

On Hrushovski's proof of the Manin-Mumford conjecture

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Abstract

The Manin-Mumford conjecture in characteristic zero was first proved by Raynaud. Later, Hrushovski gave a different proof using model theory. His main result from model theory, when applied to abelian varieties, can be rephrased in terms of algebraic geometry. In this paper we prove that intervening result using classical algebraic geometry alone. Altogether, this yields a new proof of the Manin-Mumford conjecture using only classical algebraic geometry.

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1 Introduction

The Manin-Mumford conjecture is the following statement.

Theorem 1.1 *Let A be an abelian variety defined over $\overline{\mathbb{Q}}$ and X a closed subvariety of A . Denote by $\text{Tor}(A)$ the set of torsion points of A . Then*

$$X \cap \text{Tor}(A) = \bigcup_{i \in I} X_i \cap \text{Tor}(A),$$

where I is a finite set and each X_i is the translate by an element of A of an abelian subvariety of A , immersed in X .

This conjecture has a long history of proofs. A first partial proof was given by Bogomolov in [2], who proved the statement when $\text{Tor}(A)$ is replaced by its ℓ -primary part for a prime number ℓ ; he applies results of Serre, Tate and Raynaud (see [11]) on the existence of Hodge-Tate module structures on the Tate module of abelian varieties over discrete valuation rings. A full proof of the conjecture was then given by Raynaud in [9] (see [8] for the case $\dim(X) = 1$); his proof follows from a study of the reduction of A modulo p^2 . A third and full proof of the conjecture was given by Hrushovski in [4]; he uses Weil's theorem on the characteristic polynomial of the Frobenius morphism on abelian varieties over finite fields in conjunction with the model theory (of mathematical logic) of the theory of fields with a distinguished automorphism. A fourth proof of the conjecture was given by Ullmo and Zhang (based on ideas of Szpiro) in [13] and [15] and goes via diophantine approximation

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and Arakelov theory. They actually prove a more general conjecture of Bogomolov which generalizes the Manin-Mumford conjecture.

This article was inspired by Hrushovski's proof. The bulk of Hrushovski's proof lies in the model theory part. It culminates in a result which, when applied to the special case of abelian varieties and stripped of model theoretic terminology, is essentially Theorem 2.1 below. In Section 2 we prove Theorem 2.1 with classical algebraic geometry alone. Neither scheme theory, nor Arakelov theory, nor mathematical logic are used. In section 3, for the sake of completeness, we show how to apply Theorem 2.1 to prove the Manin-Mumford conjecture.

In a subsequent article (cf. [7]), we shall consider the analogue of 2.1 for varieties over function fields of characteristic $p > 0$.

2 Hrushovski's theorem for abelian varieties

Let K be an algebraically closed field of characteristic zero, endowed with an automorphism σ . Let A be an abelian variety over K and X a closed subvariety of A . For ease of notation, we use the language of classical algebraic geometry; thus A and X denote the respective sets of K -valued points. We assume that $X \subset A$ are defined already over the fixed field K^σ . The automorphism of A induced by σ is again denoted by σ . Let $P(T) \in \mathbb{Z}[T]$ be a monic polynomial with integral coefficients. In [4, Cor. 4.1.13, p. 90], Hrushovski proves the generalisation of the following theorem to semi-abelian varieties.

Theorem 2.1 *Let Γ denote the kernel of the homomorphism $P(\sigma) : A \rightarrow A$. Assume that no complex root of P is root of unity. Then*

$$X \cap \Gamma = \bigcup_{i \in I} X_i \cap \Gamma,$$

where I is a finite set and each X_i is the translate by an element of A of an abelian subvariety of A , immersed in X .

Remark. If roots of unity are not excluded, the group Γ becomes too large for such a result. For example, if $T^m - 1$ divides P , all points of A over the fixed field K^{σ^m} of σ^m are contained in Γ .

Proof: Write $P(T) = \sum_{i=0}^n a_i T^i$ with $a_i \in \mathbb{Z}$ and $a_n = 1$. Let F be the endomorphism of A^n defined by the matrix

$$\begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & 1 \\ -a_0 & -a_1 & \dots & \dots & -a_{n-1} \end{pmatrix}$$

which is the companion matrix of the polynomial P , and note that $P(F) = 0$. In the obvious way σ induces an automorphism of A^n , denoted again by σ , that sends X^n to itself. Let Δ denote the kernel of the homomorphism $F - \sigma : A^n \rightarrow A^n$. By construction, there is a canonical bijection

$$\Gamma \rightarrow \Delta, \quad x \mapsto (x, \sigma(x), \dots, \sigma^{n-1}(x)).$$

Since $\sigma(X) = X$, this induces a bijection

$$X \cap \Gamma \rightarrow X^n \cap \Delta.$$

Its inverse is given by the projection to the first factor $A^n \rightarrow A$. Clearly, we are now reduced to the following theorem, applied to $X^n \subset A^n$ in place of $X \subset A$.

Theorem 2.2 *Let $F : A \rightarrow A$ be an algebraic endomorphism that commutes with σ and that satisfies $P(F) = 0$. Let Δ denote the kernel of the homomorphism $F - \sigma : A \rightarrow A$. Assume that no complex root of P is root of unity. Then*

$$X \cap \Delta = \bigcup_{i \in I} X_i \cap \Delta,$$

where I is a finite set and each X_i is the translate by an element of A of an abelian subvariety of A , immersed in X .

Remark 2.3 If K is algebraic over the fixed field K^σ , every element $a \in \Delta$ satisfies $F^m(a) = \sigma^m(a) = a$ for some $m \geq 1$. In other words, we have $a \in \text{Ker}(F^m - \text{id})$. The assumptions on F and P imply that $F^m - \text{id}$ is an isogeny; hence a is a torsion element. It follows that Δ is a torsion subgroup, and the theorem follows from the Manin-Mumford conjecture in this case. However, the scope of the above theorem is somewhat wider, and the Manin-Mumford conjecture will be deduced from it.

Proof: Let Y be the Zariski closure of $X \cap \Delta$. We claim that $\sigma(Y) = F(Y) = Y$. To see this, note first that σ commutes with F , and so $\sigma(\Delta) = \Delta$. By assumption we have $\sigma(X) = X$; hence $\sigma(X \cap \Delta) = X \cap \Delta$. Since $\sigma : A \rightarrow A$ comes from an automorphism of the underlying field, it is a homeomorphism for the Zariski topology, so we have $\sigma(Y) = Y$. On the other hand, the maps σ and F coincide on Δ , which implies $F(X \cap \Delta) = X \cap \Delta$. As F is a proper algebraic morphism, we deduce that $F(Y) = Y$. Clearly, Theorem 2.2 is now reduced to the following theorem (see [3, Th. 3] for the case where F is the multiplication by an integer $n > 1$).

Theorem 2.4 *Let A be an abelian variety over an algebraically closed field of characteristic zero. Let $F : A \rightarrow A$ be an algebraic endomorphism none of whose eigenvalues on $\text{Lie } A$ is a root of unity. Let Y be a closed subvariety of A satisfying $F(Y) = Y$. Then Y is a finite union of translates of abelian subvarieties of A .*

Proof: We proceed by induction on the dimension d of A . For $d = 0$ the statement is obvious; hence we assume $d > 0$. Next observe that $Y \subset F(A)$. Thus if F is not surjective, we can replace A by $F(A)$ and Y by $F(A) \cap Y$, and are finished by induction. Thus we may assume that F is an isogeny.

Since F is proper and $F(Y) = Y$, every irreducible component of Y is the image under F of some irreducible component of Y . Since the set of these irreducible components is finite, it is therefore permuted by F . We fix such an irreducible component Z and an integer $m \geq 1$ such that $F^m(Z) = Z$.

Any power $F^{rm} : A \rightarrow A$ is an isogeny, and since $\text{char}(K) = 0$, it is a separable isogeny. As a morphism of schemes it is therefore a finite étale Galois covering with Galois group $\text{Ker}(F^{rm})$, acting by translations on A . The same follows for the induced covering $(F^{rm})^{-1}(Z) \rightarrow Z$. As Z is irreducible, the irreducible components of $(F^{rm})^{-1}(Z)$ are transitively permuted by $\text{Ker}(F^{rm})$ and each of them has dimension $\dim(Z)$. Since $F^{rm}(Z) = Z$, we have $Z \subset (F^{rm})^{-1}(Z)$, and so Z itself is one of these irreducible components. Let G_r denote the stabilizer of Z in $\text{Ker}(F^{rm})$. Then $F^{rm} : Z \rightarrow Z$ is generically a finite étale Galois covering with Galois group G_r .

We now distinguish two cases. Let $\text{Stab}_A(Z)$ denote the translation stabilizer of Z in A , which is a closed algebraic subgroup of A .

Lemma 2.5 *If $|G_1| > 1$, then $\dim(\text{Stab}_A(Z)) > 0$.*

Proof: For any $r \geq 1$, the morphism $F^{rm} : Z \rightarrow Z$ is finite separable of degree $|G_r|$. Since degrees are multiplicative in composites, this degree is also equal to $|G_1|^r$. Thus if $|G_1| > 1$, we find that $|G_r|$ becomes arbitrarily large with r . Therefore $\text{Stab}_A(Z)$ contains arbitrarily large finite subgroups, so it cannot be finite. **Q.E.D.**

Lemma 2.6 *If $|G_1| = 1$, then $\dim(\text{Stab}_A(Z)) > 0$ or Z is a single point.*

Proof: Since Z is irreducible, the assertion is obvious when $\dim(Z) = 0$. So we assume that $\dim(Z) > 0$. The assumption $|G_1| = 1$ implies that F^m induces a finite (separable) morphism $\varphi : Z \rightarrow Z$ of degree 1. The set of fixed points of any positive power φ^r is then $Z \cap \text{Ker}(F^{rm} - \text{id})$. On the other hand the assumptions on F and P imply that $F^{rm} - \text{id}$ is an isogeny on A . Thus this fixed point set is finite for every $r \geq 1$; hence φ has infinite order.

Assume now that $\text{Stab}_A(Z)$ is finite. By Ueno's theorem [12, Thm. 3.10] Z is then of general type, in the sense that any smooth projective variety birationally equivalent to Z is of general type (see [12, Def. 1.7]). But the group of birational automorphisms of any irreducible projective variety Z of general type is finite (see [6] and also [5, Th. 10.11]) which yields a contradiction. **Q.E.D.**

If Z is a single point, it is a translate of the trivial abelian subvariety of A . Otherwise we know from the lemmas above that $\dim(\text{Stab}_A(Z)) > 0$, and we can use the induction hypothesis. Let B denote the identity component of $\text{Stab}_A(Z)$. Since $F^m(Z) = Z$, we also have $F^m(B) = B$. Set $\bar{A} := A/B$ and $\bar{Z} := Z/B$, and let \bar{F} denote the endomorphism of \bar{A} induced by F^m . Then we have $\bar{F}(\bar{Z}) = \bar{Z}$, and every eigenvalue of \bar{F} on $\text{Lie } \bar{A}$ is an eigenvalue of F^m on $\text{Lie } A$ and therefore not a root of unity. By Theorem 2.4, applied to $(\bar{A}, \bar{Z}, \bar{F})$ in place of (A, Y, F) , we now deduce that \bar{Z} is a finite union of translates of abelian subvarieties of \bar{A} . But Z is irreducible; hence so is \bar{Z} . Thus \bar{Z} itself is a translate of an abelian subvariety. That abelian subvariety is equal to A'/B for some abelian subvariety $A' \subset A$ containing B , and so Z is a translate of A' , as desired. This finishes the proof of Theorem 2.4, and thus also of Theorems 2.2 and 2.1. **Q.E.D.**

Remark. Suppose that in the statement of the preceding theorem, we replace the assumption that none of the eigenvalues of F on $\text{Lie } A$ is a root of unity by the weaker assumption that none of the eigenvalues of F on $\text{Lie } A$ is an algebraic unit. This case is sufficient for the application to the Manin-Mumford conjecture. Under this weaker assumption, the following alternative proof of Lemma 2.6 can be given; it does not use the theorems of Ueno and Matsumura but only elementary properties of cycle classes.

We work under the hypotheses of Lemma 2.6 and with the notations used above. First, everything can be defined over a countable algebraically closed subfield of K , and this subfield can be embedded into \mathbb{C} ; thus without loss of generality we may assume that $K = \mathbb{C}$. Then for every integer $i \geq 0$ we abbreviate $H^i := H^i(A, \mathbb{Z})$. Let $c := \text{codim}_A(Z)$ and let $\text{cl}(Z) \in H^{2c}$ be the cycle class of $Z \subset A$. We calculate the cycle class of $(F^m)^{-1}(Z)$ in two ways. On the one hand, we have seen that the group $\text{Ker}(F^m)$ acts transitively on the set of irreducible components of $(F^m)^{-1}(Z)$; the assumption $|G_1| = 1$ implies that it also acts faithfully. Thus the number of irreducible components is $|\text{Ker}(F^m)|$. Recall that Z is one of them. Since translation on A does not change cycle classes, we find that all irreducible components of $(F^m)^{-1}(Z)$ have cycle class $\text{cl}(Z)$; hence $\text{cl}((F^m)^{-1}(Z)) = |\text{Ker}(F^m)| \cdot \text{cl}(Z)$. On the other hand F induces a pullback homomorphism $F^* : H^i \rightarrow H^i$ for every $i \geq 0$, and by functoriality of cycle classes we have $\text{cl}((F^m)^{-1}(Z)) = F^*(\text{cl}(Z))$. As $\text{cl}(Z)$ is non-zero, we deduce that $|\text{Ker}(F^m)|$ is an eigenvalue of F^* on H^{2c} .

Let $d := \dim(A)$, which is also the codimension of any point in A . Repeating the above calculation with the cycle class of a point, we deduce that $|\text{Ker}(F^m)|$ is also an eigenvalue of F^* on the highest non-vanishing cohomology group H^{2d} . Since cup product yields isomorphisms $H^i \cong \Lambda^i H^1$ for all $i \geq 0$, which are compatible with F^* , it yields an F^* -equivariant perfect pairing $H^{2c} \times H^{2(d-c)} \rightarrow H^{2d}$. From this we deduce that 1 is an eigenvalue of F^* on $H^{2(d-c)}$. Moreover, this eigenvalue must be a product of $2(d-c)$ eigenvalues of F^* on H^1 . Now the eigenvalues of F^* on H^1 are precisely those of F on $\text{Lie } A$ and their complex conjugates. By assumption, they are algebraic integers but no algebraic units. Thus a product of such numbers can be 1 only if it is the empty product. This shows that $2(d-c) = 0$, that is, $\text{codim}_A(Z) = \dim(A)$; hence Z is a point, as desired. **Q.E.D.**

3 Proof of the Manin-Mumford conjecture

As a preparation let $q = p^r$, where p is a prime number and $r \geq 1$. Let \mathbb{F}_q be the unique field with q elements. Let A be an abelian variety defined over \mathbb{F}_q . As in classical algebraic geometry we identify A with the set of its $\overline{\mathbb{F}_q}$ -valued points. Let ℓ be a prime number different from p , and let $T_\ell(A)$ be the ℓ -adic Tate module of A , which is a free \mathbb{Z}_ℓ -module of rank $2 \dim(A)$. We denote by φ the Frobenius morphism $A \rightarrow A$, which acts on the coordinates of points by taking q -th powers. It also acts on the Tate module via its action on the torsion points. The following result, an analogue of the Riemann hypothesis, is due to Weil (see [14]):

Theorem 3.1 *Let T be an indeterminate. The characteristic polynomial of φ on $T_\ell(A) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ is a monic polynomial in $\mathbb{Z}[T]$. It is independent of ℓ , and all its complex roots have absolute value \sqrt{q} .*

Consider now an abelian variety A defined over $\overline{\mathbb{Q}}$ and a closed subvariety X of A . We choose a number field $L \subset \overline{\mathbb{Q}}$ over which both $X \subset A$ can be defined and fix their models over L . For any abelian group G we write $\text{Tor}(G)$ for the group of torsion points of G . Moreover, for any prime p we write $\text{Tor}_p(G)$ for the subgroup of torsion points of order a power of p , and $\text{Tor}^p(G)$ for the subgroup of torsion points of order prime to p . Note that $\text{Tor}(G) = \text{Tor}^p(G) \oplus \text{Tor}_p(G)$.

Choose a prime ideal \mathfrak{p} of \mathcal{O}_L where A has good reduction. Let \mathbb{F}_q be the finite field $\mathcal{O}_L/\mathfrak{p}$, where q is a power of a prime number p . The following lemma is lemma 5.0.10 in [4, p. 105]; we reproduce the proof for the convenience of the reader. We use Weil's theorem 3.1 and reduction modulo \mathfrak{p} to obtain an automorphism of $\overline{\mathbb{Q}}$ and a polynomial that we can feed in Theorem 2.1 to obtain Theorem 1.1.

Lemma 3.2 *There is an element $\sigma_{\mathfrak{p}} \in \text{Gal}(\overline{\mathbb{Q}}|L)$ and a monic polynomial $P_{\mathfrak{p}}(T) \in \mathbb{Z}[T]$ all of whose complex roots have absolute value \sqrt{q} , such that $P_{\mathfrak{p}}(\sigma_{\mathfrak{p}})(x) = 0$ for every $x \in \text{Tor}^p(A)$.*

Proof: Let $L_{\mathfrak{p}}$ be the completion of L at \mathfrak{p} . Extend the embedding $L \hookrightarrow L_{\mathfrak{p}}$ to an embedding of the algebraic closures $\overline{\mathbb{Q}} = \overline{L} \hookrightarrow \overline{L}_{\mathfrak{p}}$, and the surjection $\mathcal{O}_{L_{\mathfrak{p}}} \rightarrow \mathbb{F}_q$ to a surjection $\mathcal{O}_{\overline{L}_{\mathfrak{p}}} \rightarrow \overline{\mathbb{F}_q}$. On the prime-to- p torsion groups we then obtain natural isomorphisms

$$\text{Tor}^p(A) \cong \text{Tor}^p(A_{\overline{L}_{\mathfrak{p}}}) \cong \text{Tor}^p(A_{\overline{\mathbb{F}_q}}).$$

The second isomorphism expresses the fact that the field of definition of every prime-to- p torsion point is unramified at \mathfrak{p} .

Now, as before, let φ denote the automorphism of $\overline{\mathbb{F}_q}$ and of $\text{Tor}^p(A_{\overline{\mathbb{F}_q}})$ induced by the Frobenius morphism over \mathbb{F}_q . Then φ can be lifted to an element $\sigma_{\mathfrak{p}} \in \text{Gal}(\overline{\mathbb{Q}}|L)$

making the above isomorphisms equivariant. To see this, one first lifts φ to an element $\tau_{\mathfrak{p}}^{\text{nr}}$ of $\text{Gal}(L_{\mathfrak{p}}^{\text{nr}}|L_{\mathfrak{p}})$, where $L_{\mathfrak{p}}^{\text{nr}}$ is the maximal unramified extension of $L_{\mathfrak{p}}$. This lifting exists and is unique by [1, Th. 1, p. 26]. This element can then be lifted further to a (non-unique) element $\tau_{\mathfrak{p}}$ of $\text{Gal}(\overline{L}_{\mathfrak{p}}|L_{\mathfrak{p}})$, since $L_{\mathfrak{p}}^{\text{nr}}$ is a subfield of $\overline{L}_{\mathfrak{p}}$. By construction the action of $\tau_{\mathfrak{p}}$ on $\text{Tor}^p(A_{\overline{L}_{\mathfrak{p}}})$ corresponds to the action of φ on $\text{Tor}^p(A_{\overline{\mathbb{F}}_q})$. The restriction of $\tau_{\mathfrak{p}}$ to $\overline{\mathbb{Q}}$ gives the desired element $\sigma_{\mathfrak{p}}$.

Let $P_{\mathfrak{p}}(T)$ be the characteristic polynomial of φ on $T_{\ell}(A_{\overline{\mathbb{F}}_q}) \otimes_{\mathbb{Z}_{\ell}} \mathbb{Q}_{\ell}$ for any prime number $\ell \neq p$. By construction, we then have $P_{\mathfrak{p}}(\varphi) = 0$ on $\text{Tor}_{\ell}(A_{\overline{\mathbb{F}}_q})$. By Weil's result quoted above, the same equation holds for every prime $\ell \neq p$; so it holds on $\text{Tor}^p(A_{\overline{\mathbb{F}}_q})$. From the construction of $\sigma_{\mathfrak{p}}$ we deduce that $P_{\mathfrak{p}}(\sigma_{\mathfrak{p}}) = 0$ on $\text{Tor}^p(A)$. Finally, Weil's result also describes the complex roots of $P_{\mathfrak{p}}$. **Q.E.D.**

Let now $L^p, L_p \subset \overline{\mathbb{Q}}$ be the fields generated over L by the coordinates of all points in $\text{Tor}^p(A)$, resp. in $\text{Tor}_p(A)$. Both are infinite Galois extensions of L . Their intersection is known to be finite over L by Serre [10, pp. 33–34, 56–59]. Thus after replacing L by $L^p \cap L_p$, we may assume that L^p and L_p are linearly disjoint over L . The subfield of $\overline{\mathbb{Q}}$ generated by the coordinates of all points in $\text{Tor}(A) = \text{Tor}^p(A) \oplus \text{Tor}_p(A)$ is then canonically isomorphic to $L^p \otimes_L L_p$.

Let \mathfrak{p}' be a second place of good reduction of A , of residue characteristic different from p . Let $\sigma_{\mathfrak{p}}, \sigma_{\mathfrak{p}'}$ and $P_{\mathfrak{p}}, P_{\mathfrak{p}'}$ be the automorphisms and polynomials provided by Lemma 3.2, applied to \mathfrak{p} , resp. to \mathfrak{p}' . The automorphism of $L^p \otimes_L L_p$ induced by $\sigma_{\mathfrak{p}} \otimes \sigma_{\mathfrak{p}'}$ then extends to some automorphism σ of $\overline{\mathbb{Q}}$ over L . Since $P_{\mathfrak{p}}(\sigma_{\mathfrak{p}})$ vanishes on $\text{Tor}^p(A)$, so does $P_{\mathfrak{p}}(\sigma)$. Similarly, $P_{\mathfrak{p}'}(\sigma_{\mathfrak{p}'})$ vanishes on $\text{Tor}_p(A) \subset \text{Tor}^{\mathfrak{p}'}(A)$; hence so does $P_{\mathfrak{p}'}(\sigma)$. Thus with $P(T) := P_{\mathfrak{p}}(T)P_{\mathfrak{p}'}(T)$ we deduce that $P(\sigma)$ vanishes on $\text{Tor}(A)$. In other words, we have $\text{Tor}(A) \subset \Gamma := \text{Ker } P(\sigma)$. With $K = \overline{\mathbb{Q}}$ Theorem 1.1 now follows directly from Theorem 2.1. **Q.E.D.**

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