(Asymptotic) integer partitions and random matrix theory

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Motivation

Let

$$Z_U(\theta) = \prod_{j=1}^{N} \left(1 - e^{\mathrm{i}(\theta_j - \theta)}\right)$$

be the characteristic polynomial of $U \in \mathrm{U}(N)$, with eigenvalues $\left\{e^{\mathrm{i}\theta_j}\right\}$

Conjecture (Keating-Snaith)

$$\lim_{T\to\infty} \frac{1}{T(\log T)^{k^2}} \int_0^T \left|\zeta(1/2 + \mathrm{i}t)\right|^{2k} \,\mathrm{d}t = a_k g_{2k},$$

with a_k a (known) product over primes and

$$g_{2k} = \lim_{N\to\infty} \frac{1}{N^{k^2}} \int_{\mathrm{U}(N)} \left|Z_U(0)\right|^{2k} \mathrm{d}U = \lim_{N\to\infty} \frac{\left\langle \left|Z_U(0)\right|^{2k} \right\rangle_{\mathrm{U}(N)}}{N^{k^2}}.$$

“$\zeta(\frac{1}{2} + \mathrm{i}t)$ is modeled here by characteristic polynomials of unitary matrices, under Haar measure.”

Motivation (II)

Theorem (Hall) If we know asymptotics in T for

$$\int_0^T \left|\zeta(1/2+\mathrm{i}t)\right|^{2k-2h} \left|\zeta'(1/2+\mathrm{i}t)\right|^{2h} \,\mathrm{d}t,$$

with $0 \leq h \leq k$, then lower bounds on gaps between zeroes of ζ can be obtained.

That is, he can give Λ such that

$$\Lambda \leq \limsup_{n\to\infty} \frac{t_{n+1}-t_n}{\frac{2\pi}{\log t_n}},$$

with t_n the imaginary part of the nth nontrivial zero of ζ.

This method manages to give bounds that are unconditional (not even on RH!). It only depends on knowledge of the moments, not on the RMT that follows.

Motivation (III)

Conjecture (Hughes) For $0 \le h \le k$, with $k, h \in \mathbb{N}$,

$$\lim_{T\to\infty} \frac{1}{T(\log T)^{k^2+2h}} \int_0^T \left|\zeta\left(\frac{1}{2}+\mathrm{i}t\right)\right|^{2k-2h} \left|\zeta'\left(\frac{1}{2}+\mathrm{i}t\right)\right|^{2h} \mathrm{d}t = a_k \tilde{g}(2k,2h),$$

with a_k a (known) product over primes and

$$\tilde{g}(2k,2h) := \lim_{N\to\infty} \frac{1}{N^{k^2+2h}} \left\langle |Z_U(0)|^{2k} \left|\frac{Z'_U(0)}{Z_U(0)}\right|^{2h} \right\rangle_{\mathrm{U}(N)}.$$

We are curious about the structure of the $\tilde{g}(2k,2h)$, which we study first through the

$$g(2k,r) := \lim_{N\to\infty} \frac{1}{N^{k^2+r}} \left\langle |Z_U(0)|^{2k} \left(\frac{Z'_U(0)}{Z_U(0)}\right)^{r} \right\rangle_{\mathrm{U}(N)},$$

for $0 \le r \le 2h$ (because $\mathfrak{Im}\frac{Z'_U(0)}{Z_U(0)} = -\frac{N}{2}$).

Results

Theorem For each $r \in \mathbb{N}$, we have

$$(2\imath)^r \frac{g(2k,r)}{g(2k,0)} = \sum_{\mu \vdash r} m(\mu) \frac{P_\mu(k)}{Q_\mu(k)},$$

which makes clear some properties guessed by Hughes.

There exist polynomials $q_n(r)$ such that for all $r \in \mathbb{N}$, the asymptotic expansion in $\frac{1}{k}$ of $(2\imath)^r \frac{g(2k,r)}{g(2k,0)}$ is

$$1 + q_1(r)\frac{1}{k^2} + q_2(r)\frac{1}{k^4} + \cdots$$

Moreover, much information is known about the $q_n(r)$. For instance, they have leading term $\frac{1}{n!}\left(\frac{-1}{8}\right)^n$ and a closed form expression is obtained (efficiently).

Illustration

With $u = 4k^2$, starting at $r = 0$, $\frac{g(2k,r)}{g(2k,0)}$ equals

$$1, 1, \frac{u-2}{u-1}, \frac{u-4}{u-1}, \frac{u^2-16u+66}{u^2-10u+9}, \frac{u^2-20u+114}{u^2-10u+9},$$

$$\frac{u^4-51u^3+864u^2-5554u+4860}{u^4-36u^3+294u^2-484u+225}, \frac{u^4-57u^3+1134u^2-8758u+8520}{u^4-36u^3+294u^2-484u+225},$$

$$\frac{u^5-113u^4+4620u^3-86332u^2+682844u-765660}{u^5-85u^4+2058u^3-14890u^2+23941u-11025},$$

$$\frac{u^5-121u^4+5460u^3-115564u^2+1053964u-1457820}{u^5-85u^4+2058u^3-14890u^2+23941u-11025},$$

$$\frac{u^7-220u^6+18897u^5-831010u^4+20196928u^3-260164440u^2+1428629724u-2060092440}{u^7-175u^6+10437u^5-262075u^4+2864323u^3-13020525u^2+18445239u-8037225},$$

$$\frac{u^7-230u^6+20997u^5-996820u^4+26447168u^3-374214600u^2+2270621484u-3994446960}{u^7-175u^6+10437u^5-262075u^4+2864323u^3+-13020525u^2+18445239u-8037225},$$

$$\frac{u^9-363u^8+52929u^7-4083011u^6+183649422u^5-4906031274u^4+73323636100u^3+\cdots}{u^9-297u^8+31908u^7-1556564u^6+36100350u^5-394179006u^4+1953532372u^3+\cdots},$$

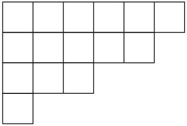
$$\frac{u^9-375u^8+57141u^7-4663655u^6+224398746u^5-6467410170u^4+105010072036u^3+\cdots}{u^9-297u^8+31908u^7-1556564u^6+36100350u^5-394179006u^4+1953532372u^3+\cdots},$$

$$\frac{u^{11}-582u^{10}+141344u^9-18977780u^8+1571817537u^7-84339778978u^6+2962887441370u^5+\cdots}{u^{11}-491u^{10}+93751u^9-9001541u^8+472885066u^7-13974129806u^6+230948238286u^5+\cdots},$$

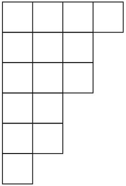
r	$g(2k,r)/g(2k,0)$, with $u=4k^2$					
0	1					
1	1					
2	1	$-\frac{1}{u}$	$-\frac{1}{u^2}$	$-\frac{1}{u^3}$	$-\frac{1}{u^4}$	$+O\left(\frac{1}{u^5}\right)$
3	1	$-\frac{3}{u}$	$-\frac{3}{u^2}$	$-\frac{3}{u^3}$	$-\frac{3}{u^4}$	$+O\left(\frac{1}{u^5}\right)$
4	1	$-\frac{6}{u}$	$-\frac{3}{u^2}$	$+\frac{24}{u^3}$	$+\frac{267}{u^4}$	$+O\left(\frac{1}{u^5}\right)$
5	1	$-\frac{10}{u}$	$+\frac{5}{u^2}$	$+\frac{140}{u^3}$	$+\frac{1355}{u^4}$	$+O\left(\frac{1}{u^5}\right)$
6	1	$-\frac{15}{u}$	$+\frac{30}{u^2}$	$+\frac{420}{u^3}$	$+\frac{3675}{u^4}$	$+O\left(\frac{1}{u^5}\right)$
7	1	$-\frac{21}{u}$	$+\frac{84}{u^2}$	$+\frac{924}{u^3}$	$+\frac{6699}{u^4}$	$+O\left(\frac{1}{u^5}\right)$
8	1	$-\frac{28}{u}$	$+\frac{182}{u^2}$	$+\frac{1652}{u^3}$	$+\frac{7847}{u^4}$	$+O\left(\frac{1}{u^5}\right)$
9	1	$-\frac{36}{u}$	$+\frac{342}{u^2}$	$+\frac{2484}{u^3}$	$+\frac{1287}{u^4}$	$+O\left(\frac{1}{u^5}\right)$
10	1	$-\frac{45}{u}$	$+\frac{585}{u^2}$	$+\frac{3105}{u^3}$	$-\frac{23040}{u^4}$	$+O\left(\frac{1}{u^5}\right)$
11	1	$-\frac{55}{u}$	$+\frac{935}{u^2}$	$+\frac{2915}{u^3}$	$-\frac{79750}{u^4}$	$+O\left(\frac{1}{u^5}\right)$
12	1	$-\frac{66}{u}$	$+\frac{1419}{u^2}$	$+\frac{924}{u^3}$	$-\frac{187176}{u^4}$	$+O\left(\frac{1}{u^5}\right)$
r	1	$-\frac{r(r-1)}{2u}$	$\frac{r(r-1)(r^2-5r+2)}{8u^2}$	$\cdots$	$\cdots$	

Definitions

A partition λ is a nonincreasing sequence of integers $(\lambda_1, \cdots, \lambda_{l(\lambda)}, 0, 0, \cdots)$. The length $l(\lambda)$ is the number of strictly positive values and the weigth $|\lambda|$ is the sum $\sum \lambda_i$ of the values in the sequence.

We identify a partition with its Young diagram. The diagram of $(6,5,3,1)$, for instance, is  .

A natural operation on diagram is transposition (or conjugation): $(6,5,3,1)^{\mathrm{t}} = (4,3,3,2,2,1)$, with diagram

 .

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Definitions (II)

The Schur polynomials are symmetric polynomials in countably many variables. They are defined in many different, equivalent, ways and are indexed by partitions. Jacobi-Trudi:

$$\mathfrak{s}_\lambda(\overline{X}) = \det \left| \mathfrak{e}_{\lambda^{\mathfrak{t}}_i + i - j}(\overline{X}) \right|_{M \times M},$$

where $\mathfrak{e}_r$ is the elementary symmetric polynomial of degree r ($\mathfrak{e}_0 = 1$) and M is large. This can be expressed simply using $\mathfrak{e}_\mu := \prod_i \mathfrak{e}_{\mu_i}$.

Definitions (II)

They can also be seen as polynomials in finitely many variables, and then form compatible families:

$$\mathfrak{s}_\lambda(x_1, \cdots, x_N) = \mathfrak{s}_\lambda(x_1, \cdots, x_N, 0) = \mathfrak{s}_\lambda(x_1, \cdots, x_N, 0, 0, \cdots).$$

There is a reduction property:

$$\mathfrak{s}_\lambda(x_1, \cdots, x_N) = 0 \text{ if } l(\lambda) > N.$$

Apart from those cases, we get irreducible characters of $\mathrm{U}(N)$ (with the convention $\mathfrak{s}_\lambda(U) := \mathfrak{s}_\lambda(e^{\mathrm{i}\theta_1}, \cdots, e^{\mathrm{i}\theta_N})$), and, under Haar measure,

$$\left\langle \mathfrak{s}_\lambda(U)\overline{\mathfrak{s}_\mu(U)} \right\rangle_{\mathrm{U}(N)} = \begin{cases} \delta_{\lambda\mu} & \text{if } N \geq l(\lambda) \\ 0 & \text{if } l(\lambda) > N \end{cases}.$$

"For N large, the $\mathfrak{s}_\lambda$ are orthonormal on $\mathrm{U}(N)$."

The Bump-Gamburd method

Bump and Gamburd compute

$$\left\langle \left|Z_U(0)\right|^{2k} \right\rangle_{\mathrm{U}(N)}$$

via the dual Cauchy identity.
Let $\overline{X} = \{x_m : m \in M\}$, $\overline{Y} = \{y_n : n \in N\}$. Then,

$$\sum_{\lambda} \mathfrak{s}_\lambda(\overline{X}) \mathfrak{s}_{\lambda^t}(\overline{Y}) = \prod_{m,n}^{M,N} (1 + x_m y_n).$$

We plug in eigenvalues of U, and copies of 1 and get

$$\sum_{\lambda} \mathfrak{s}_\lambda\left(\{1\}^k\right) \mathfrak{s}_{\lambda^t}(U) \quad = \quad \det(\mathrm{Id} + U)^k$$

or, replacing U by $-U$

$$Z_U(0)^k \quad = \quad \sum_{\lambda} (-1)^{|\lambda|} \mathfrak{s}_\lambda\left(\{1\}^k\right) \mathfrak{s}_{\lambda^t}(U).$$

The Bump-Gamburd method (II)

$$|Z_U(0)|^{2k} = \sum_{\lambda,\mu}(-1)^{|\lambda|+|\mu|}\mathfrak{s}_\lambda\left(\{1\}^k\right)\mathfrak{s}_\mu\left(\{1\}^k\right)\mathfrak{s}_{\lambda^t}(U)\overline{\mathfrak{s}_{\mu^t}(U)}.$$

So

$$\left\langle|Z_U(0)|^{2k}\right\rangle_{\mathrm{U}(N)} = \sum_{\substack{\lambda \\ \lambda_1\leq k}}\mathfrak{s}_\lambda\left(\{1\}^k\right)\mathfrak{s}_\lambda\left(\{1\}^k\right),$$

which can be evaluated in a combinatorial way, and gives multiple different expressions, including analytic continuations.

Evaluation

0	1	2	3	4	5
−1	0	1	2	3	
−2	−1	0			
−3					

9	7	6	4	3	1
7	5	4	2	1	
4	2	1			
1					

Figure: The contents and hooklengths of $(6, 5, 3, 1)$.

We need mostly here the following hook-content formula:

$$\begin{aligned} \mathfrak{s}_\lambda\left(\{1\}^K\right) &= \prod_{\square\in\lambda} \frac{K + c(\square)}{\mathrm{H}(\square)} \\ &= \frac{K \uparrow \lambda}{H(\lambda)} \end{aligned}$$

The notation $K \uparrow \lambda$ is meant to generalize the Pochhammer symbol. Note that this is a polynomial in K of degree $|\lambda|$.

Evaluation (II)

$$\begin{aligned} \mathfrak{s}_\lambda\left(\{1\}^K\right) &= \prod_{\square\in\lambda} \frac{K+c(\square)}{\mathrm{H}(\square)} \\ &= \frac{K\uparrow\lambda}{H(\lambda)} \end{aligned}$$

- We recover that $\mathfrak{s}_\lambda$ vanishes when $K \leq l(\lambda)$.
- There are several possible groupings of boxes (rows, columns, hooks).
- $(-k)\uparrow\lambda = (-1)^{|\lambda|}k\uparrow\lambda^{\mathrm{t}}$

Moments of derivatives

Back to our original problem, computing

$$(2\mathrm{i})^r \left\langle |Z_U(0)|^{2k} \left(\frac{Z_U'(0)}{Z_U(0)} \right)^r \right\rangle_{\mathrm{U}(N)}$$

(reminder: $Z_U(\theta) = \prod_{j=1}^N \left(1 - e^{\mathrm{i}(\theta_j - \theta)}\right)$), which will lead us up to what we really want later.

A theorem of Okounkov and Olshanski

Generalized binomial formula Let $\mathfrak{s}^*_\mu$ be the "shifted Schur functions", then

$$\frac{\mathfrak{s}_\lambda(1+a_1,\cdots,1+a_n)}{\mathfrak{s}_\lambda(\{1\}^n)} = \sum_{\substack{\mu \\ l(\mu)\leq n}} \frac{\mathfrak{s}^*_\mu(\lambda_1,\cdots,\lambda_n)\mathfrak{s}_\mu(a_1,\cdots,a_n)}{n\uparrow\mu}.$$

(The case $n=1$ is Newton's binomial theorem).

We also know that these admit nice expressions. For instance,

$$\mathfrak{s}^*_\mu(\{N\}^k) \quad = \quad (-1)^{|\mu|}\frac{(-N\uparrow\mu)(k\uparrow\mu)}{H(\mu)},$$

and so $\mathfrak{s}_{\langle N^k\rangle}(1+a_1,\cdots,1+a_n)$ can be expressed as a sum

over partitions.

"Taylor series of a Schur function at the identity"

Via Okounkov-Olshanski, we get by an extension of Bump-Gamburd

Proposition: For $0 \leq r \leq 2k$,

$$\lim_{N\to\infty} \frac{1}{N^r}(2\mathrm{i})^r \frac{\left\langle |Z_U(0)|^{2k} \left(\frac{Z_U'(0)}{Z_U(0)}\right)^r \right\rangle_{\mathrm{U}(N)}}{\left\langle |Z_U(0)|^{2k} \right\rangle_{\mathrm{U}(N)}} = \sum_{\mu\vdash r} \frac{r!}{H(\mu)^2} \frac{2^r(k\uparrow\mu)}{(2k)\uparrow\mu},$$

a rational function of k.

The term $\frac{r!}{H(\mu)^2}$ determines a probability measure on partitions of r, called the Plancherel measure.

As a consequence, the limit of the RHS when $k \to \infty$ is 1.

We have thus proved the first column (and the odd ones...)

r	$g(2k,r)/g(2k,0)$, with $u=4k^2$					
0	1					
1	1					
2	1	$-\frac{1}{u}$	$-\frac{1}{u^2}$	$-\frac{1}{u^3}$	$-\frac{1}{u^4}$	$+O\left(\frac{1}{u^5}\right)$
3	1	$-\frac{3}{u}$	$-\frac{3}{u^2}$	$-\frac{3}{u^3}$	$-\frac{3}{u^4}$	$+O\left(\frac{1}{u^5}\right)$
4	1	$-\frac{6}{u}$	$-\frac{3}{u^2}$	$+\frac{24}{u^3}$	$+\frac{267}{u^4}$	$+O\left(\frac{1}{u^5}\right)$
5	1	$-\frac{10}{u}$	$+\frac{5}{u^2}$	$+\frac{140}{u^3}$	$+\frac{1355}{u^4}$	$+O\left(\frac{1}{u^5}\right)$
6	1	$-\frac{15}{u}$	$+\frac{30}{u^2}$	$+\frac{420}{u^3}$	$+\frac{3675}{u^4}$	$+O\left(\frac{1}{u^5}\right)$
7	1	$-\frac{21}{u}$	$+\frac{84}{u^2}$	$+\frac{924}{u^3}$	$+\frac{6699}{u^4}$	$+O\left(\frac{1}{u^5}\right)$
8	1	$-\frac{28}{u}$	$+\frac{182}{u^2}$	$+\frac{1652}{u^3}$	$+\frac{7847}{u^4}$	$+O\left(\frac{1}{u^5}\right)$
r	1	$-\frac{r(r-1)}{2u}$	$\frac{r(r-1)(r^2-5r+2)}{8u^2}$	$\dots$	$\dots$	

Via Okounkov-Olshanski, we get by an extension of Bump-Gamburd

Proposition: For $0 \le r \le 2k$,

$$\lim_{N\to\infty} \frac{1}{N^r}(2\mathrm{i})^r \frac{\left\langle |Z_U(0)|^{2k} \left(\frac{Z_U'(0)}{Z_U(0)}\right)^r \right\rangle_{\mathrm{U}(N)}}{\left\langle |Z_U(0)|^{2k} \right\rangle_{\mathrm{U}(N)}} = \sum_{\mu\vdash r} \frac{r!}{H(\mu)^2} \frac{2^r (k\uparrow\mu)}{(2k)\uparrow\mu},$$

a rational function of k.

The term $\frac{r!}{H(\mu)^2}$ determines a probability measure on partitions of r, called the Plancherel measure.

As a consequence, the limit of the RHS when $k\to\infty$ is 1.

Plancherel measure

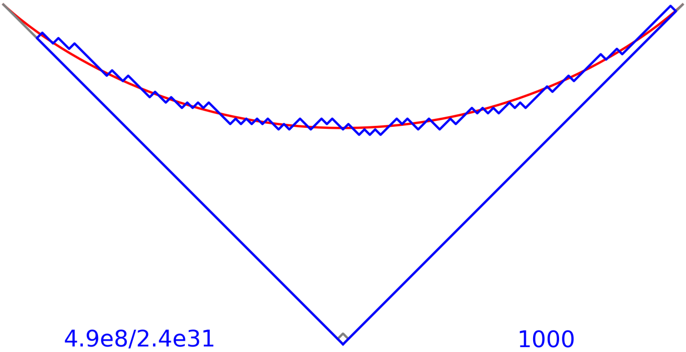
$$\dim\lambda := \dim\chi_\lambda^{\mathcal{S}_{|\lambda|}} = \frac{|\lambda|!}{H(\lambda)} = \#\text{paths to } \lambda \text{ in Young graph.}$$

Let

$$m(\lambda) = \frac{\dim(\lambda)^2}{|\lambda|!} = \frac{|\lambda|!}{H(\lambda)^2},$$

so that

$$\sum_{\lambda \text{ with } |\lambda| = \text{cst}} m(\lambda) = 1.$$

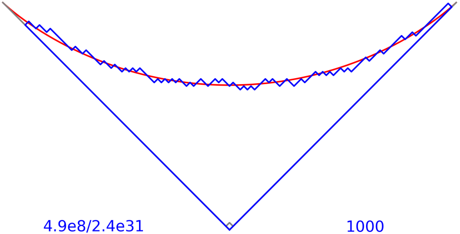


Large partitions under the Plancherel measure tend to all look the same after renormalization $\tilde{\lambda}(u) := \frac{1}{\sqrt{r}}\lambda(ru)$ (when $\lambda \vdash r$), in the sense that

$$\lim_{r\to\infty} \sum_{\lambda\vdash r} m(\lambda) \int_{-\infty}^{\infty} \tilde{\lambda}(u) g(u)\,\mathrm{d}u = \int_{-\infty}^{\infty} \Omega(u) g(u)\,\mathrm{d}u,$$

with $\Omega(u) = \frac{2}{\pi}(x \arcsin \frac{x}{2} + \sqrt{4 - x^2})$ when $|x| \leq 2$.

Kerov's central limit theorem



Set $U_p(u)$ the p^{th} Chebyshev polynomial (2nd kind) on $[-2,2]$ and define

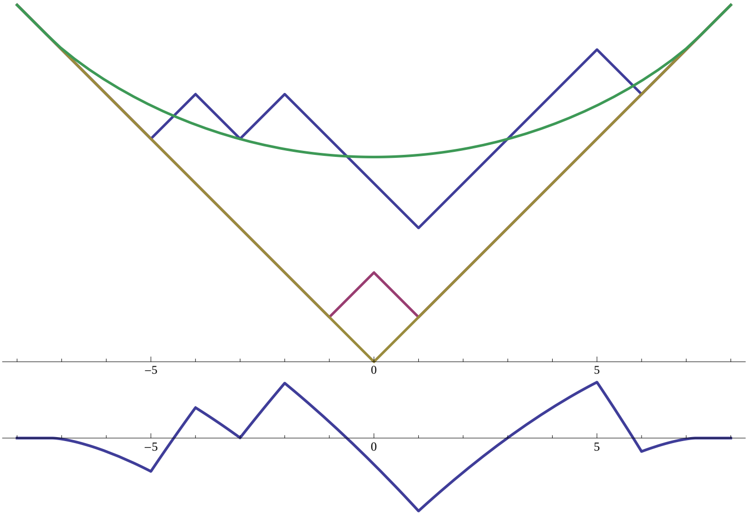
$$\Delta^{\lambda}(u)=\frac{\sqrt{r}}{2}\left(\tilde{\lambda}(u)-\Omega(u)\right), \quad \alpha_p^{\lambda}=\int_{\mathbb{R}} U_p(u)\Delta^{\lambda}(u)\,\mathrm{d}u.$$

Then,

$$\left\{\alpha_p^{(r)}\right\}_{p\geq 1} \xrightarrow{d} \left\{\frac{\xi_{p+1}}{\sqrt{p+1}}\right\}_{k\geq 1} \text{ as } r\to\infty,$$

with ξ_p iid $\mathcal{N}(0,1)$.

Kerov's central limit theorem (II)



movie...

Asymptotic information

Proposition. For $0 \leq r \leq 2k$,

$$\lim_{N\to\infty} \frac{1}{N^r}(2\mathrm{i})^r \frac{\left\langle |Z_U(0)|^{2k} \left(\frac{Z_U'(0)}{Z_U(0)}\right)^r \right\rangle_{\mathrm{U}(N)}}{\left\langle |Z_U(0)|^{2k} \right\rangle_{\mathrm{U}(N)}} = \sum_{\lambda \vdash r} m(\lambda) \frac{2^r (k \uparrow \lambda)}{(2k) \uparrow \lambda}.$$

If we expand this into series in $\frac{1}{k}$ using integrals against the outlines, we can use asymptotic information on partitions to determine in

$$1\frac{1}{k^0} + f_1(r)\frac{1}{k^2} + f_2(r)\frac{1}{k^4} + \cdots$$

the behaviour of the f_i for large r.

This is a refinement of a method presented in *Seiberg-Witten Theory and Random Partitions* (Nekrasov-Okounkov).

We have thus found the highest order term for each polynomial on the last row.

r	$g(2k,r)/g(2k,0)$, with $u=4k^2$					
0	1					
1	1					
2	1	$-\frac{1}{u}$	$-\frac{1}{u^2}$	$-\frac{1}{u^3}$	$-\frac{1}{u^4}$	$+O\left(\frac{1}{u^5}\right)$
3	1	$-\frac{3}{u}$	$-\frac{3}{u^2}$	$-\frac{3}{u^3}$	$-\frac{3}{u^4}$	$+O\left(\frac{1}{u^5}\right)$
4	1	$-\frac{6}{u}$	$-\frac{3}{u^2}$	$+\frac{24}{u^3}$	$+\frac{267}{u^4}$	$+O\left(\frac{1}{u^5}\right)$
5	1	$-\frac{10}{u}$	$+\frac{5}{u^2}$	$+\frac{140}{u^3}$	$+\frac{1355}{u^4}$	$+O\left(\frac{1}{u^5}\right)$
6	1	$-\frac{15}{u}$	$+\frac{30}{u^2}$	$+\frac{420}{u^3}$	$+\frac{3675}{u^4}$	$+O\left(\frac{1}{u^5}\right)$
7	1	$-\frac{21}{u}$	$+\frac{84}{u^2}$	$+\frac{924}{u^3}$	$+\frac{6699}{u^4}$	$+O\left(\frac{1}{u^5}\right)$
8	1	$-\frac{28}{u}$	$+\frac{182}{u^2}$	$+\frac{1652}{u^3}$	$+\frac{7847}{u^4}$	$+O\left(\frac{1}{u^5}\right)$
r	1	$-\frac{r(r-1)}{2u}$	$\frac{r(r-1)(r^2-5r+2)}{8u^2}$	$\ldots$	$\ldots$	

$$1, -\frac{r^2}{8k^2}, \frac{r^4}{2\cdot 8^2\cdot k^4}, -\frac{r^6}{6\cdot 8^3\cdot k^6, \cdots}$$

Stanley's theorem

Theorem (Stanley, 2009) We know $\langle \mathfrak{e}_\mu(\overline{X}) : \mu \rangle = \Lambda(\overline{X})$. Then,

$$\sum_{\lambda \vdash r} m(\lambda) \mathfrak{e}_\mu \left(\{ c(\square) : \square \in \lambda \} \right) = \sum_j A_\mu(j) \binom{r}{j},$$

with the $A_\mu(j)$ explicit and the sum over j effectively finite.

We compute

$$\frac{2^r (k \uparrow \lambda)}{(2k) \uparrow \lambda} = \sum_{\substack{\mu \\ \text{hook}}} - \left(-\frac{1}{2} \right)^{\mu_1} \mathfrak{s}_\mu(\{ c(\square) : \square \in \lambda \}) \frac{1}{k^{|\mu|}}$$

$$= \sum_\mu w_\mu \mathfrak{e}_\mu \left(\{ c(\square) : \square \in \lambda \} \right) \frac{1}{k^{|\mu|}},$$

so we can figure out the w_μ (using J-T), and obtain

$$\sum_{\lambda \vdash r} m(\lambda) \frac{2^r (k \uparrow \lambda)}{(2k) \uparrow \lambda} = \sum_\mu w_\mu \sum_j A_\mu(j) \binom{r}{j} \frac{1}{k^{|\mu|}}.$$

Summary

Remember

$$g(2k,r) := \lim_{N\to\infty} \frac{1}{N^{k^2+r}} \left\langle |Z_U(0)|^{2k} \left(\frac{Z_U'(0)}{Z_U(0)}\right)^r \right\rangle_{\mathrm{U}(N)},$$

r	$g(2k,r)/g(2k,0)$, with $u = 4k^2$					
0	1					
1	1					
2	1	$-\frac{1}{u}$	$-\frac{1}{u^2}$	$-\frac{1}{u^3}$	$-\frac{1}{u^4}$	$+O\left(\frac{1}{u^5}\right)$
3	1	$-\frac{3}{u}$	$-\frac{3}{u^2}$	$-\frac{3}{u^3}$	$-\frac{3}{u^4}$	$+O\left(\frac{1}{u^5}\right)$
4	1	$-\frac{6}{u}$	$-\frac{3}{u^2}$	$+\frac{24}{u^3}$	$+\frac{267}{u^4}$	$+O\left(\frac{1}{u^5}\right)$
5	1	$-\frac{10}{u}$	$+\frac{5}{u^2}$	$+\frac{140}{u^3}$	$+\frac{1355}{u^4}$	$+O\left(\frac{1}{u^5}\right)$
6	1	$-\frac{15}{u}$	$+\frac{30}{u^2}$	$+\frac{420}{u^3}$	$+\frac{3675}{u^4}$	$+O\left(\frac{1}{u^5}\right)$
7	1	$-\frac{21}{u}$	$+\frac{84}{u^2}$	$+\frac{924}{u^3}$	$+\frac{6699}{u^4}$	$+O\left(\frac{1}{u^5}\right)$
8	1	$-\frac{28}{u}$	$+\frac{182}{u^2}$	$+\frac{1652}{u^3}$	$+\frac{7847}{u^4}$	$+O\left(\frac{1}{u^5}\right)$
r	1	$-\frac{r(r-1)}{2u}$	$\frac{r(r-1)(r^2-5r+2)}{8u^2}$	$\cdots$	$\cdots$	

Real moments

Theorem (Hall) If we know asymptotics in T for

$$\int_0^T |\zeta(1/2+\mathrm{i}t)|^{2k-2h} \left|\zeta'(1/2+\mathrm{i}t)\right|^{2h} \,\mathrm{d}t,$$

with $0 \le h \le k$, then lower bounds on gaps between zeroes of ζ can be obtained.

Could be translated to information about similar moments for $\mathcal{Z}$, the Hardy function, with

$$\mathcal{Z}(t) = \sqrt{\chi\left(\frac{1}{2}-\mathrm{i}t\right)}\zeta\left(\frac{1}{2}+\mathrm{i}t\right) = e^{\mathrm{i}\vartheta(t)}\zeta\left(\frac{1}{2}+\mathrm{i}t\right)$$

and

$$\vartheta(t) = \mathfrak{Im}\log\left(\pi^{-\mathrm{i}t/2}\Gamma\left(\frac{1}{4}+\frac{1}{2}\mathrm{i}t\right)\right).$$

Warning: We have $|\mathcal{Z}(t)| = |\zeta(\frac{1}{2}+\mathrm{i}t)|$ but $|\mathcal{Z}'(t)| \neq |\zeta'(\frac{1}{2}+\mathrm{i}t)|$.

Real moments

Let

$$V_U(\theta) = e^{iN(\theta+\pi)/2} e^{-i\sum_j \theta_j/2} Z_U(\theta).$$

Then $V_U(\theta)$ is a real function of θ and $|V_U(\theta)| = |Z_U(\theta)|$, which will model $\mathcal{Z}$.

Conjecture (Hughes)

$$\lim_{T\to\infty} \frac{1}{T(\log T)^{k^2+2h}} \int_0^T \left|\mathcal{Z}\left(\frac{1}{2}+\mathrm{i}t\right)\right|^{2k-2h} \left|\mathcal{Z}'\left(\frac{1}{2}+\mathrm{i}t\right)\right|^{2h} \mathrm{d}t = a_k \hat{g}(2k,2h),$$

with a_k a (known) product over primes and

$$\hat{g}(2k,2h) = \lim_{N\to\infty} \frac{1}{N^{k^2+2h}} \left\langle |V_U(0)|^{2k-2h} |V_U'(0)|^{2h} \right\rangle_{\mathrm{U}(N)}.$$

Getting the $\hat{g}(2k,2h)$ now involves only summing linear combinations with binomial coefficients of the $g(2k,r)$.

Remember

$$(2\mathrm{i})^r \frac{g(2k,r)}{g(2k,0)} = \sum_{\lambda \vdash r} m(\lambda) \frac{2^r (k \uparrow \lambda)}{(2k) \uparrow \lambda} = \sum_{\mu} w_\mu \sum_j A_\mu(j) \binom{r}{j} \frac{1}{k^{|\mu|}}.$$

This will even simply the final answer for the $\hat{g}(2k,2h)$!