The geometric group law on a tropical elliptic curve

Bachelor Thesis written by Nina Otter

under the supervision of Prof. Richard Pink.

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Contents

Acknoledgements

I wish to express my thanks to Professor Pink for supervising my thesis. I have been extremely lucky to have had a supervisor who dedicated so much time and care to my work and could provide such critical insight into the art of mathematical writing. Thanks to Professor Pink I have learned that one of the causes of bad mathematical writing is the lack of a clear idea about what is to be written. I think that this is the most important thing that I learned while writing this thesis; for this I am very grateful to Professor Pink.

> “Les pays exotiques m’apparaissaient comme le contrepied des nôtres, le terme d’antipodes trouvait dans ma pensée un sens plus riche et plus naïf que son contenu littéral. On m’eût fort étonné en disant qu’une espèce animale ou végétale pouvait avoir le même aspect des deux côtés du globe. Chaque animal, chaque arbre, chaque brin d’herbe, devait être radicalement différent, afficher au premier coup d’œil sa nature tropicale. Le Brésil s’esquissait dans mon imagination comme des gerbes de palmiers contournés, dissimulant des architectures bizarres, le tout baigné dans une odeur [. . .] (de) parfum brûlé.” [1]
>
> *Claude Lévi-Strauss,* Tristes tropiques

0. Introduction

Tropical mathematics is an area of theoretical computer science which saw its beginning in the 1970s. It was concerned with the study of min-plus semirings - semirings in which the operations are given by taking addition and minimum on certain sets as the set of natural numbers or the ordinal numbers smaller than a certain cardinal [**10**]. Among its pioneers was the Brazilian mathematician Imre Simon, in honour of whom several French mathematicians - Dominique Perrin [**10**] and Christian Choffrut [**12**] among others - began to call these semirings “tropical semirings”. To use the words of Sturmfels and Speyer, the adjective tropical *‘simply stands for the French view of Brazil.’* [**13**]

Tropical geometry is an area of algebraic geometry which is concerned with the study of varieties over the tropical semiring of real numbers. Tropical varieties are rational polyhedral complexes satisfying a certain equilibrium condition on the vertices.

Given an algebraically closed field K with valuation v and a non-zero polynomial in two variables over K, it is possible to assign to an algebraic curve $C = \{(x, y) \in K^2 \mid f(x, y) = 0\}$ a tropical variety $\{(v(x), v(y)) \in \mathbb{R}^2 \mid (x, y) \in C\}$ which preserves many properties of the algebraic curve. Since tropical varieties are combinatorial objects, this method is widely used to translate algebraic-geometric problems into combinatorial ones, for which a solution may be easier to find.

A lot of work is being done to translate the language of algebraic geometry into tropical geometry. Often a translation is justified by its correct use in the tropical setting rather than by why it is the correct translation.

[1] “Tropical countries, as it seemed to me, must be the exact opposite of our own, and the name of Antipodes had for me a sense at once richer and more ingenuous than its literal derivation. I should have been astonished to hear it said that any species, whether animal or vegetable, could have the same appearance on both sides of the globe. Every animal, every tree, every blade of grass, must be completely different and give immediate notice, as it were, of its tropical character. I imagined Brazil as a tangled mass of palm-leaves, with glimpses of strange architecture in the middle distance, and an all-permeating smell of burning perfume.”(Translation by John Russell)

In classical algebraic geometry an elliptic curve is defined as a smooth projective cubic curve of genus one together with a fixed point $\mathcal{O}$. A procedure defined by means of chords and tangents - the chord-tangent law - yields an operation on the elliptic curve which induces a group structure with identity element the point $\mathcal{O}$. An equivalent group structure can be defined algebraically: there is a bijection between the group of divisors of degree zero and the elliptic curve.
In [**2**] Vigeland investigates the algebraic group structure on a tropical elliptic curve and alludes briefly to a geometric group structure, in analogy with the classical case. Taking as point of departure Vigeland's paper [**2**], in this thesis we analyze the geometric group law on a tropical elliptic curve.

The thesis is organised as follows: in section 1 we state a self-contained theory of tropical curves. In section 2 we recall some notions of tropical intersection theory necessary to the development of the thesis. In section 3 we give a brief exposition of tropical elliptic curves and the associated group law induced by the Jacobian, mainly recalling results and definitions given in Vigeland's paper. Section 4 is the core of the thesis: we analyze the group law induced by a geometric addition defined on the tropical elliptic curve, prove that it is isomorphic to the algebraic group structure and investigate the geometric properties of torsion points of order 2 and 3. Our interest in these particular torsion points is motivated by their importance in the theory of classical elliptic curves.
There is not yet an unanimous consensus about many notions in tropical geometry. Since the thesis is based upon Vigeland's paper, we decided to adopt the definitions and conventions therefrom whenever we judged them suitable for our purpose.

1. Tropical curves

Definition 1. A *unitary semiring* is a set R together with binary operations $+$ and $\cdot$ satisfying the following properties:
(i) $(R, +)$ is a commutative monoid
(ii) $(R, \cdot)$ is a monoid
(iii) $\cdot$ is distributive over $+$
(iv) $\forall\ r \in R$: $0 \cdot r = r \cdot 0 = 0$, where 0 is the additive identity. *(Absorption law).*
A semiring is *idempotent* if addition is idempotent, that is to say, if for all r in R we have that $r + r = r$. A semiring is *commutative* if $(R, \cdot)$ is a commutative monoid.

Remark 2. Any unitary ring is a semiring. Unlike in a semiring, in a ring cancellation with respect to addition holds: for any elements a, b, c of the ring, if $a + b = a + c$, then $b = c$. This implies that idempotent rings are necessarily trivial and that the absorption law (iv) can be deduced from the definition of a ring.

Definition 3. A *semifield* is a unitary commutative semiring in which every non-zero element has a multiplicative inverse. A semifield is idempotent if addition is idempotent.

Example 4. Let $\mathbb{R}$ be the set of real numbers and define on it the following operations:

$$\oplus : \mathbb{R} \times \mathbb{R} \to \mathbb{R} : (a, b) \mapsto max\{a, b\}$$

and

$$\odot : \mathbb{R} \times \mathbb{R} \to \mathbb{R} : (a, b) \mapsto a + b$$

where $+$ denotes the usual addition of real numbers.
In order to have a neutral element with respect to $\oplus$ we extend the set of real numbers by an element $-\infty$ such that for all r in $\mathbb{R} : -\infty \oplus r = r \oplus -\infty = r$. Furthermore we define for all r in $\mathbb{R} \cup \{-\infty\}$: $-\infty \odot r = r \odot -\infty = -\infty$.

Lemma 5. *$\mathbb{R}_{tr} = (\mathbb{R} \cup \{-\infty\}, \oplus, \odot)$ is an idempotent semifield with additive and multiplicative identities $-\infty$ and 0 respectively. For all r in $\mathbb{R}$ the multiplicative inverse is $r^{-1} = -r$ where $-r$ is the additive inverse in the field $(\mathbb{R}, +, \cdot)$.*

Remark 6. We make use of the convention that $\odot$ has precedence on $\oplus$, that is to say, that $a \oplus b \odot c = a \oplus (b \odot c)$.
Furthermore we write a^r for $\underbrace{a \odot \cdots \odot a}_{r\text{-times}}$.

Definition 7. The *action of a unitary commutative semiring* $(R, +, \cdot)$ *on a monoid* $(M, \circ)$ is a map

$$\rho : R \times M \to M$$

satisfying the following requirements for all m, n in M for all r, s in R:
(i) $\rho(1, m) = m$, where 1 is the multiplicative identity of R
(ii) $\rho(rs, m) = \rho(r, \rho(s, m))$
(iii) $\rho(r, m \circ n) = \rho(r, m) \circ \rho(r, n)$
(iv)$\rho(r + s, m) = \rho(r, m) \circ \rho(s, m)$.

Definition 8. A *semimodule over a commutative unitary semiring* R is a commutative monoid endowed with an action of R.

Example 9. Let $\mathbb{R}^n_{tr} = (\mathbb{R}^n \cup \{-\infty\}^n, \oplus, \odot)$ be the set of n-tuples of real numbers together with componentwise addition $(x_1, \ldots, x_n) \oplus (y_1, \ldots, y_n) = (x_1 \oplus y_n, \ldots, x_n \oplus y_n)$. The semiring $\mathbb{R}_{tr}$ acts on $\mathbb{R}^n_{tr}$ in the natural way:

$$\begin{gathered} \mathbb{R}_{tr} \times \mathbb{R}^n_{tr} \longrightarrow \mathbb{R}^n_{tr} \\ (r, (x_1, \ldots, x_n)) \mapsto r \odot (x_1, \ldots, x_n) := (r \odot x_1, \ldots, r \odot x_n) \end{gathered}$$

making $\mathbb{R}^n_{tr}$ into a semimodule over $\mathbb{R}_{tr}$.

Definition 10. *Tropical projective space* is defined as $\mathbb{P}^{n-1}_{tr} = (\mathbb{R}^n_{tr} \smallsetminus \{-\infty\}^n)/_\sim$ where $(x_1, \ldots, x_n) \sim (y_1, \ldots, y_n) :\Leftrightarrow \exists\, a \in \mathbb{R} : (x_1, \ldots, x_n) = a \odot (y_1, \ldots, y_n)$.

As we would expect, as a topological space $\mathbb{P}^n_{tr}$ is compact:

Lemma 11. *The space $\mathbb{P}^n_{tr}$ is homeomorphic to the simplex of dimension n.*

Definition 12. Define the set of formal linear combinations

$$\mathbb{R}_{tr}[X_1^{\pm 1}, \ldots, X_n^{\pm 1}] = \left\{ \bigoplus_{k \in I} a_k \odot X_1^{k_1} \odot \cdots \odot X_n^{k_n} \;\middle|\; \begin{array}{c} I \subset \mathbb{Z}^n \text{finite indexing set,} \\ k = (k_1, \ldots, k_n), \\ a_k \in \mathbb{R} \end{array} \right\} \cup \{-\infty\}.$$

We call the finite indexing set I the *support of f*.

Using the notation $f = \bigoplus_{k \in I} a_k \odot X_1^{k_1} \odot \cdots \odot X_n^{k_n}$ and $g = \bigoplus_{k \in J} b_k \odot X_1^{k_1} \odot \cdots \odot X_n^{k_n}$ define the following operations:

$$\begin{aligned} \oplus : \mathbb{R}_{tr}[X_1^{\pm 1}, \ldots, X_n^{\pm 1}] \times \mathbb{R}_{tr}[X_1^{\pm 1}, \ldots, X_n^{\pm 1}] &\to \mathbb{R}_{tr}[X_1^{\pm 1}, \ldots, X_n^{\pm 1}] \\ (f, g) &\mapsto f \oplus g = \bigoplus_{k \in I \cup J} (\widetilde{a_k} \oplus \widetilde{b_k}) \odot X_1^{k_1} \odot \cdots \odot X_n^{k_n} \end{aligned}$$

where

$$\widetilde{a_k} = \begin{cases} a_k, & \text{if } k \in I \\ -\infty, & \text{otherwise} \end{cases} \qquad \widetilde{b_k} = \begin{cases} b_k, & \text{if } k \in J \\ -\infty, & \text{otherwise} \end{cases}$$

and

$$\begin{aligned} \odot : \mathbb{R}_{tr}[X_1^{\pm 1}, \ldots, X_n^{\pm 1}] \times \mathbb{R}_{tr}[X_1^{\pm 1}, \ldots, X_n^{\pm 1}] &\to \mathbb{R}_{tr}[X_1^{\pm 1}, \ldots, X_n^{\pm 1}] \\ (f, g) &\mapsto f \odot g = \bigoplus_{k \in I} \bigoplus_{j \in J} a_k \odot b_j \odot X_1^{k_1 + j_1} \odot \cdots \odot X_n^{k_n + j_n} \end{aligned}$$

Furthermore define $f \oplus -\infty = -\infty \oplus f = f$ and $f \odot -\infty = -\infty \odot f = -\infty$. The elements of $\mathbb{R}_{tr}[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$ are called *tropical (Laurent) polynomials.*

From now a semiring will be a unitary, commutative and idempotent semiring.

Definition 13. A *semiring homomorphism* is a homomorphism of the underlying abelian monoids which preserves the additive and multiplicative identities.
If R is a semiring, an *R-semialgebra S* is a semiring S together with a homomorphism of semirings $R \to S$.
If S and U are R-semialgebras, an *R-semialgebra homomorphism $S \to U$* is a homomorphism of semirings making the following diagram commute:

$$\begin{tikzcd} R \arrow[r] \arrow[dr] & S \arrow[d] \\ & U \end{tikzcd}$$

Lemma 14. *The set $\mathbb{R}_{tr}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$ together with the above defined operations is a semiring with additive and multiplicative identities respectively $-\infty$ and 0. It is called* semiring of tropical polynomials *and is in a natural way an $\mathbb{R}_{tr}$-semialgebra.*

From now on we will use the notation $\mathbb{R}_{tr}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$ to denote the semiring of tropical polynomials.

Definition 15. Let $f \in \mathbb{R}_{tr}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$ be a tropical polynomial and I its support. The polynomial f has *degree d* if $max_{k \in I}\{k_1 + \cdots + k_n\} = d$.
If $I \subset \{k \in \mathbb{Z}^n \mid k_1 + \cdots + k_n = d\}$ then f is *homogeneous of degree d.*

Every tropical polynomial $f \in \mathbb{R}_{tr}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$ determines an evaluation function in the sense that we make precise in the following.
Let $\mathcal{F}(\mathbb{R}_{tr}^n, \mathbb{R}_{tr})$ denote the set of functions from $\mathbb{R}_{tr}^n$ to $\mathbb{R}_{tr}$. We put on it a semiring structure by defining the following operations:

$$\begin{aligned} \forall x \in \mathbb{R}_{tr}^n \ \forall \phi, \psi \in \mathcal{F}(\mathbb{R}_{tr}^n, \mathbb{R}_{tr}) : & (\phi \oplus \psi)(x) = \phi(x) \oplus \psi(x) \\ & (\phi \odot \psi)(x) = \phi(x) \odot \psi(x). \end{aligned}$$

The identity elements for $\oplus$ and $\odot$ are then the constant functions sending every element to $-\infty$ and 0 respectively.
Thus $\mathcal{F}(\mathbb{R}_{tr}^n, \mathbb{R}_{tr})$ is a semiring. We make it into an $\mathbb{R}_{tr}$-semialgebra by the homomorphism

$$\mathbb{R}_{tr} \to \mathcal{F}(\mathbb{R}_{tr}^n, \mathbb{R}_{tr}) : r \mapsto (x \mapsto r).$$

Lemma 16. *The evaluation map*

$$ev : \mathbb{R}_{tr}[X_1^{\pm 1}, \dots, X_n^{\pm 1}] \to \mathcal{F}(\mathbb{R}_{tr}^n, \mathbb{R}_{tr})$$
$$\bigoplus_{k \in I} a_k \odot X_1^{k_1} \odot \cdots \odot X_n^{k_n} \mapsto ((x_1, \dots, x_n) \mapsto \bigoplus_{k \in I} a_k \odot x_1^{k_1} \odot \cdots \odot x_n^{k_n})$$
$$-\infty \mapsto ((x_1, \dots x_n) \mapsto -\infty)$$

is an $\mathbb{R}_{tr}$-semialgebra homomorphism.

Definition 17. Let $f = \bigoplus_{k \in I} a_k \odot X_1^{k_1} \odot \cdots \odot X_n^{k_n} \in \mathbb{R}_{tr}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$.

We say that f *satisfies P at $x \in \mathbb{R}_{tr}^n$* if

$$(\exists k \neq j \in I)(ev(f)(x) = ev(a_k \odot X_1^{k_1} \odot \cdots \odot X_n^{k_n})(x) = ev(a_j \odot X_1^{j_1} \odot \cdots \odot X_n^{j_n})(x)).$$

Definition 18. Let $f \in \mathbb{R}_{tr}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$. The *affine corner locus of f* is

$$T(f) = \{x \in \mathbb{R}_{tr}^n \mid f \text{ satisfies } P \text{ at } x\}.$$

Remark 19. If $f \in \mathbb{R}_{tr}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$ is $-\infty$ or consists of one monomial, then $T(f) = \{\emptyset\}$.

Definition 20. An *affine tropical curve* is the affine corner locus of a tropical polynomial in two variables.

In order to give the definition of a tropical curve in projective space we proceed analogously to the classical case.
Unlike in the affine case, we cannot define an evaluation map from the semiring $\mathbb{R}_{tr}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$ to the set of functions from $\mathbb{P}_{tr}^{n-1}$ to $\mathbb{R}_{tr}$, since for all a in $\mathbb{R}_{tr}$ and for all x in $\mathbb{R}_{tr}^n$ in general $ev(f)(a \odot x) \neq ev(f)(x)$. However, if $f \in \mathbb{R}_{tr}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$ is a homogeneous polynomial of degree d, then for all a in $\mathbb{R}_{tr}$ and for all x in $\mathbb{R}_{tr}^n$ we have $ev(f)(a \odot x) = a^d \odot ev(f)(x)$, hence the property that f satisfies P at $x \in (x_1 : \cdots : x_n) \in \mathbb{P}_{tr}^{n-1}$ depends only on the equivalence class of x. Thus we can define:

Definition 21. Let $f \in \mathbb{R}_{tr}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$ be a homogeneous polynomial. The *projective corner locus of f* is

$$T(f) = \left\{(x_1 : \ldots : x_n) \in \mathbb{P}_{tr}^{n-1} \middle| \, f \text{ satisfies } P \text{ at } (x_1 : \ldots : x_n)\right\}.$$

Definition 22. A *projective tropical curve* is the projective corner locus of a homogeneous tropical polynomial in three variables.

Definition 23 (Newton polytope)**.** The convex hull in $\mathbb{R}^n$ of the support of a tropical polynomial $f \in \mathbb{R}_{tr}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$ is called *Newton polytope associated to f* and denoted by Δ_f.

Definition 24. The *translate of Δ_f by $x_0 \in \mathbb{R}^n$* is $\{y \in \mathbb{R}^n \mid y = x + x_0, \ x \in \Delta_f\}$ and denoted by $\Delta_f + x_0$.

Different tropical polynomials may have the same corner locus. In the following we will examine in which instances this occurs.

Definition 25. We define the following equivalence relation for all $f, g \in \mathbb{R}_{tr}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$:

$f \sim g :\Longleftrightarrow$ there exists an integer $m \in \mathbb{Z}_{\geq 0}$, there exist tropical polynomials $f_0, \dots, f_m$ such that $f_0 = f$, $f_m = g$ and for $i = 1, \dots, m$ the polynomial f_i is obtained from f_{i-1} in one of the following three ways:
(i) There exists an a in $\mathbb{R}$ such that $f_i = a \odot f_{i-1}$.
(ii) There exists a k in $\{1, \dots, n\}$ such that $f_i = f_{i-1} \odot X_k$.
(iii) There is a monomial of f_{i-1} at which the maximum is never attained, and f_i is obtained from f_{i-1} by omission of this monomial.

Lemma 26. *Let $f, g \in \mathbb{R}_{tr}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$ be tropical polynomials. Then*

$$f \sim g :\Leftrightarrow T(f) = T(g).$$

Proof. [**9**, Remark 3.7] □

Lemma 27. *For all tropical polynomials f the assignment $T(f) \mapsto \Delta_f + \mathbb{R}^n$ is well-defined.*

Proof. Let f and g be tropical polynomials in $\mathbb{R}_{tr}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$ such that $T(f) = T(g)$. Suppose that $f_0, \dots, f_m$ with $f_0 = f$ and $f_m = g$ are the tropical polynomials satisfying the conditions for f and g to be equivalent. Suppose that f_i is obtained from f_{i-1} via one of the three cases of Definition 25. The first case clearly does not affect the convex hull of the support of f_{i-1}. In the second case the convex hull of the support of f_{i-1} is translated by $x = (0, \dots, 0, 1, 0, \dots, 0)$, where the 1 is in the k-th component. In the third case the element of the support corresponding to the monomial at which the maximum is never attained lies in the interior of the convex hull and therefore the convex hull is not affected by its removal.

□

We can now define the degree of a tropical curve. This definition is well-defined in virtue of Lemma 27.

Definition 28. We denote by Δ_d the simplex $\{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$ and $x_1 + x_2 + x_3 = d\}$ and by T_d the triangle $\{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \geq 0, x_2 \geq 0$ and $x_1 + x_2 \leq d\}$.
A projective tropical curve $T(f)$ has *degree d* if there exists an x in $\mathbb{R}^3$ such that $(\Delta_f + x) \subset \Delta_d$ and there does not exist an x in $\mathbb{R}^3$ such that $(\Delta_f + x) \subset \Delta_{d-1}$. It is said to have *degree d with full support* if there exists an x in $\mathbb{R}^3$ such that $(\Delta_f + x) = \Delta_d$.
Analogously an affine tropical curve $T(f)$ has *degree d* if there exists an x in $\mathbb{R}^2$ such that

$(\Delta_f + x) \subset T_d$ and there does not exist an x in $\mathbb{R}^2$ such that $(\Delta_f + x) \subset T_{d-1}$. It is said to have *degree d with full support* if there exists an x in $\mathbb{R}^2$ such that $(\Delta_f + x) = T_d$.

Remark 29. A tropical affine (resp. projective) curve has degree 0 iff it is the corner locus of a tropical monomial in two (resp. three) variables and is therefore empty.

We can associate to a tropical polynomial in n variables a subdivision of the Newton polytope which is in a certain sense dual to its corner locus.
Take the convex hull of $\widehat{\Delta} = \{(k,t) \in \mathbb{R}^n \times \mathbb{R} \mid k \in I, t \leq a_k\}$ and project the bounded closed faces of it to $\mathbb{R}^n$ by deleting the last coordinate. We thus obtain a subdivision of the Newton polytope [**9**, Section 3.4].

Definition 30. The Newton polytope together with the resulting subdivision is called the *subdivision associated to f* and denoted $Subdiv_f$.

Lemma 31. *The minimum area of a lattice triangle is $\frac{1}{2}$* [**3**, Chapter 4 Section 9].

Definition 32. Let $T(f)$ be an affine or projective tropical curve. We say that $Subdiv_f$ is *maximal* if every cell is a triangle with area $\frac{1}{2}$.

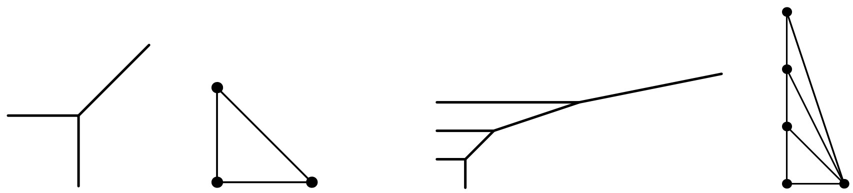
Proposition 33. *An affine or projective tropical curve $T(f)$ is a connected graph with bounded and unbounded edges.*
Let $\mathcal{E}$ denote the set of bounded edges of $T(f)$, let $\mathcal{U}$ denote the set of unbounded edges of $T(f)$ and $\mathcal{V}$ the set of vertices of $T(f)$. Furthermore let $\mathcal{I}$ denote the set of interior edges of $Subdiv_f$, let $\mathcal{D}$ denote the set of boundary edges of $Subdiv_f$ and $\mathcal{C}$ the set of 2-cells of $Subdiv_f$. There are bijections

$$\begin{aligned} \mathcal{E} &\leftrightarrow \mathcal{I} \\ \mathcal{U} &\leftrightarrow \mathcal{D} \\ \mathcal{V} &\leftrightarrow \mathcal{C}. \end{aligned}$$

such that corresponding edges are relatively orthogonal and the edges adjacent to a given vertex correspond to the edges of the cell corresponding to that vertex.

Proof. See [**4**, Section 1.4] for affine curves, [**11**, Proposition 3.5] for projective curves. $\square$

Example 34. Some tropical affine curves together with their subdivisions:



(a) (b)

The dots represent lattice points. Note that curve (a) has degree 1 with full support, while curve (b) has degree 3.

Lemma 35. *Let* $U_i = \left\{(x_1 : \cdots : x_n) \in \mathbb{P}^{n-1}_{tr} \middle| x_i \neq -\infty\right\}$. *Then* $\mathbb{P}^{n-1}_{tr} = \cup_{i=1}^{n} U_i$.

Proposition 36. *For all i in $\{1, \ldots, n\}$ the map*

$$\phi_i : U_i \to \mathbb{R}^{n-1}_{tr} : (x_1 : \cdots : x_i : \cdots : x_n) \mapsto \left(\frac{x_1}{x_i}, \ldots, \frac{\hat{x}_i}{x_i}, \ldots, \frac{x_n}{x_i}\right),$$

where the term $\frac{\hat{x_i}}{x_i}$ is omitted, is a bijection with inverse

$$\phi_i^{-1} : \mathbb{R}^{n-1}_{tr} \to U_i : (x_1, \ldots, x_{n-1}) \mapsto (x_1 : \cdots : x_{i-1} : 0 : x_i : \cdots : x_{n-1}).$$

Definition 37. Let $f = \bigoplus_{k \in I} a_k \odot X_1^{k_1} \odot \ldots X_n^{k_n}$ be a tropical polynomial of degree d in $\mathbb{R}_{tr}[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$. Let x_0 be a point in $\mathbb{R}^n$ such that $\Delta_f + x_0 \subset \Delta_d$ and let $\widetilde{I} = \{y \in \mathbb{Z}^n | y = x + x_0 \text{ for all } x \text{ in } I\}$. The *homogenization of* f is a homogeneous polynomial of degree d in $\mathbb{R}_{tr}[X_1^{\pm 1}, \ldots, X_{n+1}^{\pm 1}]$ given by:

$$\bigoplus_{k \in \widetilde{I}} a_k \odot X_1^{k_1} \odot \cdots \odot X_n^{k_n} \odot (X_{n+1})^{d-k_1-\ldots-k_n}.$$

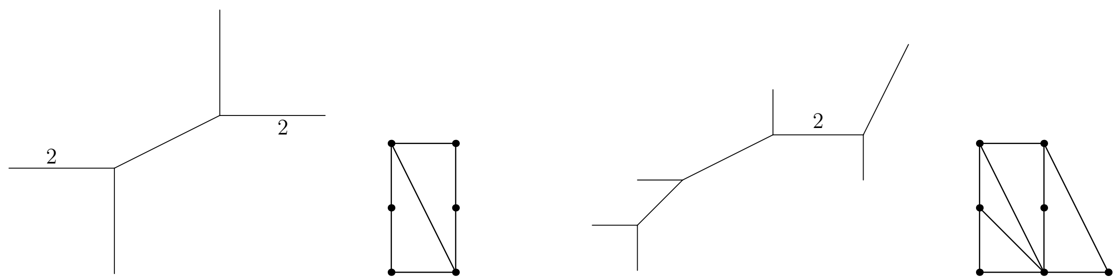
Definition 38. Let $T(f) \subset \mathbb{R}^2_{tr}$ be an affine tropical curve.
For a choice of i in $\{1, 2, 3\}$ we call

$$\phi_i^{-1}(T(f)) \cup \left\{(x_1 : x_2 : x_3) \in \mathbb{P}^2_{tr} \middle| \begin{array}{c} x_i = -\infty \text{ and the homogenization of } f \\ \text{satisfies } P \text{ at } (x_1 : x_2 : x_3) \end{array}\right\}$$

the *projective closure of* $T(f)$.

Definition 39. Let $T(f)$ be an affine tropical curve. Let E be an edge in $T(f)$ and Δ' the corresponding edge in $Subdiv_f$. The *weight* of E is defined as $|\mathbb{Z}^2 \cap \Delta'| - 1$ (i.e. 1 + number of interior lattice points of Δ').
Analogously the *weight* of an edge E of a projective tropical curve $T(f)$ is $|\mathbb{Z}^3 \cap \Delta'| - 1$.

Example 40. Some affine curves together with their subdivisions and with edges labeled with the corresponding weights (we use the convention that edges without label have weight 1):



Definition 41. A subset G of $\mathbb{P}^2_{tr}$ (or $\mathbb{R}^2_{tr}$) is a *weighted rational graph* if it is a connected finite union of rays and segments having rational slopes, rational endpoints and positive weights. Let V be any vertex of a weighted rational graph G and let m be the number of edges adjacent to V. For i=1,..,m let E_i be an edge adjacent to V with weight ω_i and ν_i be a primitive integer vector starting at V and pointing in direction E_i. The weighted rational graph G *satisfies the balancing condition at* V if $\sum_{i=1}^{m} \omega_i \nu_i = 0$.
A weighted rational graph is *balanced* if it satisfies the balancing condition at every vertex.

We now can formulate a purely geometric characterization of tropical curves.

Characterization of tropical curves. *The tropical projective (resp. affine) curves are the balanced rational weighted graphs in $\mathbb{P}^2_{tr}$ (resp. $\mathbb{R}^2_{tr}$). The graph has d unbounded rays counting weights in each coordinate direction if and only if the curve has degree d with full support.*

PROOF. [**9**, Corollary 3.16][**11**, Theorem 3.6] □

2. Tropical intersection theory

From now on we will restrict our attention to projective tropical curves. Whenever we write tropical curve a tropical projective curve will be understood. The theory that we are going to develop can be translated to affine tropical curves via the bijection of Proposition 36.

Definition 42. Let V be the vertex of a tropical projective curve.
The *valence* of V is the number of edges adjacent to V. If V has valence 3 one defines its multiplicity as follows:
let $\omega_1,\ \omega_2,\ \omega_3$ be the weights of the edges and ν_1, ν_2, ν_3 the primitive integer vectors in their direction. The *multiplicity of* V is

$$\omega_1\omega_2 \left|\det\begin{pmatrix}\nu_{11} & \nu_{12} & \nu_{13}\\ \nu_{21} & \nu_{22} & \nu_{23}\\ 1 & 1 & 1\end{pmatrix}\right| \overset{(\star)}{=} \omega_2\omega_3 \left|\det\begin{pmatrix}\nu_{21} & \nu_{22} & \nu_{23}\\ \nu_{31} & \nu_{32} & \nu_{33}\\ 1 & 1 & 1\end{pmatrix}\right| \overset{(\star)}{=} \omega_1\omega_3 \left|\det\begin{pmatrix}\nu_{11} & \nu_{12} & \nu_{13}\\ \nu_{31} & \nu_{32} & \nu_{33}\\ 1 & 1 & 1\end{pmatrix}\right|$$

$(\star)$since $\sum_{i=1}^{3}\omega_i\nu_i = 0$

Remark 43. The multiplicity is a positive integer, since ν_1, ν_2, ν_3 are elements of $\mathbb{Z}^3/_\sim \subset \mathbb{P}^2_{tr}$, where $x \sim y$ iff there exists an a in $\mathbb{Z}$ such that $x = a \odot y$.

Definition 44. A tropical curve is *smooth* if every vertex is 3-valent and has multiplicity 1.

Lemma 45. *A tropical curve $T(f)$ is smooth iff $Subdiv_f$ is maximal.*

In graph theory it is known that the first Betti number of a connected graph with v vertices and e (bounded) edges is $1 - v + e$. Since a tropical curve is a connected graph with bounded and unbounded edges, we can define its genus as follows[**4**]:

Definition 46. The *genus* of a smooth tropical curve with v vertices and e bounded edges is $1 - v + e$.

We have the following geometric characterization for smooth tropical curves:

Lemma 47. *A smooth tropical curve has genus g iff $Subdiv_f$ has g interior lattice points.*

Proposition 48 (Degree-genus formula)**.** *The genus of a smooth tropical curve of degree d with full support is $\frac{1}{2}(d-1)(d-2)$.*

Proof. [**4**, Section 2.2] $\square$

Definition 49. Two tropical curves C and D are said to *intersect transversally* if no vertex of C lies on D and viceversa.
If two tropical curves intersect transversally then we define the intersection multiplicity as follows:
let P be an intersection point and let E_1 and E_2 be the edges meeting at P with weights

respectively ω_1 and ω_2, primitive integer direction vectors ν_1 and ν_2. The *intersection multiplicity at P* is

$$\mu_P = \omega_1\omega_2 \det \begin{pmatrix} \nu_{11} & \nu_{12} & \nu_{13} \\ \nu_{21} & \nu_{22} & \nu_{23} \\ 1 & 1 & 1 \end{pmatrix}$$

this is different from zero 0 since ν_1, ν_2 are elements of $\mathbb{Z}^3/_\sim$ and $\nu_1 \nparallel \nu_2$ as they intersect.

Definition 50. The *translate by $x \in \mathbb{P}^2_{tr}$* of a projective tropical curve $T(f)$ is $\left\{y \in \mathbb{P}^2_{tr} \middle| y = x \odot z, z \in T(f)\right\}$.

Remark 51. If two tropical curves $T(f)$ and $T(f')$ differ by a translation, then $Subdiv_f = Subdiv_{f'}$.

Lemma 52. *Let C and D be tropical curves. Let $C_0 := C$, $D_0 := D$ and for $\epsilon > 0$ let C_ϵ and D_ϵ be nearby translations of C and D such that C_ϵ and D_ϵ intersect transversally. The number N of intersection points, counted with multiplicity, of C_ϵ and D_ϵ is independent of the choice of translations. Furthermore the limit $\lim_{\epsilon\to 0} C_\epsilon \cap D_\epsilon$ is a well-defined subset of N points, counted with multiplicity, of the intersection of C and D.*

Proof. [**11**, theorem 4.3].

□

Definition 53 (Stable intersection)**.** The *stable intersection* of two tropical curves C and D is

$$C \cap_{st} D := \lim_{\epsilon\to 0} C_\epsilon \cap D_\epsilon$$

Theorem 54. [Tropical Bézout] *Let C and D be two tropical curves of degree respectively c and d. If at least one of the curves has full support, then their stable intersection consists of exactly cd points counted with multiplicities.*

Proof. [**2**, Theorem 3.16]

□

Remark 55. If neither of the curves has full support, the conclusion of the theorem is not valid in general [**2**, Example 3.17].

3. Tropical elliptic curves and the algebraic group law

In this section we recall the main results from [**2**, Section 4 and 5].

3.1. Divisors and the Jacobian on a smooth projective tropical curve.

Definition 56. Let C be a smooth projective tropical curve. The *group of divisors on* C is the free abelian group on the set of points of C and an element of $Div(C)$ is a *divisor on* C.
Let x be a divisor on C. Then x is a formal sum $\sum_{P \in C} a_P P$ with a_P in $\mathbb{Z}$ and almost all a_P equal to zero. The sum of the coefficients $\sum_{P \in C} a_P$ is called the *degree of* x.

Lemma 57. *The elements of degree* 0 *in* $Div(C)$ *form a subgroup, denoted by* $Div^0(C)$.

Definition 58. Let f be a tropical homogeneous polynomial such that $T(f)$ has full support. The *divisor associated to* f is the formal sum of points in $C \cap_{st} T(f)$ each counted with the respective intersection multiplicity. It is denoted by $div(f)$. If $T(f)$ and $T(g)$ are projective tropical curves with full support of the same degree, then $div(f) - div(g)$ is called *principal divisor*.

Lemma 59. *Every principal divisor has degree 0.*

Proof. Suppose $div(f) - div(g)$ is a principal divisor and that $T(f)$ and $T(g)$ have degree d with full support and let c be the degree of C. By Bézout's Theorem the cardinality of $C \cap_{st} T(f)$ as well as of $C \cap_{st} T(g)$ is cd, thus the degree of $div(f) - div(g)$ is zero. $\square$

Define the following equivalence relation on $Div(C)$: $D_1 \sim D_2 \iff D_1 - D_2$ is principal.

As in the classical case, one defines the Jacobian $Jac(C)$ of C to be the kernel of the group homomorphism

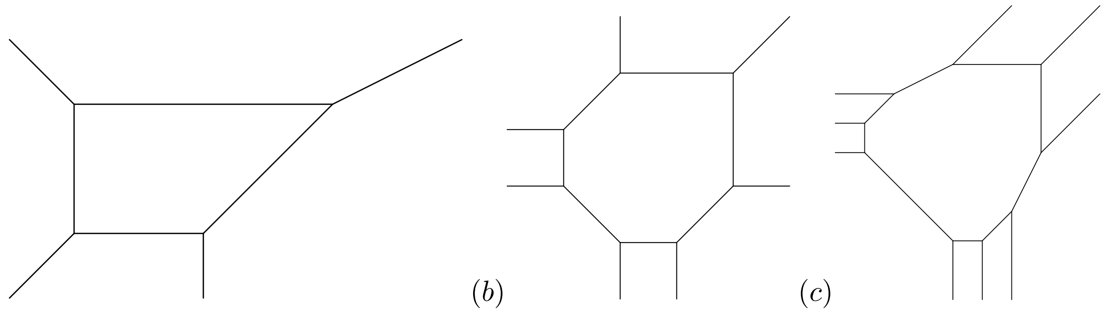
$$deg : Div(C)/_{\sim} \to \mathbb{Z}.$$

Lemma 60. *The Jacobian of* C *is* $Div^0(C)/_{\sim}$.

3.2. Tropical elliptic curves.

Definition 61. A *tropical elliptic curve* is a smooth tropical curve of degree 3 and genus 1.

Example 62. In the picture below some tropical elliptic curves are depicted. Only curve (c) has full support.



(a) (b) (c)

Notation 63. The cycle in C is denoted by $\bar{C}$.

Definition 64. Each connected component of $C \setminus \bar{C}$ is a *tentacle of* C.

Proposition 65. *If P and Q are points on the closure of the same tentacle, then $P \sim Q$.*

PROOF. [**2**, Proposition 5.2]. □

Proposition 66. *Let $\mathcal{O}$ be a fixed point of $\bar{C}$ and for any point $P \in \bar{C}$ let (P) denote the equivalence class of P in ${}^{Div(C)}/_{\sim}$.*
The map $\tau_{\mathcal{O}} : \bar{C} \to {}^{Div^0(C)}/_{\sim} : P \mapsto (P - \mathcal{O})$ is a bijection of sets.

PROOF. [**2**, Lemma 5.4, Proposition 5.5]. □

Thus the cycle $\bar{C}$ has an induced group structure:

$$+_{\bar{C}} : \bar{C} \times \bar{C} \to \bar{C} : (P, Q) \mapsto P +_{\bar{C}} Q := \tau_{\mathcal{O}}^{-1}(\tau_{\mathcal{O}}(P) + \tau_{\mathcal{O}}(Q))$$

where $\tau_{\mathcal{O}}^{-1}$ denotes the inverse of $\tau_{\mathcal{O}}$ and $+$ denotes the addition of $Jac(C)$.
The neutral element with respect to the induced group structure is $\mathcal{O}$.

Remark 67. In classical algebraic geometry the j-invariant of a smooth elliptic curve is an invariant that determines the isomorphism classes of smooth elliptic curves. The correct notion in tropical algebraic geometry seems to be that of the cycle length L of the tropical elliptic curve [**7**].

Proposition 68. *The cycle $\bar{C}$ and the unit circle are isomorphic as groups.*

PROOF. [**2**, Corollary 5.9]. □

4. The geometric group law

In [**2**, Remark 5.8] Vigeland gives a partial description of a geometric group law on $\bar{C}$. In this section we investigate the geometric group law further and prove that the geometric group law and the group law induced from the Jacobian coincide. In the classical case, in order to prove that the chord-tangent group law and the group law induced by the Jacobian coincide, one needs to assume that the neutral element of the group law on the elliptic curve is an inflection point and that the elliptic curve is in Weierstrass form. For our purpose we assume that whenever three points P, Q, R on $\bar{C}$ lie on a tropical line, then $P +_{\bar{C}} Q +_{\bar{C}} R = \mathcal{O}$. This assumption is justified by the following

Proposition 69. *Let $\bar{C}$ be the cycle of a tropical elliptic curve with fixed point $\mathcal{O}$ and let λ denote the isomorphism of Proposition 68. There exists a point $\widetilde{\mathcal{O}}$ on the cycle such that if we replace $\mathcal{O}$ by $\widetilde{\mathcal{O}}$ then for all tropical lines L which intersect $\bar{C}$ in three distinct points P, Q and R:*

$$\lambda(P) + \lambda(Q) + \lambda(R) = 0.$$

Proof. Let $L = T(f)$ and $L' = T(f')$ be any pair of distinct tropical lines which intersect $\bar{C}$ in three distinct points and let $L \cap \bar{C} = \{P, Q, R\}$ and $L' \cap \bar{C} = \{P', Q', R'\}$. Hence $P + Q + R - (P' + Q' + R') = div(f) - div(f')$ and thus $P + Q + R \sim P' + Q' + R'$. Therefore we have

$$(P - \mathcal{O}) + (Q - \mathcal{O}) + (R - \mathcal{O}) = (P' - \mathcal{O}) + (Q' - \mathcal{O}) + (R' - \mathcal{O}) = (S - \mathcal{O})$$

with $S \in \bar{C}$ independent of L and L'. Choose $\widetilde{\mathcal{O}} \in \bar{C}$ with $3(\widetilde{\mathcal{O}} - \mathcal{O}) = (S - \mathcal{O})$. Then

$$(P - \widetilde{\mathcal{O}}) + (Q - \widetilde{\mathcal{O}}) + (R - \widetilde{\mathcal{O}}) = (S - \mathcal{O}) - 3(\widetilde{\mathcal{O}} - \mathcal{O}) = 0.$$

□

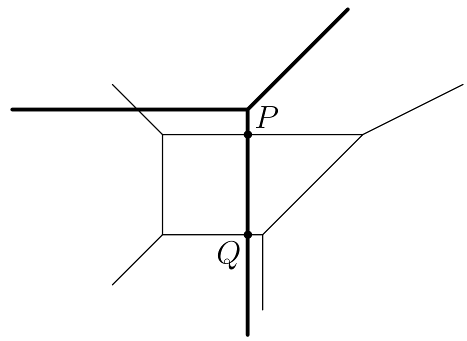
4.1. Description of the geometric addition.

Definition 70. A *tropical line* is a tropical curve of degree 1.

From now on we assume a tropical line to have full support. By Theorem 54 we need to make this assumption in order to describe the geometric group law for a general tropical elliptic curve.

Definition 71. Two points P and Q on $\bar{C}$ are in *general position* if there exists a tropical line intersecting $\bar{C}$ transversally in P and Q and in a third point different from P and Q.

Note that the last condition forbids a situation like the following where the third intersection point lies on a tentacle:



In particular, we require that the vertex of the tropical line lies in the interior of $\bar{C}$.

Let P and Q be points on $\bar{C}$ in general position and let l be the tropical line through P and Q. Denote the third point of intersection by R. Suppose that R and $\mathcal{O}$ are in general position. Let l' be the tropical line through R and $\mathcal{O}$. The third point of intersection of l' with $\bar{C}$ is the geometric sum of P and Q.

Now assume that P and Q are not in general position. We translate P and Q along $\bar{C}$ with constant speed $v(t)$ in opposite directions. We denote the translates of P and Q at time t by P_t and Q_t. Let t be a time at which P_t and Q_t are in general position. A proof of the existence of t is given in the second step of the proof of Lemma 5.4 in [**2**]. Denote the tropical line through P_t and Q_t by l_t.

Lemma 72. *The third point of intersection of l_t with $\bar{C}$ does not depend on t.*

Proof. For some T in $\mathbb{R}/\mathbb{Z}$ we have by construction $\lambda(P_t) = \lambda(P) - T$ and $\lambda(Q_t) = \lambda(Q) + T$, where λ denotes the isomorphism of Proposition 68.

Let R_t denote the third point of intersection of l_t with $\bar{C}$. By our standing assumption

$$P_t +_{\bar{C}} Q_t +_{\bar{C}} R_t = \mathcal{O}$$

and by Proposition 68

$$\lambda(P_t +_{\bar{C}} Q_t +_{\bar{C}} R_t) = \lambda(P_t) + \lambda(Q_t) + \lambda(R_t) = \lambda(\mathcal{O}) = 0$$

Thus

$$\lambda(P_t) + \lambda(Q_t) + \lambda(R_t) = \lambda(P) + \lambda(Q) + \lambda(R_t) = 0$$

hence

$$\lambda(P) + \lambda(Q) = -\lambda(R_t).$$

Now let t' be another time instant, different from t, such that $(P_{t'}, Q_{t'})$ are in general position. Let $R_{t'}$ be the third point of intersection of $l_{t'}$ with $\bar{C}$. By an analogous argument

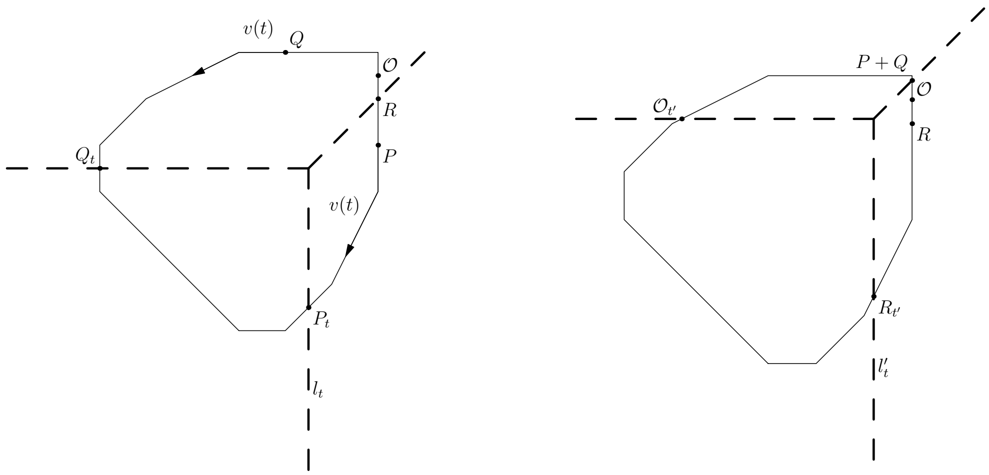
$$\lambda(P) + \lambda(Q) = -\lambda(R_{t'})$$

hence $\lambda(R_t) = \lambda(R_{t'})$, which is equivalent to $R_t = R_{t'} \mathrm{mod} L$, where L denotes the lattice length of the cycle.

□

Let R be the third point of intersection of l_t with $\bar{C}$. If R and $\mathcal{O}$ are in general position the third point of intersection on the line through R and $\mathcal{O}$ is the geometric sum of P and Q. Otherwise translate R and $\mathcal{O}$ as described above to points $R_{t'}$ and $\mathcal{O}_{t'}$ in general position. The third point of intersection of the line through $R_{t'}$ and $\mathcal{O}_{t'}$ with $\bar{C}$ is the geometric sum of P and Q.

Figure 73. The geometric addition.



In order to add a point P to itself, apply the method to add two points not in general position to the pair (P, P).

We denote the geometric addition on $\bar{C}$ by $\boxplus$.

4.2. The group structure induced by the geometric addition.

Proposition 74. *For all points P, Q on $\bar{C}$ the equality*

$$\lambda(P \boxplus Q) = \lambda(P) + \lambda(Q)$$

is satisfied.

Proof. We assume that all pairs of points that we are going to consider are not in general position, since this case can be recovered by choosing the time $t = 0$.
Let P_t and S_t be the translates of P and S in general position and let A be the third point of intersection with the cycle of the tropical line passing through P_t and S_t. By our standing assumption and by Proposition 68

$$\lambda(P) + \lambda(S) + \lambda(A) = \lambda(P_t) + \lambda(S_t) + \lambda(A) = 0 = \lambda(A) + \lambda(\mathcal{O}) + \lambda(P \boxplus S)$$

Now let $(P \boxplus S)_{t'}$ and $R_{t'}$ be the translates of $P \boxplus S$ and R in general position and let B be the third point of intersection of the tropical line through $(P \boxplus S)_{t'}$ and $R_{t'}$ and $\bar{C}$. Then by the same argument

$$\lambda(P \boxplus S)+\lambda(R)+\lambda(B) = \lambda((P \boxplus S)_{t'})+\lambda(R_{t'})+\lambda(B) = 0 = \lambda(B)+\lambda(\mathcal{O})+\lambda((P \boxplus S)\boxplus R)$$

Thus

$$\lambda(P \boxplus S) = \lambda(P) + \lambda(S).$$

$\square$

Theorem 75. *There is a bijection $\eta : \bar{C} \to \bar{C}$ such that for all $P, Q \in \bar{C}$*

$$\eta(P +_{\bar{C}} Q) = \eta(P) \boxplus \eta(Q).$$

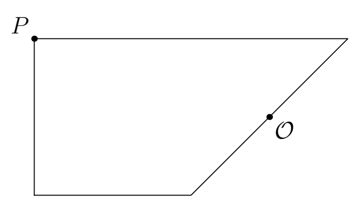
In particular $\bar{C}$ together with the geometric addition is a group with neutral element $\mathcal{O}$ and this group structure is isomorphic to the group structure induced from the Jacobian.

Proof. Immediate by Proposition 68 and Lemma 74. $\square$

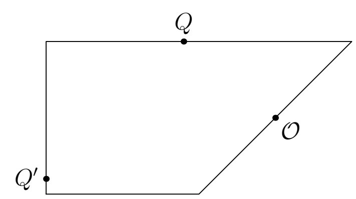
4.3. Torsion points. By Proposition 68 we have the following

Lemma 76. *The subgroup of torsion points on $\bar{C}$ is isomorphic to $\mathbb{Q}/\mathbb{Z}$.*

Example 77. We fix a point $\mathcal{O}$ on the curve (a) from example 62.



The point of order 2.



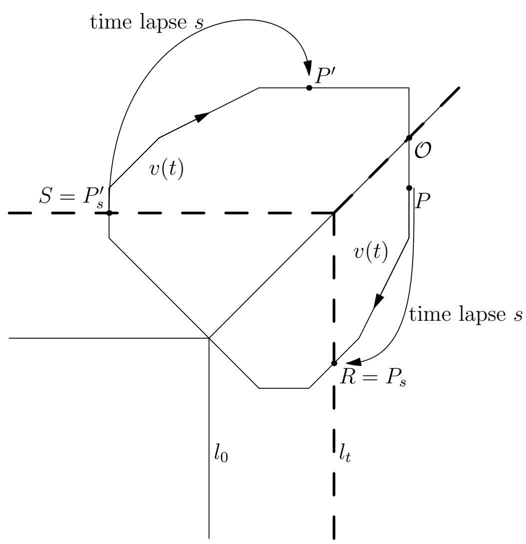
The two points of order 3.

In the following we describe a geometric method to find the inverse of a point which we will then use to give a geometric description of torsion points of order 2. The method is illustrated in Figure 78.

Let P be a point on the cycle $\bar{C}$. Let l be the tropical line, unique up to translation, passing through $\mathcal{O}$ and intersecting the cycle $\bar{C}$ in two other distinct points. Translate l such that it passes through $\mathcal{O}$ and intersects the cycle in only one other point. Denote this translate by l_0 and denote by l_t the translate of l_0 at time t that passes through $\mathcal{O}$. Now suppose that P and $\mathcal{O}$ are not in general position. Let $t > 0$ and let R and S be the other two points of intersection of l_t with $\bar{C}$. Choose one of these points, say R. Consider R as a translate of P and suppose that travelling with constant speed $v(t)$ from P to R we arrive at R after a lapse of time s. Now starting from S travel with constant speed $v(t)$ and stop

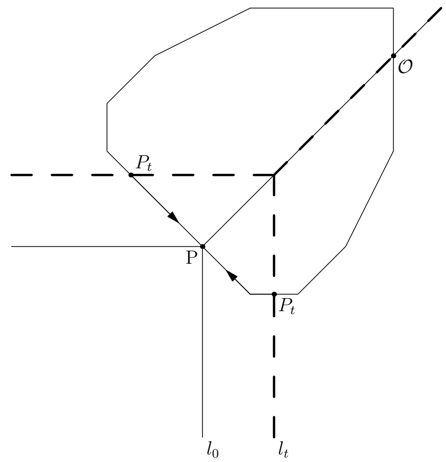
after the lapse of time s. Call the point thus reached P'.
This point is the inverse of P:
By our standing assumption $P'_s +_{\bar{C}} P_s +_{\bar{C}} \mathcal{O} = \mathcal{O}$. Therefore $\lambda(P'_s) = -\lambda(P_s)$ and since $\lambda(P'_s) + \lambda(P_s) = \lambda(P') + \lambda(P)$, we get $\lambda(P' \boxplus P) = \lambda(\mathcal{O})$.

Figure 78. A geometric method to find the inverse of a point.

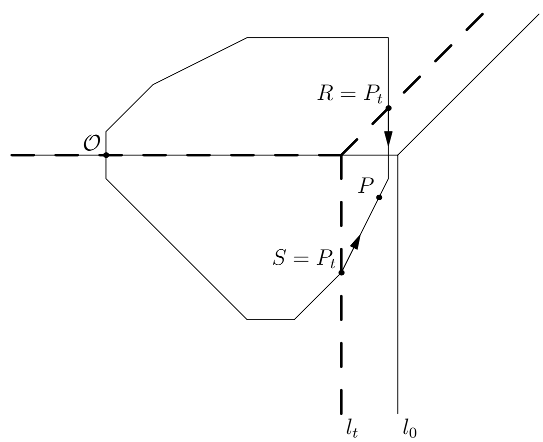


With the same method we can find the points of order 2:
let again R and S be the points of intersection (other than $\mathcal{O}$) of the translate l_t of the tropical line l_0 with $\bar{C}$. We are looking for a point P such that $P \boxplus P = \mathcal{O}$. Start at the same time from R and S and travel with constant speed in direction of the point of intersection other that $\mathcal{O}$ of l_0 with $\bar{C}$. The point at which the two trajectories meet is P.

Figure 79. How to find the points of order 2.



The previous illustration could suggest that the point of order 2 is the point of intersection of l_0 with $\bar{C}$. This is not true in general as illustrated in the following figure:



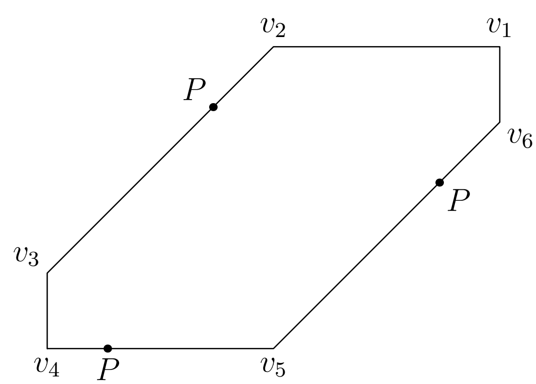
A point P is a torsion point of order 3 if and only if $3P = \mathcal{O}$ if and only if $2P \boxplus P = \mathcal{O}$, which is equivalent to $-\lambda(2P) = \lambda(P)$. Thus we see that there are exactly two points of order 3, namely the point P at lattice distance $L/3$ from $\mathcal{O}$ and the point Q at lattice distance $-L/3$ from $\mathcal{O}$ and that $Q = 2P$. For these points a partial converse to our standing assumption holds: $P, 2P$ and $3P$ add to $\mathcal{O}$ if and only if the translates of two of them and the third point lie on a tropical line.

In classical geometry it is a well-known result that there are exactly nine points of inflection on an elliptic curve. If we choose the neutral element of the group law on an elliptic curve to be an inflection point, then the points of order 3 are exactly the inflection

points. Furthermore in this situation three points of the elliptic curve lie on a line if and only if their sum is zero.
In tropical geometry an analogous notion of inflection point has not been developed until now. However, one can consider the tropicalization[2] of the nine inflection points:
in [**8**, Lemma 4.4.1] it is shown that if we assume the elliptic curve to have a particular form - called *honeycomb form* - such that its tropicalization C is dual to a Newton polygon with triangulation in equilateral triangles and consequently $\bar{C}$ is an hexagon, then the tropicalization of the nine inflection points results in three groups of three points displaced as illustrated in the following lemma.

Lemma [**8**, Lemma 4.4.2] Let $v_1, \ldots, v_6$ denote the vertices of the hexagon in counterclockwise direction and let e_i denote the edge between vertex v_i and v_{i+1}, where $v_7 = v_1$, and call l_i the lattice length of edge e_i. Fix the counterclockwise direction as positive direction. Let P be the tropicalization of an inflection point. Then one of the three following possibilities occurs:
(i) The point P lies at distance $\frac{l_2-l_1}{3}$ from v_2.
(ii) The point P lies at distance $\frac{l_4-l_3}{3}$ from v_4.
(iii) The point P lies at distance $\frac{l_6-l_5}{3}$ from v_6.

Figure 80. The three cases of [**8**, Lemma 4.4.2]



Furthermore for this hexagon [**8**, Section 4.4]

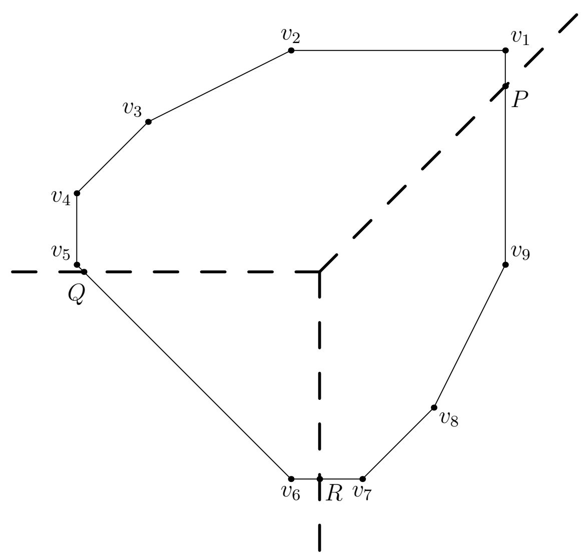
$$l_1 + l_2 = l_4 + l_5 \text{ and } l_2 + l_3 = l_5 + l_6.$$

[2]Tropicalization is a method which gives a connection between classical and tropical algebraic geometry. The tropicalization of an algebraic variety over a non-archimedean field is defined as the closure of its amoeba and a theorem by Kapranov[**6**, 3] states that the closure of the amoeba coincides with the corner locus of the tropicalization of the polynomials defining the algebraic variety. Since many properties of algebraic curves are preserved by tropicalization, this method is widely used to translate algebraic-geometric problems into combinatorial ones, for which a solution may be easier to find. For a short introduction see e.g. [**5**]. A reference for the tropicalization of inflection points on plane curves is [**1**].

With an easy calculation we get that if we choose one of these three points to be the neutral element for the group law, then the torsion points of order 3 are exactly the tropicalization of the inflection points.
In general the tropicalized inflection points do not lie on a tropical line. Restricting our attention to the tropical elliptic curve in Example 62 (c) we may ask in what case the points of order 3 lie on a tropical line. In particular, if we require from three points on the cycle to lie at equal lattice distance from each other and to lie on a tropical line, which possible dispositions do we find?

Let $v_1, \ldots, v_9$ denote the vertices of the cycle in counterclockwise direction and let e_i denote the edge between vertex v_i and v_{i+1}, where $v_{10} = v_9$, and call l_i the lattice length of edge e_i. Fix the counterclockwise direction as positive direction. Let P, Q and R be the three points in question.
With some calculations one finds the following result:
- P lies between v_9 and v_2 at distance $\frac{l_1+2l_9}{3}$ from v_9;
- Q lies between v_3 and v_5 at distance $\frac{l_4+2l_3}{3}$ from v_3;
- R lies between v_6 and v_8 at distance $\frac{l_7+2l_6}{3}$ from v_6.

Figure 81. The three points lying on a tropical line and at equal lattice distance from each other.



In analogy to the classical case we can pose the question of whether and how these points characterize the shape of the tropical elliptic curve. Could such a characterization be found, we might have a translation into tropical geometry of the notion of inflection point.

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