

EPFL and ETHZ Number Theory Days

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Abstracts

COMPUTING THE COEFFICIENTS OF A MODULAR FORM AND RELATED DIOPHANTINE PROBLEMS

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Let

$$\Delta = q \prod_{n \geq 1} (1 - q^n)^{24} = \sum_{n \geq 1} \tau(n) q^n$$

with $q = e^{2\pi iz}$ for $z \in \mathcal{H}$ (the upper half plane). This Δ is a newform of weight 12 for $\mathrm{SL}_2(\mathbb{Z})$. If one has the factorization of n , one can easily compute $\tau(n)$ in terms of the $\tau(p)$ for primes p dividing n . The purpose of these two talks is to describe an algorithm that computes $\tau(p)$ in polynomial time in $\log p$. A first funny application is to compute efficiently the number of vectors of given norm in the Leech lattice of dimension 24.

Deligne has shown the following (in 1969) : For all primes ℓ , there is a unique continuous semisimple representation

$$\rho_\ell: \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \mathrm{GL}(V_\ell)$$

where V_ℓ is a 2-dimensional \mathbb{F}_ℓ -vector space, unramified outside ℓ , such that for all primes $p \neq \ell$,

$$\det(1 - x \mathrm{Frob}_p | V_\ell) = 1 - x\tau(p) + x^2 p^{11}$$

in $\mathbb{F}_\ell[x]$. Later Deligne showed also that $|\tau(p)| < 2p^{11/2}$. (Hecke had proved $|\tau(p)| = O(p^6)$, with an explicit constant, and this will suffice for our purposes.) The image of ρ_ℓ is a transitive subgroup of $\mathrm{GL}(V_\ell)$ except for a few known small values of ℓ . An important property of the Galois module V_ℓ is that it can be embedded into the ℓ -torsion of the jacobian J_ℓ of $X_1(\ell)$:

$$V_\ell = \bigcap_{1 \leq i \leq (\ell^2 - 1)/6} \ker(T_i - \tau(i), J_\ell(\overline{\mathbb{Q}})[\ell]) :$$

We will explain how to compute V_ℓ in time polynomial in ℓ . In fact, we will compute the splitting field of V_ℓ . This number field can be given as an irreducible degree $\ell^2 - 1$ polynomial $P_\ell(x)$ with rational coefficients. This polynomial defines a degree $\ell^2 - 1$ Galois extension of \mathbb{Q} with Galois group the image of ρ_ℓ , a known subgroup of $\mathrm{GL}(V_\ell)$.

A crucial and difficult point will be to give a bound B_ℓ for the logarithmic height of P_ℓ . Arakelov theory applied to modular curves produces such a bound that is a polynomial in ℓ .

Given such a polynomial $P_\ell(x)$ and a prime integer p , algorithmic algebraic number theory provides an algorithm that tells which element in the Galois group is the Frobenius at p . This algorithm runs in time polynomial in the degree $\ell^2 - 1$ and logarithmic height B_ℓ of P_ℓ . Since B_ℓ is bounded by a polynomial in ℓ , this results in an algorithm that computes $\tau(p)$ modulo ℓ in time polynomial in ℓ . Given a prime p , we can run this algorithm for several small primes ℓ . If the product of these ℓ is bigger than $4p^{\frac{11}{2}}$, we can recover the actual value of $\tau(p)$ thanks to Chinese remainder theorem and Deligne bound.

There remains to explain how to compute $V_\ell \subset J_1(\ell)[\ell]$. Let g_ℓ be the genus of $X_1(\ell)$. Fix a degree g_ℓ rational divisor Ω on $X_1(\ell)$. A class x in V_ℓ is represented by a divisor $D_x - \Omega$ where D_x is effective of degree d . One asks how to compute such torsion divisors.

A first method computes a complex approximation of D_x for every non-zero $x \in V_\ell$. This gives a complex approximation of the (rational) coefficients of P_ℓ . Since we have a good bound on the height of these coefficients, we deduce their exact values in the field \mathbb{Q} .

Another method uses q -adic approximations instead of complex approximations.

A reference for this work is

<http://arxiv.org/abs/math/0605244>

by Bas Edixhoven, Jean-Marc Couveignes, Robin de Jong, Franz Merkl, Johan Bosman.