\mathrm{Hom}(S_{\mathbb{C}}, P_{\mathbb{C}})$, such that for some ($\Leftrightarrow$ for all) $h \in \mathcal{H}$:

(i') $\pi' \circ h: S_{\mathbb{C}} \to (P/U)_{\mathbb{C}}$ is already defined over $\mathbb{R}$.

(ii) $\pi \circ h \circ w: \mathbb{G}_{m,\mathbb{R}} \to G_{\mathbb{R}}$ is a cocharacter of the center of G that is defined over $\mathbb{Q}$.

(iii) Under the weight filtration on $\mathrm{Lie}\ P$ that is defined by $\mathrm{Ad}_P \circ h$ we have $W_{-1}(\mathrm{Lie}\ P) = \mathrm{Lie}\ W$ and $W_{-2}(\mathrm{Lie}\ P) = \mathrm{Lie}\ U$.

(iv) The Hodge structure on $\mathrm{Lie}\ P$ is of type

$$\{(-1,1),(0,0),(1,-1)\} \cup \{(-1,0),(0,-1)\} \cup \{(-1,-1)\}.$$

(v) $\mathrm{int}(\pi(h(\mathcal{A})))$ induces a Cartan involution on $G^{ad}_{\mathbb{R}}$, and G^{ad} possesses no nontrivial factors of compact type that are defined over $\mathbb{Q}$.
(vi) The center of G is an almost direct product of a $\mathbb{Q}$-split torus with a torus of compact type defined over $\mathbb{Q}$.

Then:

(a) $\mathcal{H}$ possesses a canonical $P(\mathbb{R})\cdot U(\mathbb{C})$-invariant complex structure (1.7 and 1.16).

(b) For every representation M of P we have a variation of rational mixed Hodge structures (1.7 and 1.10).

(c) The operation of $P(\mathbb{R})\cdot U(\mathbb{C})$ on $\mathcal{H}$ can be extended canonically to an operation of $P(\mathbb{R})\cdot W(\mathbb{C})$. The complex structure is still invariant under this bigger group, and $W(\mathbb{C})$ operates analytically on $\mathcal{H}$ (1.7 and 1.16).

(d) For every irreducible representation M of P that is pure of some weight n there exists a one dimensional representation of P on $\mathbb{Q}(-n)$, and a P-equivariant bilinear form $\Psi: M\times M\to\mathbb{Q}(-n)$, such that for all $h\in\mathcal{H}_W$ either Ψ or $-\Psi$ is a polarization of the corresponding Hodge structure on M (1.12 and 1.13).

(e) If $W=1$, then every connected componente of $\mathcal{H}$ is a hermitian symmetric domain (1.14).

<u>Remark</u>: We shall see later (2.19) that every connected component of $\mathcal{H}$ is (noncanonically) isomorphic to a holomorphic vector bundle on a hermitian symmetric domain.

<u>1.19. Generalization to arbitrary weights</u>: The cocharacter $\pi\circ h\circ w$ above can be considered as a weight, since it corresponds to the weight of the Hodge structure induced on any representation of G. In this sense